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Duality for mixed second-order programs with multiple arguments

N. Kailey[a], S. K. Gupta[b]

[a]*Department of Applied Sciences, Gulzar Institute of Engineering and Technology, Khanna-141 401, India*
[b]*Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee-800 013, India Ph.: +91-133-2285379*

Abstract. The duality theory is well-developed for non linear programs. Technically, a major part of such broad framework possibly be extended to mixed non linear programs, however this has demonstrated complicated, in minority as the duality theory does not integrate well with modern computational practice. In this paper, we constructed a new pair of second-order multiobjective mixed dual problems over arbitrary cones with multiple arguments, with an eye towards developing a more practical framework. Weak, strong and converse duality theorems are then established under K-η-bonvexity assumptions. Several known results are obtained as special cases.

1 Introduction

The term duality as used in our daily life means the sort of harmony of two opposite or complementary parts through which they integrate into a whole. Symmetry and inner beauty in natural phenomena are bound up with duality and, in particular, are significant in art and science. Mathematics lies at the root of duality. In the present day, the theory of duality has become a vast subject, especially due to the modern work in optimization, game theory, economic science, theoretical physics and chemistry, mathematical programming, variational analysis, nonconvex-nonsmooth analysis and control, critical point theory and in many other areas.

Mangasarian [10] introduced the concept of second-order duality for nonlinear problems. Its study is significant due to computational advantage over first-order duality as it provides tighter bounds for the value of the objective function when approximations are used. Gulati et al. [5] formulated Wolfe and Mond-Weir type second-order multiobjective symmetric dual problems over arbitrary cones and proved usual duality results under η-bonvexity/η-pseudobonvexity assumptions. Later on, Ahmad and Husain [2] points out certain omissions and inconsistencies in the earlier work of Mishra [11] and Mishra and Wang [12].

Mixed duality is a fruitful result in traditional mathematical programming and is very useful both theoretically and practically. Consequently, it quite interesting to extend the mixed duality theory to the case of multiobjective second-order programs over arbitrary cones with multiple arguments. Not surprisingly, it is difficult to develop a dual problem for multiobjective mixed second-order nonlinear programs with properties similar to those observed in the Linear Programming case. Unlike the Linear Programming

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Email addresses: kaileynavdeep21@gmail.com (N. Kailey), skgiitr@gmail.com (S. K. Gupta)

case, dual information is not easy to extract from the most commonly employed primal solution algorithms.

Xu [15] obtained usual duality theorems for mixed type duals in multiobjective fractional programming. Bector et al. [3] and Yang et al. [16] formulated mixed symmetric multiobjective differentiable and single objective nondifferentiable programming problems and then proved duality results for that programming problems. Ahmad and Husain [1] studied a pair of multiobjective mixed symmetric dual programs over arbitrary cones and established duality results under K-preinvexity/K-pseudoinvexity assumptions. Recently, Gupta and Kailey [6] achieved duality results for multiobjective second-order nondifferentiable mixed symmetric dual problems under second-order F-convexity/pseudoconvexity assumptions.

This paper is organized as follows: In Section 2, we introduce several notations of mixed duality that have appeared in the literature. In Section 3, we discuss the dual formulation of mixed second-order nonlinear programs with multiple arguments over arbitrary cones and proved the appropriate duality theorems under K-η-bonvexity assumptions. In Section 4, some special cases of our study is discussed. Finally, in Section 5, we talk about future research in this area.

2 Notations and preliminaries

Before beginning, we briefly introduce some terminology and notation.

Throughout this paper, let $N = \{1, 2, \ldots, n\}$ and $M = \{1, 2, \ldots, m\}$, let $J_1 \subseteq N, K_1 \subseteq M$ and $J_2 = N \backslash J_1$ and $K_2 = M \backslash K_1$. Let $|J_1|$ denote the number of elements in J_1. The other symbols $|J_2|, |K_1|$ and $|K_2|$ are defined similarly. Let $x^1 \in R^{|J_1|}, x^2 \in R^{|J_2|}$. Then any $x \in R^n$ can be written as (x^1, x^2). Similarly for $y^1 \in R^{|K_1|}, y^2 \in R^{|K_2|}, y \in R^m$ can be written as (y^1, y^2). It may be noted here that if $J_1 = \emptyset$, then $|J_1| = 0, J_2 = N$ and therefore $|J_2| = n$. In this case, $R^{|J_1|}, R^{|J_2|}$ and $R^{|J_1|} \times R^{|K_1|}$ will be zero-dimensional, n-dimensional and $|K_1|$-dimensional Euclidean spaces, respectively. The other situations $J_2 = \emptyset, K_1 = \emptyset$ or $K_2 = \emptyset$ can be interpret similarly.

Consider the following multiobjective programming problem:

$$\text{(P)} \qquad K\text{-minimize} \quad \phi(x)$$

$$\text{subject to} \quad x \in X^0 = \{x \in S : -g(x) \in Q\}$$

where $S \subseteq R^n$, $\phi : S \to R^k$, $g : S \to R^m$, K is closed convex pointed cone in R^k with int $K \neq \phi$ and Q is closed convex cone with nonempty interior in R^m.

In contrast to single-objective optimization, a solution to a multi-objective problem is more of a concept than a definition. Typically, there is no single global solution, and it is often necessary to determine a set of points that all fit a predetermined definition for an optimum. A number of techniques have been developed to find a compromise solution to multiobjective optimization problems. The optimality concepts in multiobjective programming are weakly efficient and efficient solution.

Definition 1 [9, 13] A point $\bar{x} \in X^0$ is a weakly efficient solution of (P) if there exists no other $x \in X^0$ such that

$$\phi(\bar{x}) - \phi(x) \in \text{int}\, K.$$

Definition 2 [9] A point $\bar{x} \in X^0$ is an efficient solution of (P) if there exists no other $x \in X^0$ such that

$$\phi(\bar{x}) - \phi(x) \in K \backslash \{0\}.$$

We can now define positive polar cone and K-η-bonvex function.

Let C_1 and C_2 be closed convex cones with nonempty interiors in R^n and R^m, respectively.

Definition 3 [9, 13] The positive polar cone C_i^* of C_i $(i = 1, 2)$ is defined as

$$C_i^* = \{z : x^T z \geqq 0, \text{ for all } x \in C_i\}.$$

Suppose that $S_1 \subseteq R^n$ and $S_2 \subseteq R^m$ are open sets such that $C_1 \times C_2 \subset S_1 \times S_2$.

Definition 4 [4] A twice differentiable function $f : S_1 \times S_2 \mapsto R^k$ is said to be K-η_1-bonvex in the first variable at $u \in S_1$ for fixed $v \in S_2$, if there exists a function $\eta_1 : S_1 \times S_1 \mapsto R^n$ such that for $x \in S_1$, $\zeta_i \in R^n$, $i = 1, 2, \ldots, k$,

$$\{f_1(x, v) - f_1(u, v) + \tfrac{1}{2}\zeta_1^T \nabla_{xx} f_1(u, v)\zeta_1 - \eta_1^T(x, u)[\nabla_x f_1(u, v) + \nabla_{xx} f_1(u, v)\zeta_1], \ldots,$$

$$f_k(x, v) - f_k(u, v) + \tfrac{1}{2}\zeta_k^T \nabla_{xx} f_k(u, v)\zeta_k - \eta_1^T(x, u)[\nabla_x f_k(u, v) + \nabla_{xx} f_k(u, v)\zeta_k]\} \in K,$$

and $f(x, y)$ is said to be K-η_2-bonvex in the second variable at $v \in S_2$ for fixed $u \in S_1$, if there exists a function $\eta_2 : S_2 \times S_2 \mapsto R^m$ such that for $y \in S_2$, $\varphi_i \in R^m$, $i = 1, 2, \ldots, k$,

$$\{f_1(u, y) - f_1(u, v) + \tfrac{1}{2}\varphi_1^T \nabla_{yy} f_1(u, v)\varphi_1 - \eta_2^T(y, v)[\nabla_y f_1(u, v) + \nabla_{yy} f_1(u, v)\varphi_1], \ldots,$$

$$f_k(u, y) - f_k(u, v) + \tfrac{1}{2}\varphi_k^T \nabla_{yy} f_k(u, v)\varphi_k - \eta_2^T(y, v)[\nabla_y f_k(u, v) + \nabla_{yy} f_k(u, v)\varphi_k]\} \in K.$$

