Filomat 28:2 (2014), 411–419
DOI 10.2298/FIL1402411H

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Weyl's theorem for algebraically quasi-paranormal operators

Young Min Han[a], Won Hee Na[b]

[a]*Department of Mathematics, College of Sciences, Kyung Hee University, Seoul 130-701, Republic of Korea*
[b]*Department of Mathematics, College of Sciences, Kyung Hee University, Seoul 130-701, Republic of Korea*

Abstract. Let T or T^* be an algebraically quasi-paranormal operator acting on Hilbert space. We prove : (i) Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$; (ii) a-Browder's theorem holds for $f(S)$ for every $S \prec T$ and $f \in H(\sigma(S))$; (iii) the spectral mapping theorem holds for the Weyl spectrum of T and for the essential approximate point spectrum of T.

1. Introduction

Throughout this note let $B(\mathcal{H})$ and $K(\mathcal{H})$ denote, respectively, the algebra of bounded linear operators and the ideal of compact operators acting on an infinite dimensional separable Hilbert space $\mathcal{H}$. If $T \in B(\mathcal{H})$ we shall write $N(T)$ and $R(T)$ for the null space and range of T. Also, let $\alpha(T) := \dim N(T)$, $\beta(T) := \dim N(T^*)$, and let $\sigma(T)$, $\sigma_a(T)$, $p_0(T)$, and $\sigma_p(T)$ denote the spectrum, approximate point spectrum, the set of poles of the resolvent of T, and point spectrum of T, respectively. For $T \in B(\mathcal{H})$, the smallest nonnegative integer p such that $N(T^p) = N(T^{p+1})$ is called the *ascent* of T and denoted by $a(T)$. If no such integer exists, we set $a(T) = \infty$. The smallest nonnegative integer q such that $R(T^q) = R(T^{q+1})$ is called the *descent* of T and denoted by $d(T)$. If no such integer exists, we set $d(T) = \infty$. An operator $T \in B(\mathcal{H})$ is called *Fredholm* if it has closed range, finite dimensional null space, and its range has finite co-dimension. The *index* of a Fredholm operator $T \in B(\mathcal{H})$ is given by

$$i(T) := \alpha(T) - \beta(T).$$

$T \in B(\mathcal{H})$ is called *Weyl* if it is Fredholm of index zero, and *Browder* if it is Fredholm of finite ascent and descent: equivalently ([12, Theorem 7.9.3]) if T is Fredholm and $T - \lambda$ is invertible for sufficiently small $\lambda \neq 0$ in $\mathbb{C}$. The essential spectrum $\sigma_e(T)$, the Weyl spectrum $\sigma_w(T)$ and the Browder spectrum $\sigma_b(T)$ of $T \in B(\mathcal{H})$ are defined by ([12],[13])

$$\sigma_e(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\},$$

$$\sigma_w(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\},$$

and

$$\sigma_b(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Browder}\},$$

2010 *Mathematics Subject Classification*. Primary 47A10, 47A53; Secondary 47B20
Keywords. Weyl's theorem, a-Weyl's theorem, algebraically quasi-paranormal operator, polaroid, single valued extension property
Received: 14 April 2011; Accepted: 08 October 2011
Communicated by Dragan Djordjevic
Research supported by the Kyung Hee University on sabbatical leave in 2010.
Email addresses: ymhan2004@khu.ac.kr, ymhan2004@hanmail.net (Young Min Han), whna@khu.ac.kr (Won Hee Na)

respectively. Evidently

$$\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T) = \sigma_e(T) \cup \operatorname{acc}\sigma(T),$$

where we write acc K for the accumulation points of $K \subseteq \mathbb{C}$. If we write iso $K = K \setminus \operatorname{acc} K$ then we let

$$\pi_{00}(T) := \{\lambda \in \operatorname{iso}\sigma(T) : 0 < \alpha(T-\lambda) < \infty\},$$
$$\pi_{00}^a(T) := \{\lambda \in \operatorname{iso}\sigma_a(T) : 0 < \alpha(T-\lambda) < \infty\},$$

and

$$p_{00}(T) := \sigma(T) \setminus \sigma_b(T).$$

We say that *Weyl's theorem holds for* $T \in B(\mathcal{H})$ (in symbols, $T \in \mathcal{W}$) if

$$\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T), \tag{1.1}$$

and that *Browder's theorem holds for* $T \in B(\mathcal{H})$ (in symbols, $T \in \mathcal{B}$) if

$$\sigma(T) \setminus \sigma_w(T) = p_{00}(T). \tag{1.2}$$

We consider the sets

$$\Phi_+(\mathcal{H}) := \{T \in B(\mathcal{H}) : R(T) \text{ is closed and } \alpha(T) < \infty\},$$

$$\Phi_+^-(\mathcal{H}) := \{T \in B(\mathcal{H}) : T \in \Phi_+(\mathcal{H}) \text{ and } i(T) \leq 0\}.$$

By definition,

$$\sigma_{ea}(T) := \cap\{\sigma_a(T+K) : K \in K(\mathcal{H})\}$$

is the essential approximate point spectrum, and

$$\sigma_{ab}(T) := \cap\{\sigma_a(T+K) : TK = KT \text{ and } K \in K(\mathcal{H})\}$$

is the Browder essential approximate point spectrum.

