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The Laplacian Eigenvalues and Invariants of Graphs

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Abstract. In this paper, we investigate some relations between the invariants (including vertex and edge connectivity and forwarding indices) of a graph and its Laplacian eigenvalues. In addition, we present a sufficient condition for the existence of Hamiltonicity in a graph involving its Laplacian eigenvalues.

1. Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V(G) = \{v_1, \cdots, v_n\}$ and edge set $E(G) = \{e_1, \cdots, e_m\}$. Denote by $d(v_i)$ the *degree* of vertex v_i. If $D(G) = diag(d_u, u \in V)$ is the diagonal matrix of vertex degrees of G and $A(G)$ is the $0-1$ *adjacency matrix* of G, the matrix $L(G) = D(G) - A(G)$ is called the *Laplacian matrix* of a graph G Moreover, the eigenvalues of $L(G)$ are called *Laplacian eigenvalues* of G. Furthermore, the Laplacian eigenvalues of G are denoted by

$$0 = \sigma_0 \leq \sigma_1 \leq \cdots \leq \sigma_{n-1},$$

since $L(G)$ is positive semi-definite. In recent years, the relations between invariants of a graph and its Laplacian eigenvalues have been investigated extensively. For example, Alon in [1] established that there are relations between an expander of a graph and its second smallest eigenvalue; Mohar in [13] presented a necessary condition foe the existence of Hamiltonicity in a graph in terms of its Laplacian eigenvalues. The reader is refereed to [3], [9] and [11] etc.

The purpose of this paper is to present some relations between some invariants of a graph and its Laplacian eigenvalues. In Section 2, the relations between the vertex and edge connectivities of a graph and its Laplacian eigenvalues are investigated. In Section 3, we present a sufficient condition for the existence of Hamiantonicity in a graph involving its Laplacian eigenvalues. In last Section, the lower bounds for forwarding indices of networks are obtained. Before finishing this section, we present a general discrepancy inequality from Chung[4], which is very useful for later.

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For a subset X of vertices in G, the *volume* $\mathrm{vol}(X)$ is defined by $\mathrm{vol}(X) = \sum_{v\in X} d_v$, where d_v is the degree of v. For any two subsets X and Y of vertices in G, denote

$$e(X, Y) = \{(x, y) : x \in X, y \in Y, \{x, y\} \in E(G)\}.$$

Theorem 1.1. *[4] Let G be a simple graph with n vertices and average degree $d = \frac{1}{n}\mathrm{vol}(G)$. If the Laplacian eigenvalues σ_i of G satisfy $|d - \sigma_i| \le \theta$ for $i = 1, 2, \cdots, n-1$, then for any two subsets X and Y of vertices in G, we have*

$$|e(X, Y) - \frac{d}{n}|X||Y| + d|X \cap Y| - \mathrm{vol}(X \cap Y)| \le \frac{\theta}{n}\sqrt{|X|(n - |X|)|Y|(n - |Y|)}.$$

2. Connectivity

The *vertex connectivity* of a graph G is the minimum number of vertices that we need to delete to make G is disconnected and denoted by $\kappa(G)$. Fiedler in [6] proved that if G is not the complete graph, then $\kappa(G)$ is at least the value of the second smallest Laplacian eigenvalue. In here, we present another bound for the vertex connectivity of a graph.

Theorem 2.1. *Let G be a simple graph of order n with the smallest degree $\delta \le \frac{n}{2}$ and average degree d. If the Laplacian eigenvalues σ_i satisfies $|d - \sigma_i| \le \theta$ for $i \ne 0$, then*

$$\kappa(G) \ge \delta - (2 + 2\sqrt{3})^2\frac{\theta^2}{\delta}.$$

Proof. Let $c = 2 + 2\sqrt{3}$. If $\theta \ge \frac{\delta}{c}$, there is nothing to show. We assume that $\theta < \frac{\delta}{c}$.

Suppose that there exists a subset $S \subset V(G)$ with $|S| < \delta - \frac{(c\theta)^2}{\delta}$ such that the induced graph $G[V \setminus S]$ is disconnected. Denote by U the set of vertices of the smallest connected component of $G[V \setminus S]$ and $W = V \setminus (S \bigcup U)$. Since the smallest degree of G is δ, $|S| + |U| > \delta$, which implies $|U| \ge \frac{(c\theta)^2}{\delta}$. Moreover, $|W| = n - (|U| + |S|) \le \frac{n-\delta}{2} \le \frac{n}{4}$. Because U and W are disjoint for two subsets of G, by 1.1, we have

$$\frac{d}{n}|U||W| \le \frac{\theta}{n}\sqrt{|U||W|(n - |U|)(n - |W|)} \le \sqrt{|U||W|}.$$