3 Mixed multiobjective second-order symmetric dual programs

Now we consider the following pair of multiobjective mixed second-order symmetric dual programs over arbitrary cones:

Primal problem (RP)

K-minimize

$$L(x_1, x_2, y_1, y_2, \lambda, p, r) = \Big[L_1\big(x_1, x_2, y_1, y_2, \lambda, p, r\big), L_2\big(x_1, x_2, y_1, y_2, \lambda, p, r\big), \ldots, L_k\big(x_1, x_2, y_1, y_2, \lambda, p, r\big)\Big]$$

subject to

$$-\sum_{i=1}^{k} \lambda_i \Big[\nabla_{y_1} f_i(x_1, y_1) + \nabla_{y_1 y_1} f_i(x_1, y_1) p_i\Big] \in C_3^*, \quad (1)$$

$$-\sum_{i=1}^{k} \lambda_i \Big[\nabla_{y_2} g_i(x_2, y_2) + \nabla_{y_2 y_2} g_i(x_2, y_2) r_i\Big] \in C_4^*, \quad (2)$$

$$(y_2)^T \sum_{i=1}^{k} \lambda_i \Big[\nabla_{y_2} g_i(x_2, y_2) + \nabla_{y_2 y_2} g_i(x_2, y_2) r_i\Big] \geqq 0, \quad (3)$$

$$\lambda^T e_k = 1, \ \lambda \in \text{int } K^*, \quad (4)$$

$$x_1 \in C_1, \ x_2 \in C_2. \quad (5)$$

Dual problem (RD)

K-maximize

$$M(u_1, u_2, v_1, v_2, \lambda, q, s) = \Big[M_1\big(u_1, u_2, v_1, v_2, \lambda, q, s\big), M_2\big(u_1, u_2, v_1, v_2, \lambda, q, s\big),$$

$$\ldots, M_k\Big(u_1, u_2, v_1, v_2, \lambda, q, s\Big)\Big]$$

subject to
$$\sum_{i=1}^{k} \lambda_i\Big[\nabla_{x_1} f_i(u_1, v_1) + \nabla_{x_1x_1} f_i(u_1, v_1)q_i\Big] \in C_1^*, \quad (6)$$

$$\sum_{i=1}^{k} \lambda_i\Big[\nabla_{x_2} g_i(u_2, v_2) + \nabla_{x_2x_2} g_i(u_2, v_2)s_i\Big] \in C_2^*, \quad (7)$$

$$(u_2)^T \sum_{i=1}^{k} \lambda_i\Big[\nabla_{x_2} g_i(u_2, v_2) + \nabla_{x_2x_2} g_i(u_2, v_2)s_i\Big] \leqq 0, \quad (8)$$

$$\lambda^T e_k = 1,\ \lambda \in \text{int}\, K^*, \quad (9)$$

$$v_1 \in C_3,\ v_2 \in C_4, \quad (10)$$

where $\lambda = \Big(\lambda_1, \lambda_2, \ldots, \lambda_k\Big)^T \in R^k$ and for $i = 1, 2, \ldots, k,$

$$L_i\Big(x_1, x_2, y_1, y_2, \lambda, p, r\Big) = \Big[f_i(x_1, y_1) - (y_1)^T \sum_{i=1}^{k} \lambda_i\Big[\nabla_{y_1} f_i(x_1, y_1) + \nabla_{y_1y_1} f_i(x_1, y_1)p_i\Big]$$
$$- \tfrac{1}{2} \sum_{i=1}^{k} \lambda_i\Big[p_i^T \nabla_{y_1y_1} f_i(x_1, y_1)p_i\Big] + g_i(x_2, y_2) - \tfrac{1}{2} r_i^T \nabla_{y_2y_2} g_i(x_2, y_2)r_i\Big]$$

$$M_i\Big(u_1, u_2, v_1, v_2, \lambda, q, s\Big) = \Big[f_i(u_1, v_1) - (u_1)^T \sum_{i=1}^{k} \lambda_i\Big[\nabla_{x_1} f_i(u_1, v_1) + \nabla_{x_1x_1} f_i(u_1, v_1)q_i\Big]$$
$$- \tfrac{1}{2} \sum_{i=1}^{k} \lambda_i\Big[q_i^T \nabla_{x_1x_1} f_i(u_1, v_1)q_i\Big] + g_i(u_2, v_2) - \tfrac{1}{2} s_i^T \nabla_{x_2x_2} g_i(u_2, v_2)s_i\Big]$$

(*i*) $f_i : R^{|J_1|} \times R^{|K_1|} \to R$ and $g_i : R^{|J_2|} \times R^{|K_2|} \to R$ are differentiable functions, $e_k = (1, \ldots, 1)^T \in R^k$,
(*ii*) $p_i \in R^{|K_1|}$, $r_i \in R^{|K_2|}$, $q_i \in R^{|J_1|}$ and $s_i \in R^{|J_2|}$,
(*iii*) $p = \Big(p_1, p_2, \ldots, p_k\Big) \in R^k$, $r = \Big(r_1, r_2, \ldots, r_k\Big) \in R^k$, $q = \Big(q_1, q_2, \ldots, q_k\Big) \in R^k$ and $s = \Big(s_1, s_2, \ldots, s_k\Big) \in R^k$.

Theorem 1 (Weak duality). *Let $(x_1, x_2, y_1, y_2, \lambda, p, r)$ be feasible for (RP) and $(u_1, u_2, v_1, v_2, \lambda, q, s)$ be feasible for (RD). Let*

(*i*) *$f(., v_1)$ be K-η_1-bonvex in the first variable at u_1 and $-f(x_1, .)$ be K-η_2-bonvex in the second variable at y_1,*
(*ii*) *$\{g_1(., v_2), g_2(., v_2), \ldots, g_k(., v_2)\}$ be K-η_3-bonvex in the first variable at u_2 and $-\{g_1(x_2, .), g_2(x_2, .), \ldots, g_k(x_2, .)\}$ be K-η_4-bonvex in the second variable at y_2,*
(*iii*) *$\eta_1(x_1, u_1) + u_1 \in C_1$, for all $x_1 \in C_1$ and $\eta_2(v_1, y_1) + y_1 \in C_3$, for all $v_1 \in C_3$,*
(*iv*) *$\eta_3(x_2, u_2) + u_2 \in C_2$, for all $x_2 \in C_2$ and $\eta_4(v_2, y_2) + y_2 \in C_4$, for all $v_2 \in C_4$,*

Then
$$L(x_1, x_2, y_1, y_2, \lambda, p, r) - M(u_1, u_2, v_1, v_2, \lambda, q, s) \notin -K\backslash\{0\}.$$

Proof Suppose, to the contrary, that

$$L(x_1, x_2, y_1, y_2, \lambda, p, r) - M(u_1, u_2, v_1, v_2, \lambda, q, s) \in -K\backslash\{0\}.$$

that is,

$$\Big\{\Big[L_1\Big(x_1, x_2, y_1, y_2, \lambda, p, r\Big), L_2\Big(x_1, x_2, y_1, y_2, \lambda, p, r\Big), \ldots,$$
$$L_k\Big(x_1, x_2, y_1, y_2, \lambda, p, r\Big)\Big] - \Big[M_1\Big(u_1, u_2, v_1, v_2, \lambda, q, s\Big),$$
$$M_2\Big(u_1, u_2, v_1, v_2, \lambda, q, s\Big), \ldots, M_k\Big(u_1, u_2, v_1, v_2, \lambda, q, s\Big)\Big]\Big\} \in -K\backslash\{0\}.$$