We say that *a-Weyl's theorem holds for* $T \in B(\mathcal{H})$ (in symbols, $T \in a\mathcal{W}$) if

$$\sigma_a(T) \setminus \sigma_{ea}(T) = \pi_{00}^a(T), \tag{1.3}$$

and that *a-Browder's theorem holds for* $T \in B(\mathcal{H})$ (in symbols, $T \in a\mathcal{B}$) if

$$\sigma_{ea}(T) = \sigma_{ab}(T). \tag{1.4}$$

It is known ([7],[13],[18]) that the following implications hold:

$$\begin{array}{ccc} \text{a-Weyl's theorem} & \Longrightarrow & \text{a-Browder's theorem} \\ \Downarrow & & \Downarrow \\ \text{Weyl's theorem} & \Longrightarrow & \text{Browder's theorem} \end{array}$$

In [20], H. Weyl proved that (1.1) holds for hermitian operators. Weyl's theorem has been extended from hermitian operators to hyponormal and Toeplitz operators ([5]), and to several classes of operators including seminormal operators ([3],[4]). Recently, R.E. Curto and Y.M. Han [6] showed that (1.1) holds for algebraically paranormal operators, and I.J. An and Y.M. Han [2] proved that every algebraically quasi-class A operator satisfies (1.1), respectively. On the other hand, V. Rakočević showed that (1.3) holds for cohyponormal operators ([18]). In this paper, we extend these results to algebraically quasi-paranormal operators using the local spectral theory.

2. Weyl's theorem for algebraically quasi-paranormal operators

Definition 2.1. (1) An operator T is said to be *class A* if

$$|T|^2 \leq |T^2|.$$

(2) T is called a *quasi-class A* operator if

$$T^*|T|^2T \leq T^*|T^2|T.$$

(3) An operator $T \in B(\mathcal{H})$ is said to be *paranormal* if

$$||Tx||^2 \leq ||T^2x||||x|| \quad \text{for all } x \in \mathcal{H}.$$

Recently, we introduced [11] the new operator class which is a common generalization of paranormal operators and quasi-class A operators.

Definition 2.2. An operator $T \in B(\mathcal{H})$ is said to be *quasi-paranormal* if

$$||T^2x||^2 \leq ||T^3x||||Tx|| \quad \text{for all } x \in \mathcal{H}.$$

We say that T is *algebraically quasi-paranormal* if there exists a nonconstant complex polynomial p such that $p(T)$ is quasi-paranormal.

In general, the following implications hold:

class A $\Longrightarrow$ quasi-class A $\Longrightarrow$ quasi-paranormal;
class A $\Longrightarrow$ paranormal $\Longrightarrow$ quasi-paranormal $\Longrightarrow$ algebraically quasi-paranormal.

The following example shows that there is a big gap between the set of paranormal operators and the set of quasi-paranormal operators. To construct the example which is quasi-paranormal but not paranormal, we need the following lemma.

Lemma 2.3. Let $T \in B(\mathcal{H})$. T is quasi-paranormal if and only if

$$T^*(T^{2*}T^2 - 2\lambda T^*T + \lambda^2)T \geq 0 \ \text{ for all } \lambda > 0.$$

Proof. Let α and β be arbitrary nonnegative real numbers. Then $(\lambda^{-\frac{1}{2}}\alpha - \lambda^{\frac{1}{2}}\beta)^2 \geq 0$ for any positive real number λ, and so we have $0 \leq 2\alpha\beta \leq \lambda^{-1}\alpha^2 + \lambda\beta^2$.

(i) If $\alpha > 0$ and $\beta > 0$, then $\lambda_0 := \frac{\alpha}{\beta} \geq 0$ so that $0 \leq 2\alpha\beta = {\lambda_0}^{-1}\alpha^2 + \lambda_0\beta^2$.

(ii) If $\alpha = 0$ or $\beta = 0$, then $0 = 2\alpha\beta = \inf_{\lambda>0}(\lambda^{-1}\alpha^2 + \lambda\beta^2)$.

