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On statistical lacunary summability of double sequences

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Abstract. In this paper, we study the notions of statistical lacunary summability and strongly θ_q-convergence for double sequences. By using four dimensional conservative matrices, we also prove an inequality related to the concept of statistical lacunary limit superior (inferior) of real bounded double sequences.

1. Introduction

The concept of statistical A-summability for a given sequence was first introduced by Edely and Mursaleen in [7]. Following this study, Mursaleen and Alotaibi introduced statistical lacunary summability and strongly θ_q-convergence and established some relations between lacunary statistical convergence, statistical lacunary summability and strongly θ_q-convergence [16]. The first main idea of this paper is to introduce and examine double analogues of these notions, and the second one is to prove an inequality related to the concept of statistical lacunary limit superior and inferior of a real bounded double sequence. Such type of inequalities are also considered in [3], [4], [5] and [14].

Firstly, recall some basic definitions and notations used in this paper.

A double sequence $x = \left(x_{jk}\right)$ of real numbers, $j, k \in \mathbb{N}$, the set of all positive integers, is said to be convergent in the Pringsheim's sense (or P-convergent) if for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\left|x_{jk} - L\right| < \epsilon$ whenever $j, k \geq N$. We shall write this as

$$P - \lim_{j,k} x_{jk} = L$$

where j and k are tending to infinity independent of each other [19].

A double sequence x is bounded if

$$\|x\|_{(\infty,2)} = \sup_{j,k} \left|x_{jk}\right| < \infty.$$

Note that in contrast to the case for single sequences, a convergent double sequence need not to be bounded. By c_2^∞, we denote the space of double sequences which are bounded convergent and by ℓ_∞^2, the space of bounded double sequences.

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Let $A = \left(a_{jk}^{mn}\right), m, n, j, k \in \mathbb{N}$, be a four dimensional matrix and $x = \left(x_{jk}\right)$ be a double sequence. Then the double (transformed) sequence, $Ax := (y_{mn})$, is denoted by

$$y_{mn} := \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{jk}^{mn} x_{jk} \tag{1}$$

where it is assumed that the summation exists as a Pringsheim limit (of the partial sums) for each $(m, n) \in \mathbb{N}\times\mathbb{N}$. Also the sum y_{mn} is called A-means of the double sequence x. We say that a sequence x is A-summable to the limit ℓ if the A -means exist for all $m, n \in \mathbb{N}$ in the sense of Pringsheim convergence, i.e.,

$$P - \lim_{p,q\to\infty} \sum_{j=1}^{p}\sum_{k=1}^{q} a_{jk}^{mn} x_{jk} = y_{mn}$$

and

$$P - \lim_{m,n\to\infty} y_{mn} = \ell.$$

A matrix $A = \left(a_{jk}^{mn}\right)$ is said to be conservative if $x \in c_2^{\infty}$ implies $Ax \in c_2^{\infty}$. In this case we write $A \in \left(c_2^{\infty}, c_2^{\infty}\right)$. If A is conservative and $P - \lim Ax = P - \lim x$ for each $x \in c_2^{\infty}$, then A is said to be RH-regular (see [9],[20]). It is known from [3] that A is conservative if and only if

$$P - \lim_{m,n} a_{jk}^{mn} = v_{jk} \text{ for each } j, k; \tag{2}$$

$$P - \lim_{m,n} \sum_{j} a_{jk}^{mn} = v; \tag{3}$$

$$P - \lim_{m,n} \sum_{j} \left|a_{jk}^{mn}\right| = v_k \text{ for each } k; \tag{4}$$

$$P - \lim_{m,n} \sum_{k} \left|a_{jk}^{mn}\right| = v_j \text{ for each } j; \tag{5}$$

$$P - \lim_{m,n} \sum_{j}\sum_{k} \left|a_{jk}^{mn}\right| \textit{ exists}; \tag{6}$$

$$\|A\| = \sup_{m,n} \sum_{j}\sum_{k} \left|a_{jk}^{mn}\right| < \infty. \tag{7}$$

It can be also seen [9, 20] that A is RH-regular if and only if (2) with $v_{jk} = 0$, (3) with $v = 1$, (4) and (5) with $v_k = v_j = 0$, and the conditions (6) and (7) hold.