Since $\lambda \in \text{int}\, K^*$ and $\lambda \neq 0$, we obtain

$$\sum_{i=1}^{k}\lambda_i\Big\{\Big[f_i(x_1,y_1)-(y_1)^T\sum_{i=1}^{k}\lambda_i(\nabla_{y_1}f_i(x_1,y_1)+\nabla_{y_1y_1}f_i(x_1,y_1)p_i)$$
$$-\tfrac{1}{2}\sum_{i=1}^{k}\lambda_i(p_i^T\nabla_{y_1y_1}f_i(x_1,y_1)p_i)+g_i(x_2,y_2)-\tfrac{1}{2}r_i^T\nabla_{y_2y_2}g_i(x_2,y_2)r_i\Big]$$
$$-\Big[f_i(u_1,v_1)-(u_1)^T\sum_{i=1}^{k}\lambda_i(\nabla_{x_1}f_i(u_1,v_1)+\nabla_{x_1x_1}f_i(u_1,v_1)q_i)$$
$$-\tfrac{1}{2}\sum_{i=1}^{k}\lambda_i(q_i^T\nabla_{x_1x_1}f_i(u,v)q_i)+g_i(u_2,v_2)-\tfrac{1}{2}s_i^T\nabla_{x_2x_2}g_i(u_2,v_2)s_i\Big]\Big\}<0. \quad (11)$$

By K-η_1-bonvexity of $f(.,v_1)$ in the first variable at u_1 and K-η_2-bonvexity of $-f(x_1,.)$ in the second variable at y_1, we have

$$\Big\{f_1(x_1,v_1)-f_1(u_1,v_1)+\tfrac{1}{2}q_1^T\nabla_{x_1x_1}f_1(u_1,v_1)q_1-\eta_1^T(x_1,u_1)[\nabla_{x_1}f_1(u_1,v_1)$$
$$+\nabla_{x_1x_1}f_1(u_1,v_1)q_1],\ldots,f_k(x_1,v_1)-f_k(u_1,v_1)+\tfrac{1}{2}q_k^T\nabla_{x_1x_1}f_k(u_1,v_1)q_k$$
$$-\eta_1^T(x_1,u_1)[\nabla_{x_1}f_k(u_1,v_1)+\nabla_{x_1x_1}f_k(u_1,v_1)q_k]\Big\}\in K,$$

and

$$\Big\{f_1(x_1,y_1)-f_1(x_1,v_1)-\tfrac{1}{2}p_1^T\nabla_{y_1y_1}f_1(x_1,y_1)p_1+\eta_2^T(v_1,y_1)[\nabla_{y_1}f_1(x_1,y_1)$$
$$+\nabla_{y_1y_1}f_1(x_1,y_1)p_1],\ldots,f_k(x_1,y_1)-f_k(x_1,v_1)-\tfrac{1}{2}p_k^T\nabla_{y_1y_1}f_k(x_1,y_1)p_k$$
$$+\eta_2^T(v_1,y_1)[\nabla_{y_1}f_k(x_1,y_1)+\nabla_{y_1y_1}f_k(x_1,y_1)p_k]\Big\}\in K.$$

Using $\lambda\in \text{int}\, K^*$, we get

$$\sum_{i=1}^{k}\lambda_i\Big\{f_i(x_1,v_1)-f_i(u_1,v_1)+\tfrac{1}{2}q_i^T\nabla_{x_1x_1}f_i(u_1,v_1)q_i$$
$$-\eta_1^T(x_1,u_1)[\nabla_{x_1}f_i(u_1,v_1)+\nabla_{x_1x_1}f_i(u_1,v_1)q_i]\Big\}\geqq 0, \quad (12)$$

and

$$\sum_{i=1}^{k}\lambda_i\Big\{f_i(x_1,y_1)-f_i(x_1,v_1)-\tfrac{1}{2}p_i^T\nabla_{y_1y_1}f_i(x_1,y_1)p_i$$
$$+\eta_2^T(v_1,y_1)[\nabla_{y_1}f_i(x_1,y_1)+\nabla_{y_1y_1}f_i(x_1,y_1)p_i]\Big\}\geqq 0. \quad (13)$$

Since $(x_1,x_2,y_1,y_2,\lambda,p,r)$ is feasible for the primal problem (RP) and $(u_1,u_2,v_1,v_2,$

$\lambda,q,s)$ is feasible for (RD), by the dual constraint (6) and so from the hypothesis (*iii*), it follows that

$$\Big[\eta_1(x_1,u_1)+u_1\Big]^T\sum_{i=1}^{k}\lambda_i\Big[\nabla_{x_1}f_i(u_1,v_1)+\nabla_{x_1x_1}f_i(u_1,v_1)q_i\Big]\geqq 0,$$

which implies

$$\eta_1^T(x_1,u_1)\sum_{i=1}^{k}\lambda_i\Big[\nabla_{x_1}f_i(u_1,v_1)+\nabla_{x_1x_1}f_i(u_1,v_1)q_i\Big]$$

$$\geqq -(u_1)^T \sum_{i=1}^{k} \lambda_i \Big[\nabla_{x_1} f_i(u_1, v_1) + \nabla_{x_1x_1} f_i(u_1, v_1) q_i\Big]. \quad (14)$$

Similarly

$$-\eta_2^T(v_1, y_1) \sum_{i=1}^{k} \lambda_i \Big[\nabla_{y_1} f_i(x_1, y_1) + \nabla_{y_1y_1} f_i(x_1, y_1) p_i\Big]$$
$$\geqq (y_1)^T \sum_{i=1}^{k} \lambda_i \Big[\nabla_{y_1} f_i(x_1, y_1) + \nabla_{y_1y_1} f_i(x_1, y_1) p_i\Big]. \quad (15)$$

Using (14) in (12) and (15) in (13), we have

$$\sum_{i=1}^{k} \lambda_i \Big[f_i(x_1, v_1) - f_i(u_1, v_1) + \tfrac{1}{2} q_i^T \nabla_{x_1x_1} f_i(u_1, v_1) q_i\Big]$$
$$\geqq -(u_1)^T \sum_{i=1}^{k} \lambda_i \Big[\nabla_{x_1} f_i(u_1, v_1) + \nabla_{x_1x_1} f_i(u_1, v_1) q_i\Big], \quad (16)$$

and

$$\sum_{i=1}^{k} \lambda_i \Big[f_i(x_1, y_1) - f_i(x_1, v_1) - \tfrac{1}{2} p_i^T \nabla_{y_1y_1} f_i(x_1, y_1) p_i\Big]$$
$$\geqq (y_1)^T \sum_{i=1}^{k} \lambda_i \Big[\nabla_{y_1} f_i(x_1, y_1) + \nabla_{y_1y_1} f_i(x_1, y_1) p_i\Big]. \quad (17)$$

Adding the above inequalities and using $\lambda^T e_k = 1$, we obtain

$$\sum_{i=1}^{k} \lambda_i \Big\{\Big[f_i(x_1, y_1) - (y_1)^T \sum_{i=1}^{k} \lambda_i \Big[\nabla_{y_1} f_i(x_1, y_1) + \nabla_{y_1y_1} f_i(x_1, y_1) p_i\Big]$$
$$- \tfrac{1}{2} \sum_{i=1}^{k} \lambda_i \Big[p_i^T \nabla_{y_1y_1} f_i(x_1, y_1) p_i\Big]\Big] - \Big[f_i(u_1, v_1) - (u_1)^T \sum_{i=1}^{k} \lambda_i \Big[\nabla_{x_1} f_i(u_1, v_1)$$
$$+ \nabla_{x_1x_1} f_i(u_1, v_1) q_i\Big] - \tfrac{1}{2} \sum_{i=1}^{k} \lambda_i \Big[q_i^T \nabla_{x_1x_1} f_i(u_1, v_1) q_i\Big]\Big]\Big\} \geqq 0. \quad (18)$$