Therefore by (i) and (ii),

$$\alpha\beta = \inf_{\lambda>0} \frac{1}{2}(\lambda^{-1}\alpha^2 + \lambda\beta^2) \ \text{ for all } \alpha, \beta \geq 0.$$

Let $x \in \mathcal{H}$ be arbitrary. Put $\alpha := ||T^3x||$ and $\beta := ||Tx||$. Then

$$\begin{aligned} ||T^3x||||Tx|| &= \inf_{\lambda>0} \frac{1}{2}(\lambda^{-1}||T^3x||^2 + \lambda||Tx||^2) \\ &= \inf_{\lambda>0} \frac{1}{2}(\langle \lambda^{-1}T^3x, T^3x\rangle + \langle \lambda Tx, Tx\rangle) \\ &= \inf_{\lambda>0} \frac{1}{2}\langle(\lambda^{-1}T^{2*}T^2 + \lambda)Tx, Tx\rangle. \end{aligned}$$

Therefore we have

$$||T^2x||^2 \leq ||T^3x||||Tx|| \Longleftrightarrow ||T^2x||^2 \leq \frac{1}{2}\langle(\lambda^{-1}T^{2*}T^2 + \lambda)Tx, Tx\rangle \text{ for all } \lambda > 0.$$

Hence T is quasi-paranormal if and only if $T^*(T^{2*}T^2 - 2\lambda T^*T + \lambda^2)T \geq 0$ for all $\lambda > 0$. □

Example 2.4. Consider the unilateral weighted shift operators as an infinite dimensional Hilbert space operator. Recall that given a bounded sequence of positive numbers $\alpha : \alpha_1, \alpha_2, \ldots$ (called weights), the unilateral weighted shift W_α associated with weight α is the operator on $\mathcal{H} = \ell_2$ defined by $W_n := \alpha_n e_{n+1}$ for all $n \geq 1$, where $\{e_n\}_{n=1}^{\infty}$ is the canonical orthonormal basis for ℓ_2. It is well known that the followings are equivalent:

(1) W_α is hyponormal;
(2) W_α is class A;
(3) W_α is paranormal;
(4) α is monotonically increasing, i.e., $\alpha_n \leq \alpha_{n+1}$ for all $n \geq 1$.

Thus hyponormality, class A, and paranormality coincide for every unilateral weighted shift. However, for quasi-paranormal operators, W_α has a very useful characterization. Using Lemma 2.3, we have

$$W_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ \alpha_1 & 0 & 0 & 0 & \\ 0 & \alpha_2 & 0 & 0 & \\ 0 & 0 & \alpha_3 & 0 & \\ 0 & 0 & 0 & \alpha_4 & \ddots \\ \vdots & & & & \ddots \end{pmatrix}$$

is quasi-paranormal if and only if $W_\alpha^*(W_\alpha^{2*}W_\alpha^2 - 2\lambda W_\alpha^* W_\alpha + \lambda^2)W_\alpha$ for all $\lambda > 0$.
On the other hand, let $\mathrm{diag}(\{\alpha_n\}_{n=1}^{\infty}) = \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3, \ldots)$ denote an infinite diagonal matrix on ℓ_2. Then

$$\begin{aligned} & W_\alpha^*(W_\alpha^{2*}W_\alpha^2 - 2\lambda W_\alpha^* W_\alpha + \lambda^2)W_\alpha \\ = & \ \mathrm{diag}(\{{\alpha_n}^2{\alpha_{n+1}}^2{\alpha_{n+2}}^2\}_{n=1}^{\infty}) - 2\lambda\ \mathrm{diag}(\{{\alpha_n}^2{\alpha_{n+1}}^2\}_{n=1}^{\infty}) + \lambda^2\ \mathrm{diag}(\{{\alpha_n}^2\}_{n=1}^{\infty}) \\ = & \ \mathrm{diag}(\{{\alpha_n}^2({\alpha_{n+1}}^2{\alpha_{n+2}}^2 - 2\lambda{\alpha_{n+1}}^2 + \lambda^2)\}_{n=1}^{\infty}). \end{aligned}$$

From the equality and Lemma 2.3 W_α is quasi-paranormal if and only if ${\alpha_n}^2({\alpha_{n+1}}^2{\alpha_{n+2}}^2 - 2\lambda{\alpha_{n+1}}^2 + \lambda^2) \geq 0$ for all $\lambda > 0$ and for every $n \geq 1$. Equivalently, $\alpha_{n+1} \leq \alpha_{n+2}$ for every $n \geq 1$. Hence W_α is quasi-paranormal if and only if $\alpha_n \leq \alpha_{n+1}$ for every $n \geq 2$.

We say that $T \in B(\mathcal{H})$ has the single valued extension property (SVEP) if for every open set U of $\mathbb{C}$ the only analytic function $f : U \longrightarrow \mathcal{H}$ which satisfies the equation

$$(T - \lambda)f(\lambda) = 0$$

is the constant function $f \equiv 0$ on U.

Before we state our main theorem (Theorem 2.8), we need several preliminary results. We begin with the following lemma which follows from Definition 2.2 and some facts about quasi-paranormal operators [11].

Lemma 2.5. Let $T \in B(\mathcal{H})$.

(1) If T is algebraically quasi-paranormal, then so is $T - \lambda$ for each $\lambda \in \mathbb{C}$.
(2) If T is algebraically quasi-paranormal and M is a closed invariant subspace under T, then $T|M$ is also algebraically quasi-paranormal.
(3) If T is algebraically quasi-paranormal, T has SVEP.
(4) Suppose T does not have dense range. Then we have:
T is quasi-paranormal $\Longleftrightarrow T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$ on $\mathcal{H} = \overline{T(\mathcal{H})} \oplus N(T^*)$,
where $A = T|\overline{T(\mathcal{H})}$ is paranormal.