Hence

$$|U| \le \frac{\theta^2 n^2}{d^2|W|} \le \frac{\theta}{d}\frac{n}{|W|}\frac{\theta n}{d} < \frac{4}{c}\frac{\theta n}{d},$$

since $\frac{\theta}{d} < \frac{\theta}{\delta} < \frac{1}{c}$. By using Corollary 4 in [4], we have

$$|2|e(U)| - \frac{d|U|(|U| - 1)}{n}| \le \frac{2\theta}{n}|U|(n - \frac{|U|}{2}).$$

Then

$$\begin{aligned} 2|e(U)| &\le 2\theta|U| + \frac{d}{n}|U|^2 \\ &\le (2\theta + \frac{d}{n}\frac{4}{c}\frac{\theta n}{d})|U| \\ &= (2 + \frac{4}{c})\theta|U|. \end{aligned}$$

Hence, by $\theta < \frac{\delta}{c}$ and $c = 2 + 2\sqrt{3}$,

$$\begin{aligned} |e(U, S)| &\ge \delta|U| - 2|e(U)| \\ &\ge (\delta - (2 + \frac{4}{c})\theta)|U| \\ &> (1 - (2 + \frac{4}{c}\frac{1}{c}))\delta|U| \\ &> (\frac{1}{2} + \frac{1}{c})\delta|U|. \end{aligned}$$

On the other hand, by 1.1 and $|S| \leq \delta$, $|U| \geq \frac{c^2\theta^2}{\delta}$ and $\frac{d}{n} \leq \frac{1}{2}$, we have

$$\begin{aligned} |e(U,S)| &\leq \frac{d}{n}|U||S| + \theta\sqrt{|U||S|} \\ &\leq (\frac{d\delta}{n} + \theta\frac{\delta\sqrt{\delta}}{c\theta})|U| \\ &= (\frac{1}{2} + \frac{1}{c})\delta|U|. \end{aligned}$$

It is a contradiction. Therefore the result holds. □

Corollary 2.2. *([10]) Let G be a $d-$regular graph of order n with $d \leq \frac{n}{2}$. Denote by λ the second largest absolute eigenvalue of $A(G)$. Then*

$$\kappa(G) \geq d - \frac{36\lambda^2}{d}.$$

Proof. Since G is a $d-$regular graph, the eigenvalues of $A(G)$ are $d-\sigma_0, d-\sigma_1, \cdots, d-\sigma_{n-1}$. Hence λ satisfies $|d-\sigma_i| \leq \lambda$ for $i \neq 0$. It follows from Theorem 2.1 that $\kappa(G) \geq d - \frac{(2+2\sqrt{3})^2 d^2}{d} \geq d - \frac{36\lambda^2}{d}$. □

From [10], for a $d-$regular graph, the lower bound for $\kappa(G)$ in Corollary 2.2 is tight up to a constant factor, which implies Theorem 2.1 is tight up to a constant factor.

It is known that the edge connectivity $\kappa'(G)$ of a graph G is the minimum number of edges that need to delete to make disconnected. In [7], Goldsmith and Entringer gave a sufficient condition for edge connectivity equal to the smallest degree. In here, we present also a sufficient condition for edge connectivity equal to the smallest degree in terms of its Laplacian eigenvalues.

Theorem 2.3. *Let G be a graph of order n with average degree d and the smallest degree δ. If the Laplacian eigenvalues satisfy $2 \leq \sigma_1 \leq \sigma_{n-1} \leq 2d-2$, then $\kappa'(G) = \delta$.*

Proof. Let U be a subset of vertices of G with $|U| \leq \frac{n}{2}$.

If $1 \leq |U| \leq \delta$, then for every vertex $u \in U$, u is adjacent to at least $\delta - |U| + 1$ vertices in $G \setminus U$. Therefore,

$$|e(U, G \setminus U)| \geq |U|(\delta - |U| + 1) \geq \delta.$$

If $\delta < |U| \leq \frac{n}{2}$, let $\theta = d-2$. Since $2 \leq \sigma_1 \leq \sigma_{n-1} \leq 2d-2$, $|d-\sigma_i| \leq \theta$ for $i \neq 0$. By Theorem 1.1,

$$\left||e(U, V\setminus U)| - \frac{d}{n}|U||V\setminus U|\right| \leq \frac{\theta}{n}|U|(n-|U|).$$

Thus,

$$|e(U, V\setminus U)| \geq \frac{d-\theta}{n}|U|(n-|U|) \geq \frac{d-\theta}{n}\delta(n-\delta) \geq \frac{2\delta(n-\delta)}{n} \geq \delta.$$

Hence there are always at least δ edges between U and $V \setminus U$. Therefore $\kappa'(G) = \delta$. □

3. Hamiltonicity and the chromatic number

In this section, we first give an upper bound for the independence number $\alpha(G)$, which is used to present a sufficient condition for a graph to have a Hamilton cycle. Moreover, a lower bound for the chromatic number of a graph is obtained. The independence number is the maximum cardinality of a set of vertices of G no two of which are adjacent.