By K-η_3-bonvexity of $\{g_1(., v_2), g_2(., v_2), \ldots, g_k(., v_2)\}$ in the first variable at u_2 and K-η_4-bonvexity of $-\{g_1(x_2, .), g_2(x_2, .), \ldots, g_k(x$ in the second variable at y_2, we have

$$\Big\{g_1(x_2, v_2) - g_1(u_2, v_2) + \tfrac{1}{2}(s_1)^T \nabla_{x_2x_2} g_1(u_2, v_2) s_1 - \eta_3^T(x_2, u_2)\Big[\nabla_{x_2} g_1(u_2, v_2)$$
$$+ \nabla_{x_2x_2} g_1(u_2, v_2) s_1\Big], \ldots, g_k(x_2, v_2) - g_k(u_2, v_2) + \tfrac{1}{2}(s_k)^T \nabla_{x_2x_2} g_k(u_2, v_2) s_k$$
$$- \eta_3^T(x_2, u_2)\Big[\nabla_{x_2} g_k(u_2, v_2) + \nabla_{x_2x_2} g_k(u_2, v_2) s_k\Big]\Big\} \in K$$

and

$$\Big\{g_1(x_2, y_2) - g_1(x_2, v_2) - \tfrac{1}{2}(r_1)^T \nabla_{y_2y_2} g_1(x_2, y_2) r_1 + \eta_4^T(v_2, y_2)\Big[\nabla_{y_2} g_1(x_2, y_2)$$
$$+ \nabla_{y_2y_2} g_1(x_2, y_2) r_1\Big], \ldots, g_k(x_2, y_2) - g_k(x_2, v_2) - \tfrac{1}{2}(r_k)^T \nabla_{y_2y_2} g_k(x_2, y_2) r_k$$
$$+ \eta_4^T(x_2, y_2)\Big[\nabla_{y_2} g_k(x_2, y_2) + \nabla_{y_2y_2} g_k(x_2, y_2) r_k\Big]\Big\} \in K.$$

It follows from $\lambda \in \operatorname{int} K^*$ that

$$\sum_{i=1}^{k} \lambda_i \Big[g_i(x_2, v_2) - g_i(u_2, v_2) + \tfrac{1}{2}(s_i)^T \nabla_{x_2x_2} g_i(u_2, v_2) s_i - \eta_3^T(x_2, u_2) \Big[\nabla_{x_2} g_i(u_2, v_2) + \nabla_{x_2x_2} g_i(u_2, v_2) s_i \Big] \Big] \geqq 0, \quad (19)$$

and

$$\sum_{i=1}^{k} \lambda_i \Big[g_i(x_2, y_2) - g_i(x_2, v_2) - \tfrac{1}{2}(r_i)^T \nabla_{y_2y_2} g_i(x_2, y_2) r_i + \eta_4^T(x_2, y_2) \Big[\nabla_{y_2} g_i(x_2, y_2) + \nabla_{y_2y_2} g_i(x_2, y_2) r_i \Big] \Big] \geqq 0. \quad (20)$$

By hypothesis (iv) and the dual constraint (7), we obtain

$$\Big[\eta_3(x_2, u_2) + u_2 \Big]^T \sum_{i=1}^{k} \lambda_i \Big[\nabla_{x_2} g_i(u_2, v_2) + \nabla_{x_2x_2} g_i(u_2, v_2) s_i \Big] \geqq 0,$$

which implies

$$\eta_3^T(x_2, u_2) \sum_{i=1}^{k} \lambda_i \Big[\nabla_{x_2} g_i(u_2, v_2) + \nabla_{x_2x_2} g_i(u_2, v_2) s_i \Big] \geqq -(u_2)^T \sum_{i=1}^{k} \lambda_i \Big[\nabla_{x_2} g_i(u_2, v_2) + \nabla_{x_2x_2} g_i(u_2, v_2) s_i \Big] \geqq 0. \text{ (By the dual constraint (8))} \quad (21)$$

Using (21) in (19), we have

$$\sum_{i=1}^{k} \lambda_i \Big[g_i(x_2, v_2) - g_i(u_2, v_2) + \tfrac{1}{2}(s_i)^T \nabla_{x_2x_2} g_i(u_2, v_2) s_i \Big] \geqq 0. \quad (22)$$

Similarly, from (20) and using hypothesis (iv) along with the primal constraints (2) and (3), we get

$$\sum_{i=1}^{k} \lambda_i \Big[g_i(x_2, y_2) - g_i(x_2, v_2) - \tfrac{1}{2}(r_i)^T \nabla_{y_2y_2} g_i(x_2, y_2) r_i \Big] \geqq 0. \quad (23)$$

Adding equations (22) and (23), we obtain

$$\sum_{i=1}^{k} \lambda_i \Big[g_i(x_2, y_2) - g_i(u_2, v_2) + \tfrac{1}{2}(s_i)^T \nabla_{x_2x_2} g_i(u_2, v_2) s_i - \tfrac{1}{2}(r_i)^T \nabla_{y_2y_2} g_i(x_2, y_2) r_i \Big] \geqq 0. \quad (24)$$

Equations (18) and (24) together yields

$$\sum_{i=1}^{k} \lambda_i \Big\{ \Big[f_i(x_1, y_1) - (y_1)^T \sum_{i=1}^{k} \lambda_i (\nabla_{y_1} f_i(x_1, y_1) + \nabla_{y_1y_1} f_i(x_1, y_1) p_i) - \tfrac{1}{2} \sum_{i=1}^{k} \lambda_i (p_i^T \nabla_{y_1y_1} f_i(x_1, y_1) p_i) + g_i(x_2, y_2) - \tfrac{1}{2} r_i^T \nabla_{y_2y_2} g_i(x_2, y_2) r_i \Big] - \Big[f_i(u_1, v_1) - (u_1)^T \sum_{i=1}^{k} \lambda_i (\nabla_{x_1} f_i(u_1, v_1) + \nabla_{x_1x_1} f_i(u_1, v_1) q_i) - \tfrac{1}{2} \sum_{i=1}^{k} \lambda_i (q_i^T \nabla_{x_1x_1} f_i(u, v) q_i) + g_i(u_2, v_2) - \tfrac{1}{2} s_i^T \nabla_{x_2x_2} g_i(u_2, v_2) s_i \Big] \Big\} \geqq 0,$$

which contradicts (11). Hence the result. □

If the variable λ in the problems (RP) and (RD) is fixed to be $\bar{\lambda}$, we shall denote these problems by $(RP)_{\bar{\lambda}}$ and $(RD)_{\bar{\lambda}}$.

Theorem 2 (Strong duality). *Let $f : R^{|J_1|} \times R^{|K_1|} \to R^k$ and $g : R^{|J_2|} \times R^{|K_2|} \to R^k$ be differentiable functions and let $(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2, \bar{\lambda}, \bar{p}, \bar{r})$ be a weak efficient solution of (RP). Suppose that*

(*i*) *the matrices $\nabla_{y_1y_1} f_i(\bar{x}_1, \bar{y}_1)$ are non singular for $i = 1, 2, \ldots, k$,*
(*ii*) *the matrices $\nabla_{y_2y_2} g_i(\bar{x}_2, \bar{y}_2)$ are non singular for $i = 1, 2, \ldots, k$,*
(*iii*) *the vectors $\nabla_{y_1} f_1(\bar{x}_1, \bar{y}_1), \nabla_{y_1} f_2(\bar{x}_1, \bar{y}_1), \ldots, \nabla_{y_1} f_k(\bar{x}_1, \bar{y}_1)$ are linearly independent,*
(*iv*) *the set $\{\nabla_{y_2} g_i(\bar{x}_2, \bar{y}_2) + \nabla_{y_2y_2} g_i(\bar{x}_2, \bar{y}_2)\bar{r}_i,\ i = 1, 2, \ldots, k\}$ is linearly independent,*
(*v*) $\sum_{i=1}^{k} \bar{\lambda}_i(\nabla_{y_1}(\nabla_{y_1y_1} f_i(\bar{x}_1, \bar{y}_1)\bar{p}_i))\bar{p}_i \notin span\{\nabla_{y_1} f_1(\bar{x}_1, \bar{y}_1), \ldots, \nabla_{y_1} f_k(\bar{x}_1, \bar{y}_1)\}\backslash\{0\}$,
(*vi*) $\sum_{i=1}^{k} \bar{\lambda}_i(\nabla_{y_1}(\nabla_{y_1y_1} f_i(\bar{x}_1, \bar{y}_1)\bar{p}_i))\bar{p}_i = 0$ *implies* $\bar{p}_i = 0\ \ \forall\ i$, *and*
(*vii*) *K is closed convex pointed cone with $R^k_+ \subseteq K$.*

Then $\bar{q}_i = 0$ and $\bar{s}_i = 0$, $i = 1, 2, \ldots, k$ such that $(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2, \bar{\lambda}, \bar{q}_1 = \bar{q}_2 = \ldots = \bar{q}_k = 0, \bar{s}_1 = \bar{s}_2 = \ldots = \bar{s}_k = 0)$ is feasible for $(RD)_{\bar{\lambda}}$, and the objective function values of (RP) and $(RD)_{\bar{\lambda}}$ are equal. Furthermore, if the hypotheses of Theorem 1 are satisfied for all feasible solutions of (RP) and $(RD)_{\bar{\lambda}}$, then $(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2, \bar{\lambda}, \bar{q}_1 = \bar{q}_2 = \ldots = \bar{q}_k = 0, \bar{s}_1 = \bar{s}_2 = \ldots = \bar{s}_k = 0)$ is an efficient solution for $(RD)_{\bar{\lambda}}$.