In [6], R.E. Curto and Y.M. Han proved that quasinilpotent algebraically paranormal operators are nilpotent. We now establish a similar result for algebraically quasi-paranormal operators.

Lemma 2.6. Let T be a quasinilpotent algebraically quasi-paranormal operator. Then T is nilpotent.

Proof. We first assume that T is quasi-paranormal. We consider two cases:
Case(I) : Suppose T has dense range. Then clearly, it is paranormal. But every quasinilpotent paranormal operator is a zero operator, hence T is nilpotent.
Case(II) : Suppose T does not have dense range. Then by Lemma 2.5, we can represent T as the upper triangular matrix

$$T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \text{ on } \mathcal{H} = \overline{T(\mathcal{H})} \oplus N(T^*),$$

where $A := T|\overline{T(\mathcal{H})}$ is an paranormal operator. Since T is quasinilpotent, $\sigma(T) = \{0\}$. But $\sigma(T) = \sigma(A) \cup \{0\}$, hence $\sigma(A) = \{0\}$. Since A is paranormal, $A = 0$ and therefore T is nilpotent. Thus if T is a quasinilpotent quasi-paranormal operator, then it is nilpotent. Now, we suppose T is algebraically quasi-paranormal. Then there exists a nonconstant polynomial p such that $p(T)$ is quasi-paranormal. If $p(T)$ has dense range, then $p(T)$ is paranormal. So T is algebraically paranormal, and hence T is nilpotent by [6, Lemma 2.2]. If $p(T)$ does not have dense range, then by Lemma 2.5, we can represent $p(T)$ as the upper triangular matrix

$$p(T) = \begin{pmatrix} C & D \\ 0 & 0 \end{pmatrix} \text{ on } \mathcal{H} = \overline{p(T)(\mathcal{H})} \oplus N(p(T)^*),$$

where $C := p(T)|\overline{p(T)(\mathcal{H})}$ is paranormal. Since T is quasinilpotent, $\sigma(p(T)) = p(\sigma(T)) = \{p(0)\}$. But $\sigma(p(T)) = \sigma(C) \cup \{0\}$, hence $\sigma(C) \cup \{0\} = \{p(0)\}$. So $p(0) = 0$, and hence $p(T)$ is quasinilpotent. Since $p(T)$ is quasi-paranormal, by the previous argument $p(T)$ is nilpotent. On the other hand, since $p(0) = 0$, $p(z) = cz^m(z - \lambda_1)(z - \lambda_2)\cdots(z - \lambda_n)$ for some natural number m. Therefore $p(T) = cT^m(T - \lambda_1)(T - \lambda_2)\cdots(T - \lambda_n)$. Since $p(T)$ is nilpotent, T is nilpotent. This completes the proof. □

An operator $T \in B(\mathcal{H})$ is called *isoloid* if every isolated point of $\sigma(T)$ is an eigenvalue of T and an operator $T \in B(\mathcal{H})$ is called *polaroid* if iso $\sigma(T) \subseteq p_0(T)$. In general, if T is polaroid then it is isoloid. However, the converse is not true. Consider the following example. Let $T \in B(\ell_2)$ be defined by

$$T(x_1, x_2, x_3, \cdots) = (\frac{1}{2}x_2, \frac{1}{3}x_3, \cdots).$$

Then T is a compact quasinilpotent operator with $\alpha(T) = 1$, and so T is isoloid. However, since $a(T) = \infty$, T is not polaroid.

It is well known that every algebraically paranormal operator is isoloid. We now extend this result to algebraically quasi-paranormal operators. We can prove more:

Lemma 2.7. Let T be algebraically quasi-paranormal operator. Then T is polaroid.

Proof. Suppose T is algebraically quasi-paranormal. Then $p(T)$ is quasi-paranormal for some nonconstant polynomial p. Let $\lambda \in$ iso $\sigma(T)$. Using the spectral projection $P = \frac{1}{2\pi i}\int_{\partial D}(\mu - T)^{-1}d\mu$, where D is a closed disk of center λ which contains no other points of $\sigma(T)$, we can represent T as the direct sum

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \text{ where } \sigma(T_1) = \{\lambda\} \text{ and } \sigma(T_2) = \sigma(T) \setminus \{\lambda\}.$$

Since T_1 is algebraically quasi-paranormal, $T_1 - \lambda$ is algebraically quasi-paranormal. But $\sigma(T_1 - \lambda) = \{0\}$, it follows from Lemma 2.6 that $T_1 - \lambda$ is nilpotent. Therefore $T_1 - \lambda$ has finite ascent and descent. On the other hand, since $T_2 - \lambda$ is invertible, clearly it has finite ascent and descent. Therefore $T - \lambda$ has finite ascent and descent, and hence λ is a pole of the resolvent of T. Thus $\lambda \in$ iso $\sigma(T)$ implies $\lambda \in p_0(T)$, and so iso $\sigma(T) \subseteq p_0(T)$. Hence T is polaroid. □

In the following theorem, recall that $H(\sigma(T))$ is the space of functions analytic in an open neighborhood of $\sigma(T)$.

