

SVEP and local spectral radius formula for unbounded operators

Pietro Aiena^a, Camillo Trapani^b, Salvatore Triolo^c

^a *Dip. DEIM, Università di Palermo, I-90128 Palermo, Italy*

^b *Dip. di Matematica e Informatica Università di Palermo Italy*

^c *Dip. DEIM, Università di Palermo, I-90128 Palermo, Italy*

Abstract. In this paper we study the localized single valued extension property for an unbounded operator T . Moreover, we provide sufficient conditions for which the formula of the local spectral radius holds for these operators.

1. Introduction and notations

As it is well-known, one of the most useful tools in the local spectral theory of bounded linear operators on Banach spaces is the so-called single valued extension property. This property is enjoyed by several classes of operators as the decomposable operators as well as other classes of operators.

The single valued extension property, SVEP, for short was introduced first by Dunford [8], [9] and has received a systematic treatment in Dunford-Schwartz [10]. It also plays an important role in the book of Colojoară and Foiaş [7] and in the more recent monographs of Laursen and Neumann [16] and Aiena [1].

In this work we shall consider the localized version of this property for an unbounded operator T in Hilbert space \mathcal{H} . This property, in the case of bounded operators defined on a Banach space, has been studied by several authors, in particular in [5], [4], [2] and [3]. In this paper we extend some of the results established in the bounded case to an unbounded linear operator. In particular our paper concerns the local spectral radius, together with a characterization of localized SVEP and a characterization of some local spectral subspaces, as the analytic core introduced in the bounded case by Vrbová [23], or Mbekhta [17]. First we begin with some preliminary notations and remarks.

Let $(T, D(T))$ be a (possibly unbounded) closed linear operator in \mathcal{H} . Clearly we define $D(T^2) := \{x \in D(T) : Tx \in D(T)\}$ and, in general, for $n \geq 2$ we put $D(T^n) := \{x \in D(T^{n-1}) : T^{n-1}x \in D(T)\}$ and $T^n x = T(T^{n-1}x)$. It is worth mentioning that nothing guarantees, in general, that $D(T^k)$ does not reduce to the null subspace $\{0\}$, for some $k \in \mathbb{N}$. For this reason powers of an unbounded operator could be of little use in many occasions.

Throughout this paper if D is linear subspace of \mathcal{H} a function $f : \Omega \rightarrow D$ is analytic if $f : \Omega \rightarrow \mathcal{H}$ is analytic and $f^{(n)}(x) \in D$ for every $x \in \Omega$, and $n \in \mathbb{N}$.

Let $(T, D(T))$ be a closed linear operator in \mathcal{H} . As usual, the spectrum of $(T, D(T))$ is defined as the set

$$\sigma(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not a bijection of } D(T) \text{ onto } \mathcal{H}\}.$$

2010 *Mathematics Subject Classification.* Primary 47A10, 47A11; Secondary 47A53, 47A55

Keywords. Localized SVEP, local spectral radius formula

Received: 17 March 2013; Accepted: 17 September 2013

Communicated by Vladimir Rakocevic

Research supported by FFR, University of Palermo

Email addresses: pietro.aiena@unipa.it (Pietro Aiena), camillo.trapani@unipa.it (Camillo Trapani), salvatore.triolo@unipa.it (Salvatore Triolo)

The set $\rho(T) = \mathbb{C} \setminus \sigma(T)$ is called the resolvent set of $(T, D(T))$, while the map $R(\lambda, T) : \rho(T) \ni \lambda \mapsto (\lambda I - T)^{-1}$ is called the resolvent of $(T, D(T))$.

The spectral radius of $(T, D(T))$ is defined by

$$r(T) := \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

It is well known that, if T is a bounded everywhere defined operator, $\sigma(T)$ is a compact subset of the complex plane. On the other hand, the converse is not true: there exist closed unbounded operators whose spectrum is a bounded subset of \mathbb{C} . Thus, the spectral radius of an unbounded operator can be finite.

Definition 1.1. Let $(T, D(T))$ be a closed operator in \mathcal{H} .

- A point $\lambda \in \mathbb{C}$ is said to be in the *local resolvent set* of $x \in \mathcal{H}$, denoted by $\rho_T(x)$, if there exist an open neighborhood \mathcal{U} of λ in \mathbb{C} and an analytic function $f : \mathcal{U} \rightarrow D(T)$ which satisfies

$$(\lambda I - T)f(\lambda) = x \text{ for all } \lambda \in \mathcal{U}. \tag{1}$$

- The *local spectrum* $\sigma_T(x)$ of T at $x \in \mathcal{H}$ is the set defined by $\sigma_T(x) := \mathbb{C} \setminus \rho_T(x)$.
- The *local spectral radius* of $(T, D(T))$ at $x \in \mathcal{H}$ is defined by

$$r_T(x) := \sup\{|\lambda| : \lambda \in \sigma_T(x)\}.$$

It is clear that $r_T(x) \leq r(T)$ for every $x \in \mathcal{H}$. Moreover, $\sigma_T(x) \subseteq \sigma(T)$, and $\sigma_T(x)$ is a closed subset of \mathbb{C} . If T is an everywhere defined, bounded operator with the single-valued extension property, then the following local spectral radius formula holds [1]:

$$\max\{|\lambda| : \lambda \in \sigma_T(x)\} = \limsup_{n \rightarrow \infty} \|T^n x\|^{1/n}. \tag{2}$$

Much less seems to be known, however, for unbounded operators. For closed operators, we are not aware of results in that direction.