For a conservative matrix A we can define the functional

$$\Gamma(A) = v - \sum_{j}\sum_{k} v_{jk}.$$

In the case A is RH-regular, $\Gamma(A) = 1$.

The concepts of double natural density and statistical convergence of double sequences were introduced by Mursaleen and Edely [13]: Let $K \subset \mathbb{N}\times\mathbb{N}$ be a two-dimensional set of positive integers and let $K(m, n)$ be the numbers of (j, k) in K such that $j \leq m$ and $k \leq n$. In case the sequence $(K(m, n)/mn)$ has a limit in Pringsheim's sense then we say that K has a double natural density and is defined as

$$P - \lim_{m,n} \frac{K(m, n)}{mn} = \delta_2(K).$$

A real double sequence $x = \left(x_{jk}\right)$ is said to be statistically convergent to the number L if for each $\epsilon > 0$, the set

$$\left\{(j,k), j \leq m, k \leq n : \left|x_{jk} - L\right| \geq \epsilon\right\}$$

has double natural density zero. In this case we write $st_2 - \lim x = L$.

Following Freedman et al. (see [8]) and Fridy and Orhan (see [10]), the ideas of lacunary sequence and lacunary statistical convergence were extended to double sequences by Savas and Patterson in [18, 22]. But the concept of double lacunary density has been recently introduced by Çakan et al. in [4].

The double sequence $\theta = \{\theta_{r,s}\} = \{(k_r, l_s)\}$ is called double lacunary if there exist two increasing sequences of positive integers such that

$$k_0 = 0, \quad h_r = k_r - k_{r-1} \to \infty \text{ as } r \to \infty$$

and

$$l_0 = 0, \quad \overline{h}_s = l_s - l_{s-1} \to \infty \text{ as } s \to \infty \ .$$

Let $h_{r,s} = h_r\overline{h}_s$, $I_r = \{j : k_{r-1} < j \leq k_r\}$, $I_s = \{k : l_{s-1} < j \leq l_s\}$ and $I_{r,s} = \{(j,k) : k_{r-1} < j \leq k_r \ \& \ l_{s-1} < k \leq l_s\}$.

Then for a given double lacunary sequence $\theta = \{(k_r, l_s)\}$, a set $E \subset \mathbb{N} \times \mathbb{N}$ has double lacunary density $\delta_2^\theta(E)$ if

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \left|\{j \in I_r,\ k \in I_s : (j,k) \in E\}\right|$$

exists. For example, let $k_r = 2^{r-1}$ and $l_s = 3^{s-1}$ and $E = \left\{\left(j^2, k^2\right) : j,k \in \mathbb{N}\right\}$. Then $\delta_2^\theta(E) = \delta_2(E) = 0$.

The double number sequence $x = \left(x_{jk}\right)$ is lacunary statistically convergent (or briefly st_2^θ-convergent) to L provided that for every $\epsilon > 0$, the set $\left\{(j,k) : \left|x_{jk} - L\right| \geq \epsilon\right\}$ has double lacunary density zero, or equivalently for every $\epsilon > 0$

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \left|\left\{(j,k) \in I_{r,s} : \left|x_{jk} - L\right| \geq \epsilon\right\}\right| = 0.$$

In this case we write $st_2^\theta - \lim x = L$. The concept of lacunary statistical convergence has also been studied in different spaces by many authors (for instance, see [11, 12, 15, 23].

The concepts of lacunary statistical limit superior and inferior for any real double sequence $x = (x_{jk})$, have been introduced in [4] as follows:

Let us define

$$D_x = \left\{D \in \mathbb{R} : \delta_2^\theta\left(\left\{(j,k) : x_{jk} > D\right\}\right) \neq 0\right\}$$

and

$$C_x = \left\{C \in \mathbb{R} : \delta_2^\theta\left(\left\{(j,k) : x_{jk} < C\right\}\right) \neq 0\right\},$$

where $\delta_2^\theta(E) \neq 0$ means that either $\delta_2^\theta(E) > 0$ or E does not have double lacunary density. Then, the lacunary statistical limit superior and the lacunary statistical limit inferior of x are defined as

$$st_2^\theta - \limsup x = \begin{cases} \sup D_x & ; \text{ if } D_x \neq \emptyset \\ -\infty & ; \text{ if } D_x = \emptyset \end{cases}$$

and

$$st_2^\theta - \liminf x = \begin{cases} \inf C_x & ; \text{ if } C_x \neq \emptyset \\ +\infty & ; \text{ if } C_x = \emptyset \end{cases}$$

respectively. Also from [4], we have

$$P - \liminf x \leq st_2^\theta - \liminf x \leq st_2^\theta - \limsup x \leq P - \limsup x. \tag{8}$$

for any real double sequence x.