Theorem 2.8. Suppose T or T^* is algebraically quasi-paranormal. Then $f(T) \in \mathcal{W}$ for every $f \in H(\sigma(T))$.

Proof. Suppose T is algebraically quasi-paranormal. We first show that $T \in \mathcal{W}$. Suppose $\lambda \in \sigma(T) \setminus \sigma_w(T)$. Then $T - \lambda$ is Weyl but not invertible. We claim that $\lambda \in \partial\sigma(T)$. Assume to the contrary that λ is an interior point of $\sigma(T)$. Then there exists a neighborhood U of λ such that $\dim N(T - \mu) > 0$ for all $\mu \in U$. It follows from [10, Theorem 10] that T does not have SVEP. On the other hand, since T is algebraically quasi-paranormal, it follows from Lemma 2.5 that T has SVEP. This is a contradiction. So $\lambda \in \partial\sigma(T) \setminus \sigma_w(T)$, and it follows from the punctured neighborhood theorem that $\lambda \in \pi_{00}(T)$.
Conversely, suppose $\lambda \in \pi_{00}(T)$. Using the spectral projection $P := \frac{1}{2\pi i}\int_{\partial D}(\mu - T)^{-1}d\mu$, where D is a closed disk of center λ which contains no other points of $\sigma(T)$, we can represent T as the direct sum

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \text{ where } \sigma(T_1) = \{\lambda\} \text{ and } \sigma(T_2) = \sigma(T) \setminus \{\lambda\}.$$

Since $\sigma(T_1) = \{\lambda\}$, $T_1 - \lambda$ is quasinilpotent. But T is algebraically quasi-paranormal, hence T_1 is also algebraically quasi-paranormal. It follows from Lemma 2.6 that $T_1 - \lambda$ is nilpotent. Since $\lambda \in \pi_{00}(T)$, $T_1 - \lambda$ is a finite dimensional operator. Therefore $T_1 - \lambda$ is Weyl. Since $T_2 - \lambda$ is invertible, $T - \lambda$ is Weyl. Thus $T \in \mathcal{W}$.
Now we claim that $f(\sigma_w(T)) = \sigma_w(f(T))$ for all $f \in H(\sigma(T))$. Let $f \in H(\sigma(T))$. Since $\sigma_w(f(T)) \subseteq f(\sigma_w(T))$ with no other restriction on T, it suffices to show that $f(\sigma_w(T)) \subseteq \sigma_w(f(T))$. Suppose that $\lambda \notin \sigma_w(f(T))$. Then $f(T) - \lambda$ is Weyl and

$$f(T) - \lambda = c(T - \alpha_1)(T - \alpha_2)\cdots(T - \alpha_n)g(T), \tag{2.8}$$

where $c, \alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{C}$ and $g(T)$ is invertible. Since the operators in the right side of (2.8) commute, every $T - \alpha_i$ is Fredholm. Since T is algebraically quasi-paranormal, T has SVEP by Lemma 2.5. Therefore by [1, Corollary 3.19] $i(T - \alpha_i) \leq 0$ for each $i = 1, 2, \cdots, n$. Therefore $\lambda \notin f(\sigma_w(T))$, and hence $f(\sigma_w(T)) = \sigma_w(f(T))$.
Now recall ([1, Lemma 3.89]) that if T is isoloid then

$$f\Big(\sigma(T) \setminus \pi_{00}(T)\Big) = \sigma(f(T)) \setminus \pi_{00}(f(T)) \quad \text{for every } f \in H(\sigma(T)).$$

Since T is isoloid by Lemma 2.7 and $T \in \mathcal{W}$,

$$\sigma(f(T)) \setminus \pi_{00}(f(T)) = f\Big(\sigma(T) \setminus \pi_{00}(T)\Big) = f(\sigma_w(T)) = \sigma_w(f(T)),$$