Proof Since $(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2, \bar{\lambda}, \bar{p}, \bar{r})$ is a weak efficient solution of (RP), there exist $\bar{\alpha} \in K^*$, $\bar{\beta} \in C_3$, $\bar{\gamma} \in C_4$, $\bar{\delta} \in R_+$ and $\bar{\xi} \in R$, such that the following by Fritz-John optimality conditions [13] are satisfied at $(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2, \bar{\lambda}, \bar{p}, \bar{r})$ (for simplicity, we write $\nabla_{x_1} f_i, \nabla_{y_1} f_i, \nabla_{x_2} g_i, \nabla_{y_2} g_i$ instead of $\nabla_{x_1} f_i(\bar{x}_1, \bar{y}_1), \nabla_{y_1} f_i(\bar{x}_1, \bar{y}_1)$, $\nabla_{x_2} g_i(\bar{x}_2, \bar{y}_2), \nabla_{y_2} g_i(\bar{x}_2, \bar{y}_2)$ etc.):

$$(x_1 - \bar{x}_1)^T \Big\{ \sum_{i=1}^{k} \bar{\alpha}_i \nabla_{x_1} f_i + \sum_{i=1}^{k} \bar{\lambda}_i \nabla_{x_1y_1} f_i [\bar{\beta} - (\bar{\alpha}^T e_k)\bar{y}_1] + \sum_{i=1}^{k} \bar{\lambda}_i [\nabla_{x_1}(\nabla_{y_1y_1} f_i \bar{p}_i)][\bar{\beta} - (\bar{\alpha}^T e_k)(\bar{y}_1 + \tfrac{1}{2}\bar{p}_i)] \Big\} \geqq 0, \ \text{for all}\ x_1 \in C_1, \tag{25}$$

$$(x_2 - \bar{x}_2)^T \Big\{ \sum_{i=1}^{k} \bar{\alpha}_i \Big[\nabla_{x_2} g_i - \tfrac{1}{2}\nabla_{x_2}(\nabla_{y_2y_2} g_i \bar{r}_i)\bar{r}_i\Big] + \sum_{i=1}^{k} \bar{\lambda}_i \Big[\nabla_{x_2y_2} g_i + \nabla_{x_2}(\nabla_{y_2y_2} g_i \bar{r}_i)\Big][\bar{\gamma} - \bar{\delta}\bar{y}_2] \Big\} \geqq 0, \ \text{for all}\ x_2 \in C_2, \tag{26}$$

$$(y_1 - \bar{y}_1)^T \Big\{ \sum_{i=1}^{k} \bar{\alpha}_i \nabla_{y_1} f_i + \sum_{i=1}^{k} \bar{\lambda}_i \nabla_{y_1y_1} f_i [\bar{\beta} - (\bar{\alpha}^T e_k)\bar{y}_1] + \sum_{i=1}^{k} \bar{\lambda}_i [\nabla_{y_1}(\nabla_{y_1y_1} f_i \bar{p}_i)][\bar{\beta} - (\bar{\alpha}^T e_k)(\bar{y}_1 + \tfrac{1}{2}\bar{p}_i)] - \sum_{i=1}^{k} \bar{\lambda}_i [\nabla_{y_1} f_i + \nabla_{y_1y_1} f_i \bar{p}_i](\bar{\alpha}^T e_k) \Big\} \geqq 0, \ \text{for all}\ y_1 \in R^{|K_1|}, \tag{27}$$

$$(y_2 - \bar{y}_2)^T \Big\{ \sum_{i=1}^{k} \bar{\alpha}_i \nabla_{y_2} g_i - \tfrac{1}{2}\nabla_{y_2}(\nabla_{y_2y_2} g_i \bar{r}_i)\bar{r}_i\Big] + \sum_{i=1}^{k} \bar{\lambda}_i \Big[\nabla_{y_2y_2} g_i + \nabla_{y_2}(\nabla_{y_2y_2} g_i \bar{r}_i)\Big][\bar{\gamma} - \bar{\delta}\bar{y}_2] - \bar{\delta}\sum_{i=1}^{k} \bar{\lambda}_i(\nabla_{y_2} g_i + \nabla_{y_2y_2} g_i \bar{r}_i) \Big\} \geqq 0, \ \text{for all}\ y_2 \in R^{|K_2|}, \tag{28}$$

$$\Big\{\Big[\bar\beta-(\bar\alpha^Te_k)\bar y_1\Big]\nabla_{y_1}f_i+\Big[\bar\beta-(\bar\alpha^Te_k)(\bar y_1+\tfrac{1}{2}\bar p_i)\Big]\nabla_{y_1y_1}f_i\bar p_i+\bar\xi$$
$$+\Big[\bar\gamma-\bar\delta\bar y_2\Big]\Big[\nabla_{y_2}g_i+\nabla_{y_2y_2}g_i\bar r_i\Big]\Big\}\Big(\lambda_i-\bar\lambda_i\Big)\geqq 0,\forall\ i,\ \ \text{for all}\ \ \lambda\in\text{int}\,K^*, \qquad (29)$$

$$\Big[(\bar\beta-(\bar\alpha^Te_k)(\bar y_1+\bar p_i))\bar\lambda_i\Big]^T\nabla_{y_1y_1}f_i=0,\ \ i=1,2,\ldots,k, \qquad (30)$$

$$\Big[(\bar\gamma-\bar\delta\bar y_2)\bar\lambda_i-\bar\alpha_i\bar r_i\Big]^T\nabla_{y_2y_2}g_i=0,\ \ i=1,2,\ldots,k, \qquad (31)$$

$$\bar\beta^T\Big[\sum_{i=1}^k\bar\lambda_i(\nabla_{y_1}f_i+\nabla_{y_1y_1}f_i\bar p_i)\Big]=0, \qquad (32)$$

$$\bar\gamma^T\Big[\sum_{i=1}^k\bar\lambda_i(\nabla_{y_2}g_i+\nabla_{y_2y_2}g_i\bar r_i)\Big]=0, \qquad (33)$$

$$\bar\delta(\bar y_2)^T\Big[\sum_{i=1}^k\bar\lambda_i(\nabla_{y_2}g_i+\nabla_{y_2y_2}g_i\bar r_i)\Big]=0, \qquad (34)$$

$$\bar\xi^T[\bar\lambda^Te_k-1]=0, \qquad (35)$$

$$(\bar\alpha,\bar\beta,\bar\gamma,\bar\delta,\bar\xi)\neq 0. \qquad (36)$$

Inequalities from (27)-(29) are equivalent to

$$\sum_{i=1}^k\bar\alpha_i\nabla_{y_1}f_i+\sum_{i=1}^k\bar\lambda_i\nabla_{y_1y_1}f_i[\bar\beta-(\bar\alpha^Te_k)\bar y_1]+\sum_{i=1}^k\bar\lambda_i[\nabla_{y_1}(\nabla_{y_1y_1}f_i\bar p_i)]$$
$$\times[\bar\beta-(\bar\alpha^Te_k)(\bar y_1+\tfrac{1}{2}\bar p_i)]-\sum_{i=1}^k\bar\lambda_i[\nabla_{y_1}f_i+\nabla_{y_1y_1}f_i\bar p_i](\bar\alpha^Te_k)=0, \qquad (37)$$

$$\sum_{i=1}^k\bar\alpha_i\Big[\nabla_{y_2}g_i-\tfrac{1}{2}\nabla_{y_2}(\nabla_{y_2y_2}g_i\bar r_i)\bar r_i\Big]$$
$$+\sum_{i=1}^k\bar\lambda_i\Big[\nabla_{y_2y_2}g_i+\nabla_{y_2}(\nabla_{y_2y_2}g_i\bar r_i)\Big][\bar\gamma-\bar\delta\bar y_2]-\bar\delta\sum_{i=1}^k\bar\lambda_i(\nabla_{y_2}g_i+\nabla_{y_2y_2}g_i\bar r_i)=0, \qquad (38)$$

$$\Big[\bar\beta-(\bar\alpha^Te_k)\bar y_1\Big]\nabla_{y_1}f_i+\Big[\bar\beta-(\bar\alpha^Te_k)(\bar y_1+\tfrac{1}{2}\bar p_i)\Big]$$
$$\times\nabla_{y_1y_1}f_i\bar p_i+\bar\xi+\Big[\bar\gamma-\bar\delta\bar y_2\Big]\Big[\nabla_{y_2}g_i+\nabla_{y_2y_2}g_i\bar r_i\Big]=0, \qquad (39)$$

Since $R^k_+\subseteq K\Rightarrow K^*\subseteq R^k_+$ which implies $\text{int}(K^*)\subseteq\text{int}(R^k_+)$.