2. Statistical Lacunary Summability

Let $A = \left(a_{jk}^{mn}\right)$ be a nonnegative RH-regular matrix, $x = \left(x_{jk}\right)$ be a double sequence and (y_{mn}) be defined as in (1). If (y_{mn}) is statistically convergent to L, then x is said to be statistically A-summable to L (see [1, 6]. In this section we deal with the special case of statistical A-summability by using double lacunary sequences.

Definition 2.1. *A double sequence $x = \left(x_{jk}\right)$ is said to be lacunary convergent to L if $P - \lim_{r,s} N_{rs}(x) = L$, where*

$$N_{rs} := N_{rs}(x) = \frac{1}{h_{r,s}} \sum_{(j,k)\in I_{r,s}} x_{jk}.$$

Definition 2.2. *A double sequence $x = \left(x_{jk}\right)$ is said to be statistically lacunary summable to L if for every $\epsilon > 0$ the set $K_\epsilon = \{(r,s) \in \mathbb{N} \times \mathbb{N} : |N_{rs} - L| \geq \epsilon\}$ has double natural density zero, i.e., $\delta_2(K_\epsilon) = 0$. In this case we write $\theta_{st_2} - \lim x = L$. That is, for every $\epsilon > 0$,*

$$P - \lim_{m,n} \frac{1}{mn} |\{r \leq m, s \leq n : |N_{rs} - L| \geq \epsilon\}| = 0.$$

Hence, a double sequence $x = \left(x_{jk}\right)$ is statistically lacunary summable to L if and only if the double sequence $(N_{rs}(x))$ is statistically convergent to L.

Note that, since a convergent double sequence is also statistically convergent to the same value, a lacunary convergent double sequence is also statistically lacunary summable with the same P-limit.

Definition 2.3. *A double sequence $x = \left(x_{jk}\right)$ is said to be strongly θ_q-convergent $(0 < q < \infty)$ to the number L if*

$$P - \lim_{r,s} N_{rs}\left(|x - L|^q\right) = 0.$$

In this case we write $x_{jk} \to L\ [C_\theta]_q$ and L is called $[C_\theta]_q$-limit of x. Also we denote the set of all lacunary strongly θ_q-convergent double sequences by $[C_\theta]_q$.

Theorem 2.4. *If a double sequnce $x = \left(x_{jk}\right)$ is bounded and lacunary statistically convergent to L then it is statistically lacunary summable to L but not conversely.*

Proof. Let $x = \left(x_{jk}\right)$ be bounded, say $\sup_{j,k}\left|x_{jk} - L\right| =: M$, and st_2^θ-convergent to L. Put $K_\theta(\epsilon) := \left\{(j,k) \in I_{r,s} : \left|x_{jk} - L\right| \geq \epsilon\right\}$. Then,

$$\begin{aligned} |N_{rs}(x) - L| &= \left|\frac{1}{h_{r,s}} \sum_{(j,k)\in I_{r,s}} x_{jk} - L\right| = \left|\frac{1}{h_{r,s}} \sum_{(j,k)\in I_{r,s}} \left(x_{jk} - L\right)\right| \\ &\leq \frac{1}{h_{r,s}} \sum_{(j,k)\in K_\theta(\epsilon)} \left|x_{jk} - L\right| + \frac{1}{h_{r,s}} \sum_{(j,k)\notin K_\theta(\epsilon)} \left|x_{jk} - L\right| \\ &\leq \frac{M}{h_{r,s}} |K_\theta(\epsilon)| + \epsilon \to 0 \end{aligned}$$

as $r, s \to \infty$ which implies that $P - \lim_{r,s} N_{rs}(x) = L$. Hence $st_2 - \lim_{r,s} N_r(x) = L$ and so, $\theta_{st_2} - \lim x = L$. To see that the converse is not true, consider the double lacunary sequence $\{\theta_{r,s}\} = \left\{\left(2^{r-1}, 3^{s-1}\right)\right\}$ and the double sequence $x = (x_{jk})$ defined as

$$x_{jk} = (-1)^j, \text{ for all } k.$$

Then

$$P-\lim_{r,s}\sum_{(j,k)\in I_{r,s}} x_{jk}=0$$

and hence $st_2-\lim N_{rs}(x)=0$, but obviously x is not st_2^{θ}-convergent. This completes the proof of theorem. □