which implies that $f(T) \in \mathcal{W}$.
Now suppose that T^* is algebraically quasi-paranormal. We first show that $T \in \mathcal{W}$. Suppose that $\lambda \in \sigma(T) \setminus \sigma_w(T)$. Observe that $\sigma(T^*) = \overline{\sigma(T)}$ and $\sigma_w(T^*) = \overline{\sigma_w(T)}$. So $\overline{\lambda} \in \sigma(T^*) \setminus \sigma_w(T^*)$. Since $T^* \in \mathcal{W}$, $\overline{\lambda} \in \pi_{00}(T^*)$. Therefore λ is an isolated point of $\sigma(T)$, and so $\lambda \in \pi_{00}(T)$. Conversely, suppose that $\lambda \in \pi_{00}(T)$. Then λ is an isolated point of $\sigma(T)$ and $0 < \alpha(T - \lambda) < \infty$. Since $\overline{\lambda}$ is an isolated point of $\sigma(T^*)$ and T^* is algebraically quasi-paranormal, it follows from Lemma 2.7 that $\overline{\lambda} \in p_0(T^*)$. So $\lambda \in p_0(T)$, and hence $T - \lambda$ is Weyl. Consequently, $\lambda \in \sigma(T) \setminus \sigma_w(T)$. Thus $T \in \mathcal{W}$. Now we show that $f(\sigma_w(T)) = \sigma_w(f(T))$ for each $f \in H(\sigma(T))$. Let $f \in H(\sigma(T))$. It is sufficient to show that $f(\sigma_w(T)) \subseteq \sigma_w(f(T))$. Suppose that $\lambda \notin \sigma_w(f(T))$. Then $f(T) - \lambda$ is Weyl. Since T^* is algebraically quasi-paranormal, it has SVEP. It follows from [1, Corollary 3.19] that $i(T - \alpha_i) \geq 0$ for each $i = 1, 2, \cdots, n$. Since

$$0 \leq \sum_{i=1}^{n} i(T - \alpha_i) = i(f(T) - \lambda) = 0,$$

$T - \alpha_i$ is Weyl for each $i = 1, 2, \cdots, n$. Hence $\lambda \notin f(\sigma_w(T))$, and so $f(\sigma_w(T)) \subseteq \sigma_w(f(T))$. Thus $f(\sigma_w(T)) = \sigma_w(f(T))$ for each $f \in H(\sigma(T))$. Since $T \in \mathcal{W}$ and T is isoloid, $f(T) \in \mathcal{W}$ for every $f \in H(\sigma(T))$. This completes the proof. □

From the proof of Theorem 2.8, we obtain the following useful consequence.

Corollary 2.9. Suppose T or T^* is algebraically quasi-paranormal. Then

$$\sigma_w(f(T)) = f(\sigma_w(T)) \quad \text{for every } f \in H(\sigma(T)).$$

3. a-Weyl's theorem for algebraically quasi-paranormal operators

Let $T \in B(\mathcal{H})$. It is well known that the inclusion $\sigma_{ea}(f(T)) \subseteq f(\sigma_{ea}(T))$ holds for every $f \in H(\sigma(T))$ with no restriction on T ([19, Theorem 3.3]). The next theorem shows that the spectral mapping theorem holds for the essential approximate point spectrum for algebraically quasi-paranormal operators.

Theorem 3.1. Suppose T or T^* is algebraically quasi-paranormal. Then

$$\sigma_{ea}(f(T)) = f(\sigma_{ea}(T)) \quad \text{for every } f \in H(\sigma(T)).$$

Proof. Suppose first that T is algebraically quasi-paranormal and let $f \in H(\sigma(T))$. It suffices to show that $f(\sigma_{ea}(T)) \subseteq \sigma_{ea}(f(T))$. Suppose that $\lambda \notin \sigma_{ea}(f(T))$. Then $f(T) - \lambda \in \Phi_+^-(\mathcal{H})$ and

$$f(T) - \lambda = c(T - \alpha_1)(T - \alpha_2) \cdots (T - \alpha_n)g(T), \tag{3.1}$$

where $c, \alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{C}$, and $g(T)$ is invertible. Since T is algebraically quasi-paranormal, it has SVEP by Lemma 2.5. It follows from [1, Corollary 3.19] that $i(T - \alpha_i) \leq 0$ for each $i = 1, 2, \cdots, n$. Therefore $\lambda \notin f(\sigma_{ea}(T))$, and hence $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$. Suppose now that T^* is algebraically quasi-paranormal. Then T^* has SVEP. Therefore by [1, Corollary 3.19] $i(T - \alpha_i) \geq 0$ for each $i = 1, 2, \cdots, n$. Since

$$0 \leq \sum_{i=1}^{n} i(T - \alpha_i) = i(f(T) - \lambda) \leq 0,$$

$T - \alpha_i$ is Weyl for each $i = 1, 2, \cdots, n$. Hence $\lambda \notin f(\sigma_{ea}(T))$, and so $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$. This completes the proof. □

$X \in B(\mathcal{H})$ is called a quasiaffinity if it has trivial kernel and dense range. $S \in B(\mathcal{H})$ is said to be a quasiaffine transform of $T \in B(\mathcal{H})$ (notation: $S \prec T$) if there is a quasiaffinity $X \in B(\mathcal{H})$ such that $XS = TX$. If both $S \prec T$ and $T \prec S$, then we say that S and T are quasisimilar. In general, we cannot expect that Weyl's theorem holds for operators having SVEP. Consider the following example: let $T \in B(\ell_2)$ be defined by

$$T(x_1, x_2, x_3, \cdots) = (\frac{1}{2}x_2, \frac{1}{3}x_3, \cdots).$$

Then T is quasinilpotent, and so T has SVEP. But $\sigma(T) = \sigma_w(T) = \{0\}$ and $\pi_{00}(T) = \{0\}$, hence $T \notin \mathcal{W}$. However, we have the following theorem.