In [14] the authors exhibit, using quite strong assumptions, several examples of unbounded operators on Banach spaces, having the single-valued extension property for which the local spectrum at suitable points can be determined and for which a local spectrum formula holds. In this paper we consider a similar problem for unbounded operators. In particular we provide some sufficient conditions that ensure that the local spectral radius formula is valid for closed operators. As well as for the case of bounded operators, this result closely depends on the fact that the operator under consideration enjoys the single-valued extension property. Finally, in the last section we extend to the unbounded case some results known for bounded linear operators on Banach spaces.

2. Preliminary results

Clearly, any analytic function which verifies (1) on $\rho_T(x)$ is a local extension of the analytic function $R(\lambda, T) := (\lambda I - T)^{-1}x$ defined on the resolvent set $\rho(T)$ of T . Generally, the analytic solutions of (1) are not uniquely determined. It is clear from the definition itself that, if T has SVEP at λ_o , (see Definition 3.1 below) then the analytic solution of (1) is uniquely determined in an open disc centered at λ_o .

Unbounded operators, in general, possess extensions. The local behavior at a point x in the common part of the domain clearly depends on the considered extension. Obviously, we have:

Lemma 2.1. *Let $(T, D(T))$ be a closed linear operator in \mathcal{H} . Let S be a closed extension of T defined on $D(S) \supset D(T)$. For every $x \in D(T)$ we have $\sigma_S(x) \subseteq \sigma_T(x)$.*

Let $(T, D(T))$ be a closed linear operator. A crucial role in our analysis will be played by the set $D^\infty(T)$ of C^∞ -vectors for T . More precisely, $D^\infty(T) = \bigcap_{n=0}^\infty D(T^n)$. As mentioned before, there are examples of operators in Hilbert space whose square has the null subspace as domain (Naimark [18] proved, for instance, the existence of a closed symmetric operator T with $D(T^2) = \{0\}$ and Chernoff showed the existence of semibounded closed symmetric operators exhibiting the same pathology [19]). On the other hand Schmüdgen has proved that if at least one of deficiency indices of the closed symmetric operator T is finite, then $D^\infty(T)$ is dense in \mathcal{H} (see, e.g. [11]).

Let $(T, D(T))$ be a closed operator. If $x \in D^\infty(T)$ then we put

$$v_T(x) := \limsup_{n \rightarrow \infty} \|T^n x\|^{1/n} \tag{3}$$

in the extended positive real numbers. Given an operator $(T, D(T))$, we define

$$\mathcal{H}_T(t) = \{x \in D^\infty(T) : v_T(x) \leq t\}, \quad t \in [0, +\infty].$$

Let us list some basic properties of $v_T(x)$ (see [20, Lemma 1])

Proposition 2.2. *Let $(T, D(T))$ be a closed operator and $x, y \in D^\infty(T)$. Then,*

- (i) $v_T(\alpha x + \beta y) \leq \max\{v_T(x) + v_T(y)\}$ for all $\alpha, \beta \in \mathbb{C}$ and $x, y \in D^\infty(T)$.
- (ii) $v_T(x) = v_T(Tx)$ for all $x \in D^\infty(T)$.
- (iii) For every $t \in [0, +\infty]$, $\mathcal{H}_T(t)$ is a linear subspace of $D^\infty(T)$ which is invariant for T .

Clearly the question is when the equality $v_T(x) = r_T(x)$ holds.

Proposition 2.3. *Let $(T, D(T))$ be closed linear operator in \mathcal{H} . If $x \in D^\infty(T)$, then (in the extended positive real numbers) we have*

$$\sup\{|\lambda| : \lambda \in \sigma_T(x)\} \leq \limsup_{n \rightarrow \infty} \|T^n x\|^{1/n}. \tag{4}$$

Proof. If the right hand side in (4) is finite, for $|\lambda| > v_T(x)$ the series

$$\sum_{n=0}^\infty \frac{T^n x}{\lambda^{n+1}}$$

converges to a vector $f(\lambda) \in \mathcal{H}$. Let us show that $f(\lambda) \in D(T)$. Put $f_N(\lambda) = \sum_{n=0}^N \frac{T^n x}{\lambda^{n+1}}$. Then if $N > M$ and $M \rightarrow \infty$,

$$\begin{aligned} \|Tf_N(\lambda) - Tf_M(\lambda)\| &= \left\| \sum_{n=M+1}^N \frac{T^{n+1}x}{\lambda^{n+1}} \right\| \\ &= |\lambda| \left\| \sum_{n=M+1}^N \frac{T^{n+1}x}{\lambda^{n+2}} \right\| = |\lambda| \|f_{N+1} - f_{M+1}\| \rightarrow 0. \end{aligned}$$

Hence, $f(\lambda) \in D(T)$ and

$$Tf(\lambda) = \sum_{n=0}^\infty \frac{T^{n+1}x}{\lambda^{n+1}}.$$

It is, finally, easily seen that the sum $f(\lambda)$ satisfies $(\lambda I - T)f(\lambda) = x$. Hence $\lambda \in \rho_T(x)$ for all $|\lambda| > v_T(x)$. \square

The converse inequality in (4) need not always hold, however, not even for globally defined bounded operators. It should be noted that if $D(T) = \mathcal{H}$, T is bounded, and T has the single-valued extension property, then the equality

$$\max\{|\lambda| : \lambda \in \sigma_T(x)\} = \limsup_{n \rightarrow \infty} \|T^n x\|^{1/n} \tag{5}$$

holds, see [16, Proposition 3.3.13].