Theorem 2.5. *(i) If* $0<q<\infty$ *and a sequence* $x=\left(x_{jk}\right)$ *is strongly* θ_q*-convergent to the limit* L*, then it is lacunary statistically convergent to* L*.*
(ii) If $x=\left(x_{jk}\right)$ *is bounded and lacunary statistically convergent to* L *then* $x_{jk}\to L\ [C_\theta]_q$ *.*

Proof. (i) If $0<q<\infty$ and $x_{jk}\to L\ [C_\theta]_q$, then

$$\frac{1}{h_{r,s}}\sum_{(j,k)\in I_{r,s}}\left|x_{jk}-L\right|^q\geq\frac{1}{h_{r,s}}\sum_{(j,k)\in K_\theta(\epsilon)}\left|x_{jk}-L\right|^q\geq\frac{\epsilon^q}{h_{r,s}}\left|K_\theta(\epsilon)\right|$$

where $K_\theta(\epsilon)$ is as in Theorem 2.4. Taking limit as $r,s\to\infty$ in both sides of the above inequality, we conclude that $st_2^{\theta}-\lim x=L$.

(ii) Let x be bounded and st_2^{θ}-convergent to L. Then,

$$\begin{aligned}\frac{1}{h_{r,s}}\sum_{(j,k)\in I_{r,s}}\left|x_{jk}-L\right|^q &= \frac{1}{h_{r,s}}\sum_{(j,k)\in K_\theta(\epsilon)}\left|x_{jk}-L\right|^q+\frac{1}{h_{r,s}}\sum_{(j,k)\notin K_\theta(\epsilon)}\left|x_{jk}-L\right|^q\\ &\leq \frac{M}{h_{r,s}}\left|K_\theta(\epsilon)\right|+\epsilon\to 0\end{aligned}$$

as $r,s\to\infty$. This completes the proof. □

In the next result we characterize statistically lacunary summable sequences through the lacunary convergence of subsequences. Such a type of result for statistical convergence has been proved by Šalát [21] and Mursaleen and Edely [13].

Theorem 2.6. *A sequence* $x=\left(x_{jk}\right)$ *is statistically lacunary summable to* L *if and only if there exists a set* $K=\{(r,s)\}\subset\mathbb{N}\times\mathbb{N}$ *such that* $\delta_2(K)=1$*, and* $(x_{rs})_{(r,s)\in K}$ *is lacunary convergent to* L*, i.e.,*

$$P-\lim_{\substack{r,s\\(r,s)\in K}}N_{rs}(x)=L.$$

Proof. Let x *be statistically lacunary summable to* L *and define*

$$K_p(\theta):=\left\{(r,s)\in\mathbb{N}\times\mathbb{N}:\ \left|N_{rs}(x)-L\right|\geq\frac{1}{p}\right\}$$

and

$$M_p(\theta):=\left\{(r,s)\in\mathbb{N}\times\mathbb{N}:\ \left|N_{rs}(x)-L\right|<\frac{1}{p}\right\},(p=1,2,\cdots).$$

Since x *is statistically lacunary summable to* L*, we have* $\delta_2\left(K_p(\theta)\right)=0$*. Furthermore*

$$M_1(\theta)\supset M_2(\theta)\supset\cdots\supset M_i(\theta)\supset M_{i+1}(\theta)\supset\cdots \tag{9}$$

and

$$\delta_2\left(M_p(\theta)\right)=1,\ p=1,2,3,\cdots \tag{10}$$

Assume now that for $(r,s) \in M_p(\theta)$, $(N_{rs}(x))$ *is not convergent to L. Therefore there is* $\epsilon > 0$ *such that* $|N_{rs} - L| \geq \epsilon$ *for infinitely many terms. Let*

$$M_\epsilon(\theta) := \{(r,s) \in \mathbb{N} \times \mathbb{N} : |N_{rs}(x) - L| < \epsilon\} \text{ and } \epsilon > \frac{1}{p}, (p = 1, 2, \cdots).$$