As $\bar\lambda\in\text{int}\,(K^*)$, therefore $\bar\lambda>0$.

As $\nabla_{y_1y_1}f_i$ is nonsingular and $\bar\lambda_i>0$ for $i=1,2,\ldots,k$, (30) implies

$$\bar\beta=(\bar\alpha^Te_k)(\bar y_1+\bar p_i),\ \ i=1,2,\ldots,k. \qquad (40)$$

By hypothesis (ii), equation (31) yields

$$(\bar\gamma-\bar\delta\bar y_2)\bar\lambda_i=\bar\alpha_i\bar r_i,\ \ i=1,2,\ldots,k. \qquad (41)$$

On rearranging (38), we get

$$\sum_{i=1}^{k}\Big[\bar{\alpha}_i-\bar{\delta}\bar{\lambda}_i\Big]\nabla_{y_2}g_i+\sum_{i=1}^{k}\bar{\lambda}_i\nabla_{y_2y_2}g_i\Big[\bar{\gamma}-\bar{\delta}\bar{y}_2-\bar{\delta}\bar{r}_i\Big]$$
$$+\sum_{i=1}^{k}\nabla_{y_2}(\nabla_{y_2y_2}g_i\bar{r}_i)\Big[(\bar{\gamma}-\bar{\delta}\bar{y}_2)\bar{\lambda}_i-\tfrac{1}{2}\bar{\alpha}_i\bar{r}_i\Big]=0,$$

which by equation (41) becomes

$$\sum_{i=1}^{k}\Big[\bar{\alpha}_i-\bar{\delta}\bar{\lambda}_i\Big]\Big[\nabla_{y_2}g_i+\nabla_{y_2y_2}g_i\bar{r}_i\Big]+\frac{1}{2}\sum_{i=1}^{k}\bar{\lambda}_i\nabla_{y_2}(\nabla_{y_2y_2}g_i\bar{r}_i)\Big[\bar{\gamma}-\bar{\delta}\bar{y}_2\Big]=0, \tag{42}$$

Now, let $\bar{\alpha}_j=0$ for some j. Since $\bar{\lambda}>0$, therefore relation (41) yields

$$\bar{\gamma}=\bar{\delta}\bar{y}_2, \tag{43}$$

this reduces (42) to

$$\sum_{i=1}^{k}(\bar{\alpha}_i-\bar{\delta}\bar{\lambda}_i)(\nabla_{y_2}g_i+\nabla_{y_2y_2}g_i\bar{r}_i)=0,$$

which on using hypothesis (iv), gives

$$\bar{\alpha}_i=\bar{\delta}\bar{\lambda}_i,\ \ i=1,2,\ldots,k. \tag{44}$$

Since $\bar{\lambda}_i>0$, for $i=1,2,\ldots,k$ and $\bar{\alpha}_j=0$ for some j, therefore (44) implies $\bar{\delta}=0$ and thus from (43) and (44) respectively, we get $\bar{\gamma}=0$ and $\bar{\alpha}_i=0,\ i=1,2,\ldots,k$. Further, the equation (40) and (39) gives $\bar{\beta}=0$ and $\bar{\xi}=0$, respectively. Thus $(\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\delta},\bar{\xi})=0$, a contradiction to (36).

Hence $\bar{\alpha}_i\neq 0,\ \forall i$. As $\bar{\alpha}\in K^*\subseteq R^k_+$ implies

$$\bar{\alpha}_i>0,\ i=1,2,\ldots,k. \tag{45}$$

Now, we claim that $\bar{p}_i=0$, for $i=1,2,\ldots,k$.

Using (40) and (45) in (37), we get

$$\sum_{i=1}^{k}\bar{\lambda}_i\Big[\nabla_{y_1}(\nabla_{y_1y_1}f_i\bar{p}_i)\Big]\bar{p}_i=-\frac{2}{\bar{\alpha}^Te_k}\sum_{i=1}^{k}\nabla_{y_1}f_i\Big[\bar{\alpha}_i-(\bar{\alpha}^Te_k)\bar{\lambda}_i\Big], \tag{46}$$

which by hypotheses (v) and (vi) implies

$$\bar{p}_i=0,\ i=1,2,\ldots,k, \tag{47}$$

and thus the relation (40) gives

$$\bar{\beta}=(\bar{\alpha}^Te_k)\bar{y}_1. \tag{48}$$

Equations (45)-(47), yield

$$\sum_{i=1}^{k}\nabla_{y_1}f_i\Big[\bar{\alpha}_i-(\bar{\alpha}^Te_k)\bar{\lambda}_i\Big]=0,$$

which on using hypothesis (iii) gives

$$\bar{\alpha}_i=(\bar{\alpha}^Te_k)\bar{\lambda}_i,\ i=1,2,\ldots,k. \tag{49}$$

Subtracting (34) from (33) yields

$$\left[\bar{\gamma} - \bar{\delta}(\bar{y}_2)\right]^T \sum_{i=1}^{k} \bar{\lambda}_i\left[\nabla_{y_2} g_i + \nabla_{y_2 y_2} g_i \bar{r}_i\right] = 0, \tag{50}$$

Using (41), we get

$$\sum_{i=1}^{k} \bar{\alpha}_i \bar{r}_i\left[\nabla_{y_2} g_i + \nabla_{y_2 y_2} g_i \bar{r}_i\right] = 0.$$

By the hypothesis (iv) with $\bar{\alpha}_i > 0,\ i = 1, 2, \ldots, k$, we have

$$\bar{r}_i = 0,\ i = 1, 2, \ldots, k. \tag{51}$$

As $\bar{\lambda}_i > 0,\ i = 1, 2, \ldots, k$, (41) yields

$$\bar{\gamma} = \bar{\delta}\bar{y}_2. \tag{52}$$

Using (51) and (52) in (38), we obtain

$$\sum_{i=1}^{k} (\bar{\alpha}_i - \bar{\delta}\bar{\lambda}_i)\nabla_{y_2} g_i = 0,$$

which on using hypothesis (iv) and (51) gives

$$\bar{\alpha}_i = \bar{\delta}\bar{\lambda}_i,\ i = 1, 2, \ldots, k. \tag{53}$$

From (53) and $\bar{\lambda}^T e_k = 1$, it is clear that $\bar{\alpha}^T e_k = \bar{\delta}(\bar{\lambda}^T e_k) = \bar{\delta}$. Since $\bar{\alpha}_i > 0,\ \forall\ i$ therefore

$$\bar{\delta} > 0. \tag{54}$$

Equation (52) yields

$$\bar{y}_2 = \frac{\bar{\gamma}}{\bar{\delta}} \in C_4. \tag{55}$$

Also, from (48), we have

$$\bar{y}_1 = \frac{\bar{\beta}}{\bar{\alpha}^T e_k} \in C_3. \tag{56}$$

Using (45), (47)-(49) in (25), we get

$$(x_1 - \bar{x}_1)^T \sum_{i=1}^{k} \bar{\lambda}_i \nabla_{x_1} f_i \geqq 0,\ \ \forall\ x_1 \in C_1. \tag{57}$$

Let $x_1 \in C_1$, then $x_1 + \bar{x}_1 \in C_1$ and so (57) implies

$$x_1^T \sum_{i=1}^{k} \bar{\lambda}_i \nabla_{x_1} f_i \geqq 0,\ \forall\ x_1 \in C_1.$$