3. Local spectral subspaces and SVEP

The single-valued extension property dates back to the early days of local spectral theory and appeared first in the works of Dunford ([8], [9]). The localized version of this property, considered in this article, was introduced by Finch [12], and has now developed into one of the major tools in the connection of local spectral theory and Fredholm theory for operators on Banach spaces, see the recent books [16] and [1].

Definition 3.1. The operator $(T, D(T))$ is said to have the *single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open disc \mathbf{D}_{λ_0} centered at λ_0 the only analytic function $f : \mathbf{D}_{\lambda_0} \rightarrow D(T)$ which satisfies the equation

$$(\lambda I - T)f(\lambda) = 0 \tag{6}$$

is the function $f \equiv 0$.

An unbounded linear operator $(T, D(T))$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$.

Let $(T, D(T))$ be closed linear operator in \mathcal{H} . As in the bounded case, for every subset Ω of \mathbb{C} , the *analytic spectral subspace* of T associated with Ω is the set

$$X_T(\Omega) := \{x \in \mathcal{H} : \sigma_T(x) \subseteq \Omega\}.$$

Note that $X_T(\Omega_1 \cap \Omega_2) = X_T(\Omega_1) \cap X_T(\Omega_2)$. Furthermore, if $\Omega_1 \subseteq \Omega_2$ then $X_T(\Omega_1) \subseteq X_T(\Omega_2)$.

Remark 3.2. If T is globally defined ($D(T) = \mathcal{H}$) and bounded then the SVEP may be easily characterized by means of the subspace $X_T(\emptyset)$ through the equivalence of the following statements[16]:

- (i) T has SVEP.
- (ii) If $\sigma_T(x) = \emptyset$ then $x = 0$, i.e. $X_T(\emptyset) = \{0\}$.
- (iii) $X_T(\emptyset)$ is closed.

Given a (possibly unbounded) linear operator $(T, D(T))$ and a closed set $F \subseteq \mathbb{C}$, let $\mathfrak{X}_T(F)$ consist of all $x \in \mathcal{H}$ for which there exists an analytic function $f : \mathbb{C} \setminus F \rightarrow D(T)$ that satisfies

$$(\lambda I - T)f(\lambda) = x \text{ for all } \lambda \in \mathbb{C} \setminus F. \tag{7}$$

Clearly, the identity $X_T(F) = \mathfrak{X}_T(F)$ holds for all closed sets $F \subseteq \mathbb{C}$ whenever T has SVEP.

The next proposition is similar to [16, Lemma 1.2.14]. However, since our set-up is different, we give a complete proof.

Lemma 3.3. Let $(T, D(T))$ be a closed operator and \mathcal{U} be an open subset of \mathbb{C} . If $x \in D(T)$ and f is an analytic functions from \mathcal{U} into D satisfying the equation $(\lambda I - T)f(\lambda) = x$ on \mathcal{U} , then $\sigma_T(x) = \sigma_T(f(\lambda))$ for all $\lambda \in \mathcal{U}$.

Proof. For shortness we put here $D := D(T)$. Given an arbitrary $\lambda \in \mathcal{U}$, we introduce the function $g : \mathcal{U} \rightarrow D$ by $g(\lambda) := -f'(\lambda)$ and $g(\mu) := \frac{f(\lambda) - f(\mu)}{\mu - \lambda}$ for all $\mu \in \mathcal{U} \setminus \{\lambda\}$. Since $(\mu I - T)f(\mu) = x$ for all $\mu \in \mathcal{U}$, we obtain that

$$\begin{aligned} (\mu I - T)g(\mu) &= (\mu - \lambda)^{-1}(\mu I - T)(f(\lambda) - f(\mu)) \\ &= (\mu - \lambda)^{-1}((\mu I - T)f(\lambda) - (\lambda I - T)f(\lambda)) = f(\lambda), \end{aligned}$$

for all $\mu \in \mathcal{U} \setminus \{\lambda\}$.

Since $g(\mu) \rightarrow g(\lambda)$ and $(\mu I - T)g(\mu)$ is constant, by the closability of T one gets, by taking the limit for $\mu \rightarrow \lambda$, $(\lambda I - T)g(\lambda) = f(\lambda)$. Hence, in conclusion, $(\mu I - T)g(\mu) = f(\lambda)$, for every $\mu \in \mathcal{U}$. This proves that $\mathcal{U} \subseteq \rho_T(f(\lambda))$.

If $\omega \in \rho_T(x) \setminus \mathcal{U}$, we may choose an open neighborhood \mathcal{W} of ω not containing λ and an analytic function $h : \mathcal{W} \rightarrow D$ such that $(\mu I - T)h(\mu) = x$ for all $\mu \in \mathcal{W}$. Therefore

$$k(\mu) := \frac{f(\lambda) - h(\mu)}{\mu - \lambda} \quad \mu \in \mathcal{W}$$

has the property that $(\mu I - T)k(\mu) = f(\lambda)$, for all $\mu \in \mathcal{W}$. Hence $\rho_T(x) \subseteq \rho_T(f(\lambda))$.