Then

$$\delta_2(M_\epsilon(\theta)) = 0,$$

and by (9), $M_p(\theta) \subset M_\epsilon(\theta)$. *Hence* $\delta_2\left(M_p(\theta)\right) = 0$, *which contradicts (10). Hence, for* $(r,s) \in M_p(\theta)$, (x_{rs}) *has to be lacunary convergent to L, i.e.,* $(N_{rs}(x))_{(r,s)\in M_p(\theta)}$ *is P-convergent to L.*
For conversely, suppose that there exists a set $K = \{(r,s)\} \subset \mathbb{N} \times \mathbb{N}$ *such that* $\delta_2(K) = 1$ *and L is lacunary limit of* $(x_{rs})_{(r,s)\in K}$. *Then there exists* $t_0 \in \mathbb{N}$ *such that for every* $\epsilon > 0$

$$|N_{rs}(x) - L| < \epsilon$$

for all $r, s \geq t_0$. *From the inclusion*

$$K_\epsilon(\theta) = \left\{(r,s) : \ \left|N_{r,s}(x) - L\right| \geq \epsilon\right\} \subset \mathbb{N} \times \mathbb{N} - \{(i_{t_0+1}, j_{t_0+1}), (i_{t_0+2}, j_{t_0+2}), \cdots\}$$

we conclude that $0 \leq \delta_2(K_\epsilon(\theta)) \leq 1 - 1 = 0$. *Hence x is statistically lacunary summable to L.* □

3. Some inequalities

In this section we prove an inequality related to the concepts of statistical lacunary limit superior and inferior of real double sequences. Following [2] and [4], we first define these concepts.

Let $x = (x_{jk})$ be a real double sequence and define the sets G_x and F_x as

$$G_x = \left\{g \in \mathbb{R} : \delta_2^\theta(\{(r,s) : N_{rs} > g\}) \neq 0\right\}$$

and

$$F_x = \left\{f \in \mathbb{R} : \delta_2^\theta(\{(r,s) : N_{rs} < f\}) \neq 0\right\}.$$

Then statistical lacunary limit superior and statistical lacunary limit inferior of x are defined as

$$\theta_{st_2} - \limsup x = \begin{cases} \sup G_x & ; \text{ if } G_x \neq \emptyset \\ -\infty & ; \text{ if } G_x = \emptyset \end{cases}$$

and

$$\theta_{st_2} - \liminf x = \begin{cases} \inf F_x & ; \text{ if } F_x \neq \emptyset \\ +\infty & ; \text{ if } F_x = \emptyset \end{cases}$$

respectively. We note that $\theta_{st_2} - \limsup x$ and $\theta_{st_2} - \liminf x$ are equal to $st_2^\theta - \limsup_{r,s} N_{r,s}$ and $st_2^\theta - \liminf_{r,s} N_{r,s}$, respectively.

We quote here three important results on some inequalities related to the class of conservative four dimensional matrices. The first can be obtained from Lemma 3.1 in [17], the second and third are given in [3] .

Lemma 3.1. *If* $A = \left(a_{jk}^{mn}\right)$ *is a matrix such that conditions (2), (4), (5) and (7) hold, then for any* $y \in \ell_\infty^2$ *with* $\left\|y\right\| \leq 1$ *we have*

$$P - \limsup_{m,n} \sum_j \sum_k a_{jk}^{mn} y_{jk} = P - \limsup_{m,n} \sum_j \sum_k \left|a_{jk}^{mn}\right|.$$

Lemma 3.2. *Let $A = \left(a_{jk}^{mn}\right)$ be conservative and $\lambda \in \mathbb{R}^{+}$. Then*

$$P - \limsup_{m,n} \sum_{j} \sum_{k} \left| a_{jk}^{mn} - v_{jk} \right| \leq \lambda \tag{11}$$

if and only if

$$P - \limsup_{m,n} \sum_{j} \sum_{k} \left(a_{jk}^{mn} - v_{jk} \right)^{+} \leq \frac{\lambda + |\Gamma(A)|}{2}$$

and

$$P - \limsup_{m,n} \sum_{j} \sum_{k} \left(a_{jk}^{mn} - v_{jk} \right)^{-} \leq \frac{\lambda - |\Gamma(A)|}{2}$$

where for any $\gamma \in \mathbb{R}$, $\gamma^{+} = \max\{\gamma, 0\}$ and $\gamma^{-} = \max\{-\gamma, 0\}$.