Lemma 3.1. *Let G be a graph of order n with average d. If the Lapalcian eigenvalues satisfies $|d-\sigma_i| \leq \theta$ for $i \neq 0$, then*

$$\alpha(G) \leq \frac{2n\theta + d}{d+\theta}.$$

Proof. Let U be an independent set with the seize $\alpha(G)$. By Corollary 4 in [4], we have

$$|2|e(U)| - \frac{d|U|(|U|-1)}{n}| \leq \frac{2\theta}{n}|U|(n - \frac{|U|}{2}).$$

Hence $|U| \leq \frac{2n\theta+d}{d+\theta}$. □

Lemma 3.2. *[5] Let G be a graph. If the vertex connectivity of G is at least as large as its independence number, then G is Hamiltonian.*

Theorem 3.3. *Let G be a graph of order n with average d and the smallest degree δ. If the Laplacian eigenvalues satisfies $|d - \sigma_i| \leq \theta$ for $i \neq 0$ and $\delta - (2+2\sqrt{3})^2\frac{\theta^2}{\delta} \geq \frac{2n\theta+d}{d+\theta}$, then G is Hamiltonian.*

Proof. By Theorem 2.1, G has at least $\delta - (2+2\sqrt{3})^2\frac{\theta^2}{\delta}$ vertex connected. On the other hand, by Lemma 3.1, the independence number of G is at most $\frac{2n\theta+d}{d+\theta}$. It follows from Lemma 3.2 that G is Hamiltonian. □

Theorem 3.4. *Let G be a connected graph of order n with the smallest degree δ. If $\sigma_1 \geq \frac{\sigma_{n-1}-\delta}{\sigma_{n-1}}n$, then G is Hamiltonian.*

Proof. By a theorem in [6], $\kappa(G) \geq \sigma_1$. On the other hand, by Corollary 3.3 in [15], the independence number $\alpha(G) \leq \frac{\sigma_{n-1}-\delta}{\sigma_{n-1}}n$. It follows from Lemma 3.2 that G is Hamiltonian. □

The proper coloring of the vertices of G is an assignment of colors to the vertices in such a way that adjacent vertices have distinct colors. The chromatic number, denoted by $\chi(G)$, is the minimal number od colors in a vertex coloring of G.

Theorem 3.5. *Let G be a graph of order n with the smallest degree $\delta \geq 1$. Then*

$$\chi(G) \geq \frac{\sigma_{n-1}}{\sigma_{n-1} - \delta}.$$

Moreover, if G is a d− regular bipartite graph, or a complete r−partite graph $K_{s,s,\cdots,s}$, then equality holds.

Proof. Let $V_1, V_2, \cdots, V_\chi$ denote the color class of G. Denote by e the vector with all component equal to 1. Let s_i be the restriction vector of $\frac{1}{|V_i|}e$ to V_i; that is, $(s_i)_j = \frac{1}{|V_i|}$, if $j \in V_i$; $(s_i)_j = 0$, otherwise. Thus $S = (s_1, \cdots, s_\chi)$ is an $n \times \chi$ matrix and $S^TS = I_n$. Let $B = S^TL(G)S = (b_{ij})$ and its eigenvalues $\mu_0 \leq \mu_1 \leq \cdots \leq \mu_{\chi-1}$. By eigenvalue interlacing, it is easy to see that $\mu_0 = 0$ and $\mu_{\chi-1} \leq \sigma_{n-1}$. Moreover, $b_{ii} = \frac{1}{|V_i|}\sum_{v\in V_i} d_v \geq \delta$. Hence

$$\delta\chi \leq \mathrm{tr}B = \mu_0 + \cdots \mu_{\chi-1} \leq (\chi-1)\sigma_{n-1},$$

which yields the desired inequality. If G is a $d-$ regular graph, then $\chi = 2$, $\delta = d$ and $\sigma_{n-1} = 2d$. So equality holds. If G is a complete $r-$partite graph, then $\chi = r$, $\delta = (r-1)s$ and $\sigma_{n-1} = \frac{r}{r-1}s$. Hence equality holds. □

4. Forwarding indices of graphs

In this section, we discuss some relations between the Laplacian eigenvalues of a graph and its forwarding indices.