Let $\omega \in \rho_T(f(\lambda))$ be given, and h an analytic function defined in an open neighborhood \mathcal{W} of ω , $h : \mathcal{W} \rightarrow D$ such that $(\mu I - T)h(\mu) = f(\lambda)$ for all $\mu \in \mathcal{W}$. Then

$$(\mu I - T)(\lambda I - T)h(\mu) = (\lambda I - T)f(\lambda) = x.$$

This proves that $\rho_T(f(\lambda)) \subseteq \rho_T(x)$, then $\sigma_T(f(\lambda)) = \sigma_T(x)$ for all $\lambda \in \mathcal{U}$. □

The following proposition generalizes partially the result of remark (3.2).

Theorem 3.4. *Every closed linear operator $(T, D(T))$ such that $X_T(\emptyset) = \{0\}$ has SVEP.*

Proof. Let f be an analytic function defined on an open set \mathcal{U} into D , such that

$$(\lambda I - T)f(\lambda) = 0 \quad \text{for all } \lambda \in \mathcal{U}. \tag{8}$$

Since $\sigma_T(0) = \emptyset$, by lemma 3.3 we get $\sigma_T(f(\lambda)) = \emptyset$. Thus $f(\lambda) = 0$, for every $\lambda \in \mathcal{U}$. Thus, T has SVEP. □

Let us now come back to the problem of the local spectral radius formula. Assume that $(T, D(T))$ is a closed operator that has SVEP. Let $x \in \mathcal{H}$ and suppose that $r_T(x) < \infty$. Then, there exists $f(\lambda)$ analytic in $\mathbb{C} \setminus \{|\lambda| \leq r_T(x)\}$ such that

$$(\lambda I - T)f(\lambda) = x \quad \text{for all } |\lambda| > r_T(x). \tag{9}$$

Then $f(\lambda)$ has a Laurent expansion $f(\lambda) = \sum_{n=1}^{\infty} \frac{z_n}{\lambda^n}$ where the z_n 's are vectors of \mathcal{H} given by $(R > r_T(x))$

$$z_n := \frac{1}{2\pi i} \int_{|\lambda|=R} \frac{f(\lambda)}{\lambda^{-n+1}} d\lambda.$$

Let $w \in D(T^*)$ then

$$\begin{aligned} (z_n, T^*w) &= \frac{1}{2\pi i} \int_{|\lambda|=R} \frac{(f(\lambda), T^*w)}{\lambda^{-n+1}} d\lambda \\ &= \frac{1}{2\pi i} \int_{|\lambda|=R} \frac{(Tf(\lambda), w)}{\lambda^{-n+1}} d\lambda \\ &= \frac{1}{2\pi i} \int_{|\lambda|=R} \frac{(x + \lambda f(\lambda), w)}{\lambda^{-n+1}} d\lambda \\ &= \frac{1}{2\pi i} \int_{|\lambda|=R} \frac{(x, w)}{\lambda^{-n+1}} d\lambda + \frac{1}{2\pi i} \int_{|\lambda|=R} \frac{(f(\lambda), w)}{\lambda^{-n}} d\lambda. \end{aligned}$$

If $n = 1$,

$$(x_1, T^*w) = \frac{1}{2\pi i} \int_{|\lambda|=R} \frac{(f(\lambda), w)}{\lambda^{-1}} d\lambda = (x_2, w).$$

Since the above equality holds for every $w \in D(T^*)$, then $x_1 \in D(T)$ and $Tx_1 = x_2$.

More, in general, for $n > 1$

$$(z_n, T^*w) = \frac{1}{2\pi i} \int_{|\lambda|=R} \frac{(f(\lambda), w)}{\lambda^{-n}} d\lambda = (z_{n+1}, w),$$

which implies, as before, that $z_n \in D(T)$ and $Tz_n = z_{n+1}$.

This implies that $z_1 \in D^\infty(T)$ and $z_{n+1} = Tz_n$ or $z_{n+1} = T^n z_1$. Therefore,

$$f(\lambda) = \sum_{n=1}^{\infty} \frac{T^{n-1}z_1}{\lambda^n} = \frac{1}{\lambda} \sum_{n=1}^{\infty} \frac{T^n z_1}{\lambda^n}. \tag{10}$$

Now, by (9), $T(f(\lambda)) = \lambda f(\lambda) - x$, but by (10), $\frac{1}{\lambda} \sum_0^N \frac{T^n x}{\lambda^n} \rightarrow f(\lambda)$, so

$$\frac{1}{\lambda} \sum_0^N \frac{T^{n+1}x}{\lambda^n} = \sum_0^N \frac{T^{n+1}x}{\lambda^{n+1}} \rightarrow \lambda f(\lambda) - x_1.$$

Since $(T, D(T))$ is closed we then conclude that $z_1 = x$

The previous discussion can be summarized in the following

Theorem 3.5. *Let $(T, D(T))$ be a closed linear operator in \mathcal{H} having SVEP and let $x \in \mathcal{H}$. If $r_T(x) < \infty$, then $x \in D^\infty(T)$ and*

$$f(\lambda) = \frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{T^n x}{\lambda^n} \quad \text{for all } |\lambda| > r_T(x) \tag{11}$$

Moreover, the local spectral formula

$$\sup\{|\lambda| : \lambda \in \sigma_T(x)\} = \limsup_{n \rightarrow \infty} \|T^n x\|^{1/n} \tag{12}$$

holds for all nonzero $x \in D^\infty(T)$.