Lemma 3.3. *Let $A = \left(a_{jk}^{mn}\right)$ be conservative. Then for some constant $\lambda \geq |\Gamma(A)|$ and $x \in \ell_{\infty}^{2}$, one has*

$$P - \limsup_{m,n} \sum_{j} \sum_{k} \left(a_{jk}^{mn} - v_{jk} \right) x_{jk} \leq \frac{\lambda + |\Gamma(A)|}{2} L(x) - \frac{\lambda - |\Gamma(A)|}{2} l(x)$$

if and only if (11) holds, where $L(x) = P - \limsup x$ and $l(x) = P - \liminf x$.

Theorem 3.4. *Let $A = \left(a_{jk}^{mn}\right)$ be conservative and $x \in \ell_{\infty}^{2}$. Then*

$$P - \limsup_{m,n} \sum_{j,k} \left(a_{jk}^{mn} - v_{jk} \right) x_{jk} \leq \frac{\lambda + |\Gamma(A)|}{2} N_{\theta}(x) + \frac{\lambda - |\Gamma(A)|}{2} n_{\theta}(-x) \tag{12}$$

for any $\lambda \geq |\Gamma(A)|$, if and only if (11) holds and

$$P - \limsup_{m,n} \sum_{(j,k) \in E} \left| a_{jk}^{mn} - v_{jk} \right| = 0 \tag{13}$$

for every $E \subset \mathbb{N} \times \mathbb{N}$ with $\delta_{2}^{\theta}(E) = 0$, where $N_{\theta}(x) = \theta_{st_2} - \limsup x$ and $n_{\theta}(x) = \theta_{st_2} - \liminf x$.

Proof. Let (12) holds. From definition of $L(x)$, for given $\epsilon > 0$ there exist r_0, s_0 such that

$$x_{rs} < L(x) + \epsilon$$

for $r > r_0$ and $s > s_0$. This implies that

$$P - \limsup_{r,s} N_{rs} < L(x) + \epsilon$$

for all $x \in \ell_{\infty}^{2}$. Since ϵ is arbitrary, so that $P - \limsup_{r,s} N_{rs} \leq L(x)$ and hence, by (8) we have

$$st_{2}^{\theta} - \limsup_{r,s} N_{rs} \leq P - \limsup_{r,s} N_{rs} \leq L(x),$$

i.e., $N_{\theta}(x) \leq P - \limsup x$ for all $x \in \ell_{\infty}^{2}$. Similarly, we can prove that $n_{\theta}(-x) \leq -(P - \liminf x)$. Thus, by Lemma 3.3 we get the necessity of (11). Now, for any $E \subset \mathbb{N} \times \mathbb{N}$, define the matrix $B = \left(b_{jk}^{mn}\right)$ by

$$b_{jk}^{mn} = \begin{cases} a_{jk}^{mn} - v_{jk} & ; \text{ if } (j,k) \in E, \\ 0 & ; \text{ otherwise.} \end{cases}$$

Since A is conservative , the matrix B satisfies the conditions of Lemma 3.1. Hence there exists a $y \in \ell_\infty^2$ such that $\left\|y\right\| \leq 1$ and

$$P-\limsup_{m,n}\sum_j\sum_k b_{jk}^{mn}y_{jk}=P-\limsup_{m,n}\sum_j\sum_k\left|b_{jk}^{mn}\right|.$$

Now, for the same E, let $y=\left(y_{jk}\right)$ be defined by

$$y_{jk}=\begin{cases}1 & ;\text{ if } (j,k)\in E,\\ 0 & ;\text{ otherwise.}\end{cases}$$

Then, clearly $\theta_{st_2}-\lim y=N_\theta\left(y\right)=n_\theta\left(-y\right)=0$. So, we have from (12) that

$$P-\limsup_{m,n}\sum_{(j,k)\in E}\left|b_{jk}^{mn}\right|\leq\frac{\lambda+\left|\Gamma\left(A\right)\right|}{2}N_\theta\left(y\right)+\frac{\lambda-\left|\Gamma\left(A\right)\right|}{2}n_\theta\left(-y\right)=0$$

and we get (13).