Therefore, $\sum_{i=1}^{k} \bar{\lambda}_i \nabla_{x_1} f_i \in C_1^*$. (58)

Moreover, equations (26) and (51)-(54) give

$$(x_2 - \bar{x}_2)^T \sum_{i=1}^{k} \bar{\lambda}_i \nabla_{x_2} g_i \geqq 0, \ \ \forall\, x_2 \in C_2. \qquad (59)$$

Since for each $x_2 \in C_2$, $\bar{x}_2 \in C_2$, $x_2 + \bar{x}_2 \in C_2$ as C_2 is a closed convex cone, the above inequality gives

$$x_2^T \sum_{i=1}^{k} \bar{\lambda}_i \nabla_{x_2} g_i \geqq 0, \ \forall\, x_2 \in C_2,$$

or

$$\sum_{i=1}^{k} \bar{\lambda}_i \nabla_{x_2} g_i \in C_2^*.$$

Putting $x_2 = 0$ in (59), we obtain

$$\bar{x}_2 \sum_{i=1}^{k} \bar{\lambda}_i \nabla_{x_2} g_i \leqq 0.$$

Hence $(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2, \bar{\lambda}, \bar{q} = 0, \bar{s} = 0)$ satisfies the dual constraints $(RD)_{\bar{\lambda}}$ and so it is a feasible solution for the dual problem $(RD)_{\bar{\lambda}}$.

Now, letting $x_1 = 0$ and $x_1 = 2\bar{x}_1$ in (57), we get

$$\bar{x}_1^T \sum_{i=1}^{k} \bar{\lambda}_i \nabla_{x_1} f_i = 0. \qquad (60)$$

Further, from (32), (45), (47) and (48), we obtain

$$\bar{y}_1^T \sum_{i=1}^{k} \bar{\lambda}_i \nabla_{y_1} f_i = 0. \qquad (61)$$

Therefore, using (47), (51), (60) and (61), we get

$$\Big\{ (f_1(\bar{x}_1, \bar{y}_1) - \bar{y}_1^T \sum_{i=1}^{k} \bar{\lambda}_i (\nabla_{y_1} f_i + \nabla_{y_1 y_1} f_i \bar{p}_i) - \tfrac{1}{2} \sum_{i=1}^{k} \bar{\lambda}_i (\bar{p}_i^T \nabla_{y_1 y_1} f_i \bar{p}_i) + g_1(\bar{x}_2, \bar{y}_2)$$
$$- \tfrac{1}{2} \bar{r}_1^T \nabla_{y_2 y_2} g_1 \bar{r}_1, \ldots, f_k(\bar{x}_1, \bar{y}_1) - \bar{y}_1^T \sum_{i=1}^{k} \bar{\lambda}_i (\nabla_{y_1} f_i + \nabla_{y_1 y_1} f_i \bar{p}_i) - \tfrac{1}{2} \sum_{i=1}^{k} \bar{\lambda}_i (\bar{p}_i^T \nabla_{yy} f_i \bar{p}_i)$$
$$+ g_k(\bar{x}_2, \bar{y}_2) - \tfrac{1}{2} \bar{r}_k^T \nabla_{y_2 y_2} g_k \bar{r}_k \Big\}$$
$$= \Big\{ (f_1(\bar{x}_1, \bar{y}_1) - \bar{x}_1^T \sum_{i=1}^{k} \bar{\lambda}_i (\nabla_{x_1} f_i + \nabla_{x_1 x_1} f_i \bar{q}_i) - \tfrac{1}{2} \sum_{i=1}^{k} \bar{\lambda}_i (\bar{q}_i^T \nabla_{x_1 x_1} f_i \bar{q}_i) + g_1(\bar{x}_2, \bar{y}_2)$$
$$- \tfrac{1}{2} \bar{s}_1^T \nabla_{x_2 x_2} g_1 \bar{s}_1, \ldots, f_k(\bar{x}_1, \bar{y}_1) - \bar{x}_1^T \sum_{i=1}^{k} \bar{\lambda}_i (\nabla_{x_1} f_i + \nabla_{x_1 x_1} f_i \bar{q}_i) - \tfrac{1}{2} \sum_{i=1}^{k} \bar{\lambda}_i (\bar{q}_i^T \nabla_{x_1 x_1} f_i \bar{q}_i)$$
$$+ g_k(\bar{x}_2, \bar{y}_2) - \tfrac{1}{2} \bar{s}_k \nabla_{x_2 x_2} g_k \bar{s}_k \}$$

that is, the two objective function values are equal.

Now, let $(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2, \bar{\lambda}, \bar{q} = 0, \bar{s} = 0)$ is not an efficient solution of $(RD)_{\bar{\lambda}}$ then there exist $(\bar{u}_1, \bar{u}_2, \bar{v}_1, \bar{v}_2, \bar{\lambda}, \bar{q} = 0, \bar{s} = 0)$ feasible for $(RD)_{\bar{\lambda}}$, such that,

$$\Big\{ f_1(\bar{x}_1, \bar{y}_1) - \bar{x}_1^T \sum_{i=1}^{k} \bar{\lambda}_i (\nabla_{x_1} f_i(\bar{x}_1, \bar{y}_1) + \nabla_{x_1 x_1} f_i(\bar{x}_1, \bar{y}_1) \bar{q}_i) - \tfrac{1}{2} \sum_{i=1}^{k} \bar{\lambda}_i (\bar{q}_i^T \nabla_{x_1 x_1} f_i(\bar{x}_1, \bar{y}_1) \bar{q}_i)$$

$$+ g_1(\bar{x}_2,\bar{y}_2) - \tfrac{1}{2}\bar{s}_1^T\nabla_{x_2x_2}g_1(\bar{x}_2,\bar{y}_2)\bar{s}_1,\ldots,f_k(\bar{x}_1,\bar{y}_1) - \bar{x}_1^T\sum_{i=1}^k\bar{\lambda}_i(\nabla_{x_1}f_i(\bar{x}_1,\bar{y}_1)$$

$$+ \nabla_{x_1x_1}f_i(\bar{x}_1,\bar{y}_1)\bar{q}_i) - \tfrac{1}{2}\sum_{i=1}^k\bar{\lambda}_i(\bar{q}_i^T\nabla_{x_1x_1}f_i(\bar{x}_1,\bar{y}_1)\bar{q}_i) + g_k(\bar{x}_2,\bar{y}_2) - \tfrac{1}{2}\bar{s}_k^T\nabla_{x_2x_2}g_k(\bar{x}_2,\bar{y}_2)\bar{s}_k\Big\}$$

$$-\Big\{f_1(\bar{u}_1,\bar{v}_1) - \bar{u}_1^T\sum_{i=1}^k\bar{\lambda}_i(\nabla_{x_1}f_i(\bar{u}_1,\bar{v}_1) + \nabla_{x_1x_1}f_i(\bar{u}_1,\bar{v}_1)\bar{q}_i) - \tfrac{1}{2}\sum_{i=1}^k\bar{\lambda}_i(\bar{q}_i^T\nabla_{x_1x_1}f_i(\bar{u}_1,\bar{v}_1)\bar{q}_i)$$

$$+ g_1(\bar{u}_2,\bar{v}_2) - \tfrac{1}{2}\bar{s}_1\nabla_{x_2x_2}g_1(\bar{u}_2,\bar{v}_2)\bar{s}_1,\ldots,f_k(\bar{u}_1,\bar{v}_1) - \bar{u}_1^T\sum_{i=1}^k\bar{\lambda}_i(\nabla_{x_1}f_i(\bar{u}_1,\bar{v}_1) + \nabla_{x_1x_1}$$

$$f_i(\bar{u}_1,\bar{v}_1)\bar{q}_i) - \tfrac{1}{2}\sum_{i=1}^k\bar{\lambda}_i(\bar{q}_i^T\nabla_{x_1x_1}f_i(\bar{u}_1,\bar{v}_1)\bar{q}_i) + g_k(\bar{u}_2,\bar{v}_2) - \tfrac{1}{2}\bar{s}_k\nabla_{x_2x_2}g_k(\bar{u}_2,\bar{v}_2)\bar{s}_k\Big\} \in -K\backslash\{0\}.$$