Theorem 3.2. Suppose T is algebraically quasi-paranormal and that $S \prec T$. Then $f(S) \in a\mathcal{B}$ for every $f \in H(\sigma(S))$.

Proof. Suppose T is algebraically quasi-paranormal and that $S \prec T$. We first show that S has SVEP. Let U be any open set and let $f : U \longrightarrow \mathcal{H}$ be any analytic function such that $(S - \lambda)f(\lambda) = 0$ for all $\lambda \in U$. Since $S \prec T$, there exists a quasiaffinity X such that $XS = TX$. So $X(S - \lambda) = (T - \lambda)X$ for all $\lambda \in U$. Since $(S - \lambda)f(\lambda) = 0$ for all $\lambda \in U$, $0 = X(S - \lambda)f(\lambda) = (T - \lambda)Xf(\lambda)$ for all $\lambda \in U$. But T is algebraically quasi-paranormal, hence T has SVEP. Therefore $Xf(\lambda) = 0$ for all $\lambda \in U$. Since X is a quasiaffinity, $f(\lambda) = 0$ for all $\lambda \in U$. Therefore S has SVEP. Now we show that $S \in a\mathcal{B}$. It is well known that $\sigma_{ea}(S) \subseteq \sigma_{ab}(S)$. Conversely, suppose that $\lambda \in \sigma_a(S) \setminus \sigma_{ea}(S)$. Then $S - \lambda \in \Phi_+^-(\mathcal{H})$ and $S - \lambda$ is not bounded below. Since S has SVEP and $S - \lambda \in \Phi_+^-(\mathcal{H})$, it follows from [1, Theorem 3.16] that $a(S - \lambda) < \infty$. Therefore by [19, Theorem 2.1], $\lambda \in \sigma_a(S) \setminus \sigma_{ab}(S)$.

Thus $S \in a\mathcal{B}$. Let $f \in H(\sigma(S))$ be arbitrary. Since S has SVEP, it follows from the proof of Theorem 3.1 that $\sigma_{ea}(f(S)) = f(\sigma_{ea}(S))$. Therefore

$$\sigma_{ab}(f(S)) = f(\sigma_{ab}(S)) = f(\sigma_{ea}(S)) = \sigma_{ea}(f(S)),$$

and hence $f(S) \in a\mathcal{B}$. □

An operator $T \in B(\mathcal{H})$ is called *a-isoloid* if every isolated point of $\sigma_a(T)$ is an eigenvalue of T. Clearly, if T is a-isoloid then it is isoloid. However, the converse is not true. Consider the following example: let $T = U \oplus Q$, where U is the unilateral forward shift on ℓ_2 and Q is an injective quasinilpotent operator on ℓ_2, respectively. Then $\sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ and $\sigma_a(T) = \{\lambda \in \mathbb{C} : |\lambda| = 1\} \cup \{0\}$. Therefore T is isoloid but not a-isoloid.

Suppose that T^* is algebraically quasi-paranormal. Then we can prove more:

Theorem 3.3. Suppose T^* is algebraically quasi-paranormal. Then $f(T) \in a\mathcal{W}$ for every $f \in H(\sigma(T))$.

Proof. Suppose T^* is algebraically quasi-paranormal. We first show that $T \in a\mathcal{W}$. Suppose that $\lambda \in \sigma_a(T) \setminus \sigma_{ea}(T)$. Then $T - \lambda$ is upper semi-Fredholm and $i(T - \lambda) \leq 0$. Since T^* is algebraically quasi-paranormal, T^* has SVEP. Therefore by [1, Corollary 3.19] $i(T - \lambda) \geq 0$, and hence $T - \lambda$ is Weyl. Since T^* has SVEP, it follows from [10, Corollary 7] that $\sigma(T) = \sigma_a(T)$. Also, since $T \in \mathcal{W}$ by Theorem 2.8, $\lambda \in \pi_{00}^a(T)$. Conversely, suppose that $\lambda \in \pi_{00}^a(T)$. Since T^* has SVEP, $\sigma(T) = \sigma_a(T)$. Therefore λ is an isolated point of $\sigma(T)$, and hence $\overline{\lambda}$ is an isolated point of $\sigma(T^*)$. But T^* is algebraically quasi-paranormal, hence by Lemma 2.7 that $\overline{\lambda} \in p_0(T^*)$. Therefore $\lambda \in p_0(T)$, and hence $T - \lambda$ is Weyl. So $\lambda \in \sigma_a(T) \setminus \sigma_{ea}(T)$. Thus $T \in a\mathcal{W}$. Now we show that T is a-isoloid. Let λ be an isolated point of $\sigma_a(T)$. Since T^* has SVEP, λ is an isolated point of $\sigma(T)$. But T^* is polaroid, hence T is also polaroid. Therefore it is isoloid, and hence $\lambda \in \sigma_p(T)$. Thus T is a-isoloid. Finally, we shall show that $f(T) \in a\mathcal{W}$ for every $f \in H(\sigma(T))$. Let $f \in H(\sigma(T))$. Since $T \in a\mathcal{W}$, $\sigma_{ea}(T) = \sigma_{ab}(T)$. It follows from Theorem 3.1 that