Conversely suppose that (11) and (13) hold and let $x\in\ell_\infty^2$. Write $E_1=\left\{(j,k):N_{jk}>N_\theta\left(x\right)+\epsilon\right\}$ and $E_2=\left\{(j,k):N_{jk}<n_\theta\left(x\right)-\epsilon\right\}$. Then, by Theorem 3.5 of [4], we have $\delta_2^\theta\left(E_1\right)=\delta_2^\theta\left(E_2\right)=0$; and hence $\delta_2^\theta\left(E\right)=0$ for $E=E_1\cap E_2$. Now, we can write

$$\sum_{j,k}b_{jk}^{mn}x_{jk}=\sum_{(j,k)\in E}b_{jk}^{mn}x_{jk}+\sum_{(j,k)\notin E}\left(b_{jk}^{mn}\right)^{+}x_{jk}-\sum_{(j,k)\notin E}\left(b_{jk}^{mn}\right)^{-}x_{jk}.$$

Hence

$$\begin{aligned}P-\limsup_{m,n}\sum_{j,k}b_{jk}^{mn}x_{jk} &\leq P-\limsup_{m,n}\sum_{(j,k)\in E}\left|b_{jk}^{mn}\right|\left|x_{jk}\right|\\ &+P-\limsup_{m,n}\sum_{(j,k)\notin E}\left(b_{jk}^{mn}\right)^{+}N_{jk}\\ &+P-\limsup_{m,n}\left[-\sum_{(j,k)\notin E}\left(b_{jk}^{mn}\right)^{-}N_{jk}\right]\\ &=I_1\left(x\right)+I_2\left(x\right)+I_3\left(x\right).\end{aligned}$$

From condition (13), we have $I_1\left(x\right)=0$. Let $\epsilon>0$. Then there is a set as defined above such that for $(j,k)\notin E$;

$$n_\theta\left(x\right)-\epsilon\leq N_{jk}\leq N_\theta\left(x\right)+\epsilon,\quad N_\theta\left(-x\right)-\epsilon\leq -N_{jk}\leq n_\theta\left(-x\right)+\epsilon. \tag{14}$$

Therefore from conditions (11) and (14) and Lemma 3.2, we get

$$I_2\left(x\right)\leq\frac{\lambda+\left|\Gamma\left(A\right)\right|}{2}\left(N_\theta\left(x\right)+\epsilon\right)$$

$$I_3\left(x\right)\geq\frac{\lambda-\left|\Gamma\left(A\right)\right|}{2}\left(n_\theta\left(-x\right)+\epsilon\right).$$

Hence we have

$$P-\limsup_{m,n}\sum_{j,k}b_{jk}^{mn}x_{jk}\leq\frac{\lambda+\left|\Gamma\left(A\right)\right|}{2}N_\theta\left(x\right)+\frac{\lambda-\left|\Gamma\left(A\right)\right|}{2}n_\theta\left(-x\right)+\lambda\epsilon.$$

Since ϵ is arbitrary, the proof is completed. □

In the case of $|\Gamma(A)| > 0$ and $\lambda = |\Gamma(A)|$, we have the following result.

Corollary 3.5. *Let $A = \left(a_{jk}^{mn}\right)$ be conservative and $x \in \ell_\infty^2$. Then*

$$P - \limsup_{m,n} \sum_{j,k} \left(a_{jk}^{mn} - v_{jk}\right) x_{jk} \leq |\Gamma(A)| N_\theta(x)$$

if and only if (11) holds and

$$P - \lim_{m,n} \sum_{(j,k)\in E} \left|a_{jk}^{mn} - v_{jk}\right| = |\Gamma(A)|$$

for every $E \subset \mathbb{N} \times \mathbb{N}$ with $\delta_2^\theta(E) = 0$.

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