Proof. If $r_T(x) = \infty$, then (12) follows from (4). If $r_T(x)$ is finite, then $r_T(x) = \max\{|\lambda| : \lambda \in \sigma_T(x)\} < \infty$. If $r_T(x)$ is not finite, our claim can be actually proved by the following reductio ad absurdum. If we assume $r_T < \limsup_{n \rightarrow \infty} \|T^n x\|^{1/n}$ and $\mu \in \mathbb{C}$ such that $r_T < \mu < \limsup_{n \rightarrow \infty} \|T^n x\|^{1/n}$ then $\mu \in \rho_T(x)$ therefore $f(\lambda)$ would have a convergent Laurent series which necessarily has the form (11), and this is impossible. \square

Corollary 3.6. *For every closed operator $(T, D(T))$ such that $X_T(\emptyset) = \{0\}$, the local spectral radius formula*

$$\sup\{|\lambda| : \lambda \in \sigma_T(x)\} = \limsup_{n \rightarrow \infty} \|T^n x\|^{1/n} \tag{13}$$

holds for all nonzero $x \in D^\infty(T)$.

3.1. Examples of local spectra

In the case of bounded operators the local spectrum is always bounded, since it is a subset of the spectrum. The local spectrum of an unbounded operator may be, as the spectrum, either bounded or unbounded, as the following two examples show.

Example 3.7. Let us consider the Hermite operator

$$T : D \rightarrow D \quad T := \frac{d^2}{dt^2} - t^2,$$

where $D := S(\mathbb{R})$ is the Schwartz space of rapidly decreasing C^∞ -functions, regarded as a subspace of $L^2(\mathbb{R})$. By [13, Corollary 5.4] and [14], a nonzero $f \in D$ has nonempty local spectrum $\sigma_T(f)$ and the local spectral radius formula (12) holds, in the extended positive real numbers.

The operator T can be obviously described (be a simple matrix representation) as follows

$$D(T) := \{(a_n) \in l^2 : (na_n) \in l^2\}$$

$$(T(a_1, a_2, a_3, \dots, a_n, \dots)) = (a_1, 2a_2, 3a_3, \dots, na_n, \dots).$$

This representation allows us to find easily a vector x such that $x \in D^\infty(T)$ but $r_T(x) = \infty$. Indeed, if we take $x = (e^{-n})$ we have

$$+\infty = \sup\{|\lambda| : \lambda \in \sigma_T(x)\} = \limsup_{n \rightarrow \infty} \|T^n x\|^{1/n}.$$

Actually, a computation of the right hand side gives

$$\|T^k(e^{-n})\|^{1/k} = \text{PolyLog}[(-2k), \frac{1}{e^2}]^{\frac{1}{2k}},$$

where the polylogarithm (also known as Jonquire’s function) is a special function, defined by the power series

$$\text{PolyLog}[k, z] := \sum_{s=1}^{\infty} \frac{z^s}{s^k}.$$

It is easy to check that for every sequence $x = (a_n)$ such that $a_n = 0$ if $n > k$ we have $\sigma_T(x) = \{1, 2, \dots, k\}$ and $x \in D(T^\infty)$.

Example 3.8. Let $(T, D(T))$ be the operator defined by

$$D(T) := \{u \in L^2([0, 1]) : u(x) = \int_0^x v(y)dy; u(1) = 0, v \in L^2([0, 1])\}$$

$$(Tu)(x) = v(x).$$

It is easy to check that for every $u \in L^2([0, 1])$ we have $\sigma_T(u) = \sigma(T) = \mathbb{C}$ and T has SVEP.

Let $(S, D(S))$ be the operator

$$D(S) := \{u \in L^2([0, 1]) : u(x) = u(0) + \int_0^x v(y)dy; v \in L^2([0, 1])\},$$

$$(Su)(x) = v(x).$$

It is easy to check that every $\lambda \in \mathbb{C}$ is an eigenvalue of S , so that the operator S does not have SVEP in every $\lambda \in \mathbb{C}$. Since $S = T^*$, this shows how deeply different the behavior of T and T^* can be with respect to SVEP.

Let $(A, D(A))$ be the operator

$$D(A) := \{u \in L^2([0, 1]) : u(x) = u(0) + \int_0^x v(y)dy; u(1) = u(0) v \in L^2([0, 1])\},$$

$$(Au)(x) = v(x).$$

The operator A is self-adjoint; thus it has SVEP and, for every $u \in L^2([0, 1])$ we have

$$\sigma_A(u) = \sigma(A) = \{2k\pi i; k \in \mathbb{Z}\}.$$

4. Localized SVEP

In this section we generalize to closed operators some results concerning the localized SVEP that are known for bounded operators ([5], [4] and [3]). First we define an important subspace in spectral theory, that has been introduced, in the bounded case, by Vrbová [23] and Mbektha [17].

Definition 4.1. Let $(T, D(T))$ be a closed linear operator in \mathcal{H} . The *analytical core* of T is the set $K(T)$ of all $x \in D(T)$ such that there exists a sequence $(u_n) \subset D(T)$ and $\delta > 0$ for which:

- (1) $x = u_0$, and $Tu_{n+1} = u_n$ for every $n \in \mathbb{N}$.
- (2) $\|u_n\| \leq \delta^n \|x\|$ for every $n \in \mathbb{N}$.