$$\sigma_{ab}(f(T)) = f(\sigma_{ab}(T)) = f(\sigma_{ea}(T)) = \sigma_{ea}(f(T)),$$

and hence $f(T) \in a\mathcal{B}$. So $\sigma_a(f(T)) \setminus \sigma_{ea}(f(T)) \subseteq \pi_{00}^a(f(T))$.
Conversely, suppose $\lambda \in \pi_{00}^a(f(T))$. Then λ is an isolated point of $\sigma_a(f(T))$ and $0 < \alpha(f(T) - \lambda) < \infty$. Since λ is an isolated point of $f(\sigma_a(T))$, if $\alpha_i \in \sigma_a(T)$ then α_i is an isolated point of $\sigma_a(T)$ by (3.1). Since T is a-isoloid, $0 < \alpha(T - \alpha_i) < \infty$ for each $i = 1, 2, \cdots, n$. Since $T \in a\mathcal{W}$, $T - \alpha_i$ is upper semi-Fredholm and $i(T - \alpha_i) \leq 0$ for each $i = 1, 2, \cdots, n$. Therefore $f(T) - \lambda$ is upper semi-Fredholm and $i(f(T) - \lambda) = \sum_{i=1}^{n} i(T - \alpha_i) \leq 0$. Hence $\lambda \in \sigma_a(f(T)) \setminus \sigma_{ea}(f(T))$, and so $f(T) \in a\mathcal{W}$ for each $f \in H(\sigma(T))$. This completes the proof. □

References

[1] P. Aiena, Fredholm and Local Spectral Theory, with Applications to Multipliers, Kluwer Academic Publishers, 2004.
[2] I.J. An and Y.M. Han, *Weyl's theorem holds for algebraically quasi-class A operators*, Integral Equations Operator Theory **62** (2008), 1–10.
[3] S.K. Berberian, *An extension of Weyl's theorem to a class of not necessarily normal operators*, Michigan Math. J. **16** (1969), 273–279.
[4] S.K. Berberian, *The Weyl spectrum of an operator*, Indiana Univ. Math. J. **20** (1970), 529–544.
[5] L.A. Coburn, *Weyl's theorem for nonnormal operators*, Michigan Math. J. **13** (1966), 285–288.
[6] R.E. Curto and Y.M. Han, *Weyl's theorem holds for algebraically paranormal operators*, Integral Equations Operator Theory **47** (2003), 307–314.
[7] S.V. Djordjević and Y.M. Han, *Browder's theorems and spectral continuity*, Glasgow Math. J. **42** (2000), 479–486.
[8] B.P. Duggal, *Polaroid operators satisfying Weyl's theorem*, Linear Algebra Appl. **414** (2006), 271–277.
[9] B.P. Duggal, In Ho Jeon, and In Hyoun Kim, *On Weyl's theorem for quasi-class A operators*, J. Korean Math. Soc. **43** (2006), 899–909.
[10] J.K. Finch, *The single valued extension property on a Banach space*, Pacific J. Math. **58** (1975), 61–69.
[11] Y.M. Han and W.H. Na, *A note on quasi-paranormal operators*, Mediterranean Journal of Mathematics. 10 (2013) 383-393.
[12] R.E. Harte, Invertibility and Singularity for Bounded Linear Operators, Dekker, New York, 1988.
[13] R.E. Harte and W.Y. Lee, *Another note on Weyl's theorem*, Trans. Amer. Math. Soc. **349** (1997), 2115–2124.
[14] In Ho Jeon and In Hyoun Kim, *On operators satisfying* $T^*|T^2|T \geq T^*|T|^2T$, Linear Algebra Appl. **418** (2006), 854–862.
[15] C.S. Kubrusly, Hilbert Space Operators *A Problem Solving Approach*, Birkhäuser, Boston, 2003.

[16] K.B. Laursen, *Operators with finite ascent,* Pacific J. Math. **152** (1992), 323–336.
[17] K.B. Laursen and M.M. Neumann, An Introduction to Local Spectral Theory, London Mathematical Society Monographs New Series 20, Clarendon Press, Oxford, 2000.
[18] V. Rakočević, *On the essential approximate point spectrum II,* Mat. Vesnik. **36** (1984), 89–97.
[19] V. Rakočević, *Approximate point spectrum and commuting compact perturbations,* Glasgow Math. J. **28** (1986), 193–198.
[20] H. Weyl, *Über beschränkte quadratische Formen, deren Differenz vollsteig ist,* Rend. Circ. Mat. Palermo **27** (1909), 373–392.