A routing R of a graph G of order n is a set of $n(n-1)$ paths specified for all ordered pairs u and v of vertices of G. Denote $\xi(G, R, v)$ by the number of paths of R going through v (where v is not an end vertex). The vertex forwarding index of G is defined to be

$$\xi(G) = \min_R \max_{v\in V(V)} \xi(G, R, v).$$

Denote $\pi(G, R, e)$ by the number of paths of R going through edge e. The *edge forwarding index* of G is defined to be

$$\pi(G) = \min_R \max_{e\in E(G)} \pi(G, R, e).$$

Let X be a proper subset of V. The vertex cut induced by X is $N(X) = \{y \in V \setminus X | \{x, y\} \in E(G)\}$. Moreover, denote X^+ by the complement of $X \bigcup N(X)$ in V. The *vertex expanding index* is defined by

$$\gamma(G) = \min\{\frac{|N(X)|}{|X||X^+|} \mid X \subseteq V, 1 \le |X| \le n-1, |X^+| \ge 1\},$$

where the min on a void set of X is taken to be infinite.

Theorem 4.1. *Let G be a graph of order n with average degree d. If the Laplacian eigenvalues satisfies $|d - \sigma_i| \le \theta$ for $i \neq 0$, then*

$$\gamma(G) \ge \frac{d^2 - \theta^2}{n\theta^2}.$$

Proof. Let U be a subset of G such that

$$\gamma(G) = \frac{|N(U)|}{|U||U^+|}, \quad 1 \le |U| \le n-1, \quad |U^+| \ge 1.$$

Set $W = V \setminus (U \bigcup N(U))$. By Theorem 1.1, we have

$$||e(U, W)| - \frac{d}{n}|U||W|| \le \frac{\theta}{n}\sqrt{|U|(n - |U|)|W|(n - |W|)}.$$

Hence

$$d^2|U||W| \le \theta^2(|U| + |N(U)|)(|W| + |N(W)|).$$

Then

$$\frac{|N(U)|}{|U||U^+|} = \frac{|N(U)|}{|U|(n - |W|)} \ge \frac{d^2 - \theta^2}{n\theta^2}.$$

We complete the proof. □

Theorem 4.2. *Let G be a graph of order n. If $\sigma_1 \le \frac{1}{2}$, then $\xi(G) \ge \sqrt{\frac{1-2\sigma_1}{\sigma_1}}$.*

Proof. By Lemma 2.4 in [1], we have

$$\sigma_1 \ge \frac{c^2}{4 + 2c^2},$$

where c satisfies $\frac{|N(X)|}{|X|} \ge c$ for every $|X| \le \frac{n}{2}$ and $X \subset U$. Hence

$$\gamma(G) \le c \le \sqrt{\frac{4\sigma_1}{1 - 2\sigma_1}}.$$

On the other hand, there exists a subset U such that $\gamma(G) = \frac{|N(U)|}{|U||U^+|}$. It follows from the definition of $\xi(G)$ that $2|U||U^+| \ge \xi(G)|N(U)|$, since there does not exist edges between U and U^+. Hence

$$\xi(G) \ge \frac{2|U||U^+|}{|N(U)|} = \frac{2}{\gamma(G)} \ge \sqrt{\frac{1 - 2\sigma_1}{\sigma_1}}.$$

We finish the proof. □

Lemma 4.3. *Let G be a graph of order n with average degree d and let $\beta(G) = \min\{\frac{|e(U,V\setminus U)|}{|U|(n-|U|)}, \ 1 \le |U| \le n-1\}$. If the Laplacian eigenvalues satisfy $|d - \sigma_i||le\theta$ for $i \neq 0$, then*

$$\beta(G) \le \frac{d + \theta}{n}.$$

Proof. By the definition of $\beta(G)$, there exists a subset U such that $\beta(G) = \frac{|e(U,V\setminus U)|}{|U|(n-|U|)}$. On the other hand, by Theorem 1.1, we have

$$||e(U, V \setminus U)| - \frac{d}{n}|U|(n-|U|)| le \frac{\theta}{n}|U|(n-|U|).$$

Hence $\beta(G) \le \frac{d+\theta}{n}$. □

Theorem 4.4. *Let G be a graph of order n with average degree d. If the Laplacian eigenvalues satisfy* $|d-\sigma_i| \le \theta$ *for* $i \neq 0$*, then*

$$\pi(G) \ge \frac{2n}{d+\theta}.$$

Proof. It follows from Theorem 1 $\pi(G)\beta(G) \ge 2$ in [14] and Lemma 4.3 that the result holds. □

Remark The lower bounds for $\xi(G)$ and $\pi(G)$ are tight up to a constant factor. For example, Let P_n be a path of order n. It is easy to see that $\xi(P_n) = 2(\lfloor \frac{n}{2} \rfloor(\lceil \frac{n}{2} \rceil - 1)$, $\pi(G) = 2\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$; while $\sigma_1 = 4\sin^2 \frac{\pi}{2n}$.

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