Remark 4.2. It easily follows from the definition that $K(T)$ is a linear subspace and, as in [1, Theorem 1.21], it is easy to check that $T(K(T)) = K(T)$.

As in the bounded case, the analytic core $K(T)$ admits a local spectral characterization:

Theorem 4.3. *For a closed linear operator $(T, D(T))$ we have*

$$K(T) = \{x \in D(T) : 0 \in \rho_T(x)\} = X_T(\mathbb{C} \setminus \{0\}).$$

Proof. Let $x \in K(T)$. We can suppose that $x \neq 0$. By definition of $K(T)$, there exist $\delta > 0$ and $(u_n) \subseteq D(T)$ be a sequence for which

$$x = u_0, T(u_{n+1}) = u_n, \|u_n\| \leq \delta^n \|x\|, \text{ for every } n.$$

Then the function $f : \mathbf{D}_{(0, \frac{1}{\delta})} \rightarrow D(T)$, where $\mathbf{D}_{(0, \frac{1}{\delta})}$ is the open disc centered at 0 and radius $\frac{1}{\delta}$, defined by

$$f(\lambda) := \sum_{n=1}^{\infty} \lambda^{n-1} u_n \quad \text{for all } \lambda \in \mathbf{D}_{(0, \frac{1}{\delta})}$$

is analytic, and $f(\lambda) \in D(T)$ for all $\lambda \in \mathbf{D}_{(0, \frac{1}{\delta})}$. Indeed, if we put $f_N(\lambda) = \sum_{n=1}^N \lambda^{n-1} u_n$ then if $N > M$ and $M \rightarrow \infty$,

$$\|Tf_N(\lambda) - Tf_M(\lambda)\| = \left\| \sum_{n=M+1}^N \lambda^{n-1} u_{n-1} \right\| \rightarrow 0.$$

Hence, $f(\lambda) \in D(T)$ and it is easily seen that $f(\lambda)$ satisfies $(\lambda I - T)f(\lambda) = x$. for every $\lambda \in \mathbf{D}_{(0, \frac{1}{\delta})}$. Consequently $0 \in \rho_T(x)$. The rest of the proof is analogous to that of [1, Theorem 2.18]. \square

By Theorem 3.4 if there exists an element $0 \neq x \in D(T)$ such that $\sigma_T(x) = \emptyset$, then T does not have SVEP. The next result shows a local version of this fact.

Theorem 4.4. *Let $(T, D(T))$ be a closed linear operator in \mathcal{H} . Then T does not have SVEP at 0 if and only if there exists $0 \neq x \in \ker T$ such that $\sigma_T(x) = \emptyset$.*

Proof. Suppose that there exists an element $0 \neq x_0 \in \ker T$ such that $\sigma_T(x_0) = \emptyset$. Then, by Theorem 4.3, $x_0 \in K(T)$. We can assume $\|x_0\| = 1$. By definition of $K(T)$ there exists a sequence $(u_n) \subseteq D$ such that

$$u_0 = x_0, Tu_n = u_{n-1} \text{ and } \|u_n\| \leq \delta^n,$$

for every n . Clearly, the series $\sum_{n=0}^{\infty} \lambda^n u_n$ converges for $|\lambda| < \frac{1}{\delta}$; so, the function $f(\lambda) := \sum_{n=0}^{\infty} \lambda^n u_n$ is analytic on the open disc $\mathbf{D}(0, \frac{1}{\delta})$. Let us show that $f(\lambda) \in D(T)$. Put $f_N(\lambda) = \sum_{n=0}^N \lambda^n u_n$. Then if $N > M$ and $M \rightarrow \infty$,

$$\begin{aligned} \|Tf_N(\lambda) - Tf_M(\lambda)\| &= \left\| \sum_{n=M+1}^N \lambda^n u_{n-1} \right\| \\ &\leq \frac{1}{\delta} \sum_{n=M+1}^N |\lambda|^n \delta^n \leq \frac{1}{\delta} \sum_{n=M+1}^{+\infty} |\lambda|^n \delta^n \rightarrow 0. \end{aligned}$$

Hence, $f(\lambda) \in D(T)$ and

$$(\lambda I - T) \sum_{n=0}^k \lambda^n u_n = \lambda^{k+1} u_k$$

with $\|\lambda^{k+1} u_k\| \leq \delta^k |\lambda|^{k+1}$. Furthermore, for every $|\lambda| < \frac{1}{\delta}$, $\lim_{k \rightarrow \infty} \delta^k |\lambda|^{k+1} = 0$. Thus,

$$(\lambda I - T)f(\lambda) = \lim_{k \rightarrow \infty} (\lambda I - T) \left(\sum_{n=0}^k \lambda^n u_n \right) = 0.$$

Since $f(0) = x_0 \neq 0$, it follows that T does not have SVEP at 0.