As $\bar{x}_1^T\sum_{i=1}^k\bar{\lambda}_i\nabla_{x_1}f_i(\bar{x}_1,\bar{y}_1) = \bar{y}_1^T\sum_{i=1}^k\bar{\lambda}_i\nabla_{y_1}f_i(\bar{x}_1,\bar{y}_1)$ and $\bar{p}_i = 0,\ \bar{r}_i = 0,\ \forall\ i$

$$\Big\{f_1(\bar{x}_1,\bar{y}_1) - \bar{y}_1^T\sum_{i=1}^k\bar{\lambda}_i(\nabla_{y_1}f_i(\bar{x}_1,\bar{y}_1) + \nabla_{y_1y_1}f_i(\bar{x}_1,\bar{y}_1)\bar{p}_i) - \tfrac{1}{2}\sum_{i=1}^k\bar{\lambda}_i(\bar{p}_i^T\nabla_{y_1y_1}f_i(\bar{x}_1,\bar{y}_1)\bar{p}_i)$$

$$+ g_1(\bar{x}_2,\bar{y}_2) - \tfrac{1}{2}\bar{r}_1^T\nabla_{y_2y_2}g_1(\bar{x}_2,\bar{y}_2)\bar{r}_1,\ldots,f_k(\bar{x}_1,\bar{y}_1) - \bar{y}_1^T\sum_{i=1}^k\bar{\lambda}_i(\nabla_{y_1}f_i(\bar{x}_1,\bar{y}_1)$$

$$+ \nabla_{y_1y_1}f_i(\bar{x}_1,\bar{y}_1)\bar{p}_i) - \tfrac{1}{2}\sum_{i=1}^k\bar{\lambda}_i(\bar{p}_i^T\nabla_{y_1y_1}f_i(\bar{x}_1,\bar{y}_1)\bar{p}_i) + g_k(\bar{x}_2,\bar{y}_2) - \tfrac{1}{2}\bar{r}_k^T\nabla_{y_2y_2}g_k(\bar{x}_2,\bar{y}_2)\bar{r}_k\Big\}$$

$$-\{f_1(\bar{u}_1,\bar{v}_1) - \bar{u}_1^T\sum_{i=1}^k\bar{\lambda}_i(\nabla_{x_1}f_i(\bar{u}_1,\bar{v}_1) + \nabla_{x_1x_1}f_i(\bar{u}_1,\bar{v}_1)\bar{q}_i) - \tfrac{1}{2}\sum_{i=1}^k\bar{\lambda}_i(\bar{q}_i^T\nabla_{x_1x_1}f_i(\bar{u}_1,\bar{v}_1)\bar{q}_i)$$

$$+ g_1(\bar{u}_2,\bar{v}_2) - \tfrac{1}{2}\bar{s}_1^T\nabla_{x_2x_2}g_1(\bar{u}_2,\bar{v}_2)\bar{s}_1,\ldots,f_k(\bar{u}_1,\bar{v}_1) - \bar{u}_1^T\sum_{i=1}^k\bar{\lambda}_i(\nabla_{x_1}f_i(\bar{u}_1,\bar{v}_1)$$

$$+ \nabla_{x_1x_1}f_i(\bar{u}_1,\bar{v}_1)\bar{q}_i) - \tfrac{1}{2}\sum_{i=1}^k\bar{\lambda}_i(\bar{q}_i^T\nabla_{x_1x_1}f_i(\bar{u}_1,\bar{v}_1)\bar{q}_i) + g_k(\bar{u}_2,\bar{v}_2) - \tfrac{1}{2}\bar{s}_k^T\nabla_{x_2x_2}g_k(\bar{u}_2,\bar{v}_2)\bar{s}_k\Big\} \in -K\backslash\{0\},$$

which contradicts weak duality theorem. Hence $(\bar{x}_1,\bar{x}_2,\bar{y}_1,\bar{y}_2,\bar{\lambda},\bar{q} = 0,\bar{s} = 0)$ is an efficient solution of $(RD)_{\bar{\lambda}}$. □

Theorem 3 (Converse duality). *Let $f : R^{|J_1|}\times R^{|K_1|} \to R^k$ and $g : R^{|J_2|}\times R^{|K_2|} \to R^k$ be differentiable functions and let $(\bar{u}_1,\bar{u}_2,\bar{v}_1,\bar{v}_2,\bar{\lambda},\bar{q},\bar{s})$ be a weak efficient solution of (RD). Suppose that*

(*i*) *the matrices $\nabla_{x_1x_1}f_i(\bar{u}_1,\bar{v}_1)$ are non singular for $i = 1,2,\ldots,k$,*
(*ii*) *the matrices $\nabla_{x_2x_2}g_i(\bar{u}_2,\bar{v}_2)$ are non singular for $i = 1,2,\ldots,k$,*
(*iii*) *the vectors $\nabla_{x_1}f_1(\bar{u}_1,\bar{v}_1),\nabla_{x_1}f_2(\bar{u}_1,\bar{v}_1),\ldots,\nabla_{x_1}f_k(\bar{u}_1,\bar{v}_1)$ are linearly independent,*
(*iv*) *the set $\{\nabla_{x_2}g_i(\bar{u}_2,\bar{v}_2) + \nabla_{x_2x_2}g_i(\bar{u}_2,\bar{v}_2)\bar{s}_i,\ i = 1,2,\ldots,k\}$ is linearly independent,*
(*v*) $\sum_{i=1}^k\bar{\lambda}_i(\nabla_{x_1}(\nabla_{x_1x_1}f_i(\bar{u}_1,\bar{v}_1)\bar{q}_i))\bar{q}_i \notin span\{\nabla_{x_1}f_1(\bar{u}_1,\bar{v}_1),\ldots,\nabla_{x_1}f_k(\bar{u}_1,\bar{v}_1)\}\backslash\{0\}$,
(*vi*) $\sum_{i=1}^k\bar{\lambda}_i(\nabla_{x_1}(\nabla_{x_1x_1}f_i(\bar{u}_1,\bar{v}_1)\bar{q}_i))\bar{q}_i = 0$ *implies* $\bar{q}_i = 0\ \forall\ i$, *and*
(*vii*) *K is closed convex pointed cone with $R_+^k \subseteq K$.*

Then $\bar{p}_i = 0$ and $\bar{r}_i = 0,\ i = 1,2,\ldots,k$ such that $(\bar{u}_1,\bar{u}_2,\bar{v}_1,\bar{v}_2,\bar{\lambda},\bar{p}_1 = \bar{p}_2 = \ldots = \bar{p}_k = 0,\bar{r}_1 = \bar{r}_2 = \ldots = \bar{r}_k = 0)$ is feasible for $(RP)_{\bar{\lambda}}$, and the objective function values of $(RP)_{\bar{\lambda}}$ and (RD) are equal. Furthermore, if the hypotheses of Theorem 1 are satisfied for all feasible solutions of $(RP)_{\bar{\lambda}}$ and (RD), then $(\bar{u}_1,\bar{u}_2,\bar{v}_1,\bar{v}_2,\bar{\lambda},\bar{p}_1 = \bar{p}_2 = \ldots = \bar{p}_k = 0,\bar{r}_1 = \bar{r}_2 = \ldots = \bar{r}_k = 0)$ is an efficient solution for $(RP)_{\bar{\lambda}}$.

Proof It follows on the lines of Theorem 2 . □

4 Special cases

In this section, we consider special cases of the problems studied in Section 3.

(*i*) If $J_2 = \emptyset$, $K_2 = \emptyset$ in (RP) and (RD), then the programs (WP) and (WD) of [7] are obtained.

(*ii*) If $J_1 = \emptyset$, $K_1 = \emptyset$, $K = R^k_+$, $C_2 = R^{|J_2|}_+$, $C_4 = R^{|K_2|}_+$ then (RP) and (RD) become the programs studied in [14].

5. Conclusions

A new pair of second-order multiobjective mixed symmetric dual programs over arbitrary cones has been formulated. Weak, Strong and Converse duality theorems established under K-η-bonvexity assumptions. It will be quite interesting to see whether or not the work in the present paper can be further extended to study higher-order multiobjective mixed symmetric dual programs.

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