To show the converse, suppose that for every $0 \neq x \in \ker T$ we have $\sigma_T(x) \neq \emptyset$. Consider the open disc $\mathbf{D}(0, \epsilon)$ and let $f : \mathbf{D}(0, \epsilon) \rightarrow X$ be an analytic function such that $(\lambda I - T)f(\lambda) = 0$ for every $\lambda \in \mathbf{D}(0, \epsilon)$. Then $f(\lambda) = \sum_{n=0}^{\infty} \lambda^n u_n$, for a suitable sequence $(u_n) \subset D(T)$, and clearly,

$$Tu_0 = T(f(0)) = 0,$$

so $u_0 \in \ker T$. Moreover, by (3.3) we have $\sigma_T(f(0)) = \sigma_T(u_0) = \emptyset$. From the assumption, we then conclude that $u_0 = 0$. For all $0 \neq \lambda \in \mathbf{D}(0, \epsilon)$, then we have

$$\begin{aligned} 0 &= (\lambda I - T)f(\lambda) = (\lambda I - T)\left(\sum_{n=1}^{\infty} \lambda^n u_n\right) \\ &= \lambda(\lambda I - T)\left(\sum_{n=0}^{\infty} \lambda^n u_{n+1}\right), \end{aligned}$$

and therefore

$$0 = (\lambda I - T)\left(\sum_{n=0}^{\infty} \lambda^n u_{n+1}\right) \text{ for every } 0 \neq \lambda \in \mathbf{D}(0, \epsilon).$$

By continuity this is still true for every $\lambda \in \mathbf{D}(0, \epsilon)$. At this point, by using the same argument as in the first part of the proof, it is possible to show that $u_1 = 0$ and, by iterating this procedure, we conclude that $u_2 = u_3 = \dots = 0$. This shows that $f \equiv 0$ on $\mathbf{D}(0, \epsilon)$ and therefore T has SVEP at 0. \square

Theorem 4.5. *Let $(T, D(T))$ be a closed linear operator in \mathcal{H} . Then the following conditions are equivalent:*

- (i) T has SVEP at λ_0 ;
- (ii) $\ker(\lambda_0 I - T) \cap X_T(\emptyset) = \{0\}$;
- (iii) for every $0 \neq x \in \ker(\lambda_0 I - T)$ we have $\sigma_T(x) = \{\lambda_0\}$;
- (iv) $\ker(\lambda_0 I - T) \cap K(\lambda_0 I - T) = \{0\}$.

Proof. The equivalence (i) \Leftrightarrow (ii) is Theorem 4.4.

The implication (iii) \Rightarrow (ii) is obvious. To show (ii) \Rightarrow (iii), consider an element $0 \neq x \in \ker(\lambda_0 I - T)$ and take $\mu \in \mathbb{C} \setminus \{\lambda_0\}$. Let $\mathbf{D}(\mu, \epsilon)$ be an open disc centered at μ and with radius ϵ such that $\lambda_0 \notin \mathbf{D}(\mu, \epsilon)$. Evidently, the function $f(\lambda) := \frac{x}{\lambda - \lambda_0}$ is analytic on $\mathbf{D}(\mu, \epsilon)$ and

$$(\lambda I - T)f(\lambda) = [(\lambda - \lambda_0)I + (\lambda_0 I - T)]\left(\frac{1}{\lambda - \lambda_0}x\right) = x.$$

Hence $\mu \in \rho_T(x)$ for every $\mu \neq \lambda_0$ and therefore $\sigma_T(x) \subseteq \{\lambda_0\}$. By assumption (ii) $\sigma_T(x)$ cannot be empty, so $\sigma_T(x) = \{\lambda_0\}$.

(iii) \Rightarrow (iv) We may assume without loss of generality $\lambda_0 = 0$. It suffices to prove the equality

$$\ker T \cap K(T) = \ker T \cap X_T(\emptyset).$$

If $0 \neq x \in \ker(T)$ then, by (iii), $\sigma_T(x) = \{0\}$, so $\ker(T) \subseteq X_T(\{0\})$. Therefore, by Theorem 4.3,

$$\ker T \cap K(T) = \ker T \cap X_T(\mathbb{C} \setminus \{0\}) \subseteq X_T(\{0\}) \cap X_T(\mathbb{C} \setminus \{0\}) = X_T(\emptyset).$$

Since $X_T(\emptyset) \subseteq X_T(\mathbb{C} \setminus \{0\}) = K(T)$ we then conclude that

$$\ker T \cap K(T) = \ker T \cap K(T) \cap X_T(\emptyset) = \ker T \cap X_T(\emptyset).$$

(iv) \Rightarrow (ii) By assumption and by 4.3 we have

$$\{0\} = \ker T \cap K(T) = \ker T \cap X_T(\mathbb{C} \setminus \{0\}) \supseteq \ker T \cap X_T(\emptyset).$$

Therefore $\{0\} = \ker T \cap X_T(\emptyset)$ as required. \square

Let $(T, D(T)), D := D(T)$, be a closed linear operator in \mathcal{H} such that $T^n(D) \subseteq D$. To every linear operator T on a vector space D there corresponds the two chains:

$$\{0\} = \ker T^0 \subseteq \ker T \subseteq \ker T^2 \dots$$

and

$$D = T^0(D) \supseteq T(D) \supseteq T^2(D) \dots$$

Definition 4.6. Let $(T, D(T)), D := D(T)$, be a closed linear operator in \mathcal{H} such that $T^n(D) \subseteq D$. The *hypperrange* of T is the subspace defined as

$$T^\infty(D) := \bigcap_{n \in \mathbb{N}} T^n(D).$$

The hyper-kernel of T is the subspace

$$N^\infty(T) := \bigcup_{n \in \mathbb{N}} \ker T^n.$$

Clearly, $\ker T \subseteq N^\infty(T)$, while

$$X_T(\emptyset) \subseteq X_T(\mathbb{C} \setminus \{0\}) \subseteq K(T) \subseteq T^\infty(D) \subseteq T(D). \tag{14}$$

It is obvious, from Theorem 4.5, that if $K(T) = \{0\}$ (this condition, in the bounded case, is satisfied by right weighted shifts, see [1, p. 99]) then T has SVEP at 0. If $T^n(D) \subseteq D$ then this condition may be weakened as follows:

Corollary 4.7. Suppose that $(T, D(T)), D := D(T)$, is a closed linear operator in \mathcal{H} such that $T^n(D) \subseteq D$. If T verifies one of the following conditions:

- (i) $N^\infty(\lambda_0 I - T) \cap (\lambda_0 I - T)^\infty(D) = \{0\}$;
 - (ii) $N^\infty(\lambda_0 I - T) \cap K(\lambda_0 I - T) = \{0\}$;
 - (iii) $N^\infty(\lambda_0 I - T) \cap X_T(\emptyset) = \{0\}$;
 - (iv) $N^\infty(\lambda_0 I - T) \cap (\lambda_0 I - T)(D) = \{0\}$;
- then T has SVEP in λ_0 .

Proof. The proof is a consequence of 14 and Theorem 4.5 \square

Assuming always that $T^n(D) \subseteq D$, we may define two important quantities in Fredholm theory: the *ascent* of T is the smallest positive integer $p = p(T)$, whenever it exists, such that $\ker T^p = \ker T^{p+1}$. If such p does not exist we let $p = +\infty$. Analogously, the *descent* of T is defined to be the smallest integer $q = q(T)$, whenever it exists, such that $T^{q+1}(D) = T^q(D)$. If such q does not exist we let $q = +\infty$.

Theorem 4.8. Let $(T, D(T))$ be a closed linear operator in \mathcal{H} such that $T^n(D) \subseteq D$. If there exists an element $0 \neq x \in \ker T$ such that $\sigma_T(x) = \emptyset$, then $p(T) = +\infty$.

Proof. If there exists an element $0 \neq x \in \ker T$ such that $\sigma_T(x) = \emptyset$, then, by Theorem 4.3, $x \in K(T)$. Therefore there is a sequence $(u_n) \subset D(T)$ such that $u_0 = x$ and $Tu_{n+1} = u_n$ for every $n = 0, 1, \dots$. We have

$$T^{n+1}u_n = T^n(Tu_n) = T^n u_{n-1} = \dots = Tu_0 = 0$$

and

$$T^n u_n = T^{n-1} u_{n-1} = \dots = u_0 \neq 0,$$

for every $n = 0, 1, \dots$. Hence $u_n \in \ker T^{n+1}$ whereas $u_n \notin \ker T^n$. This implies that $p(T) = +\infty$. \square

Theorem 4.8 also shows that

Corollary 4.9. *If $p(\lambda I - T) < \infty$ for every $\lambda \in \mathbb{C}$, then T has SVEP.*

Concluding remark – In the analysis performed in this paper, we have considered closed operators as single objects, regardless to any algebraic structure where they can be cast into. Some classes of closed operators such as measurable operators, exhibit, however, a nice structure (see, e.g. [21], [22] and [24]) and in some cases they can also be normed (think, for instance, of noncommutative L^p spaces). It is certainly relevant to consider the role of the single valued extension property with respect, for instance, to the local spectral radius formula in this specialized situations. We hope to discuss this aspect in a future paper.

References

- [1] P. Aiena, *Fredholm and Local Spectral Theory, With Applications to Multipliers*, (2004) Springer.
- [2] P. Aiena, M. L. Colasante, M. González. *Operators which have a closed quasi-nilpotent part*. Proc. Amer. Math. Soc. **130** (2002), 2701-2710.
- [3] P. Aiena, T. L. Miller, M. M. Neumann, *On a localized single-valued extension property*. Math. Proc. Royal Irish Ac. **104A** (1), (2004), 17–34.
- [4] P. Aiena, and O. Monsalve, *The single valued extension property and the generalized Kato decomposition property*, Acta Sci. Math. (Szeged) **67** (2001), 791–807.
- [5] P. Aiena, O. Monsalve *Operators which do not have the single valued extension property*. J. Math. Anal. Appl. **250** (2000), no. 2, 435-448
- [6] P. Aiena, and M. M. Neumann, *On the stability of the localized single-valued extension property under commuting perturbations*, Proc. Amer. Math. Soc. **141** (2013), no. 6, 20392050.
- [7] I. Colojoară, C. Foiaş *Theory of generalized spectral operators*. Gordon and Breach, New York, 1968.
- [8] N. Dunford *Spectral theory I. Resolution of the identity*. Pacific J. Math. **2** (1952), 559-614.
- [9] N. Dunford *Spectral operators*. Pacific J. Math. **4** (1954), 321-54.
- [10] N. Dunford, J. T. Schwartz *Linear operators, Part I*. (1967), Wiley, New York.
- [11] K. Schmüdgen, *Unbounded operator algebras and representation theory*, Birkhäuser Verlag, Basel, 1990.
- [12] J. K. Finch *The single valued extension property on a Banach space*. Pacific J. Math. **58** (1975), 61-9.
- [13] E. K. Berberian *Some condition on an operator implying normality*. Proc. Amer. Math. Soc. **26** (1970), 277-281.
- [14] N. B. Andersen and M. De Jeu *Local spectral radius formulas for a class of unbounded operators on Banach spaces* Journal of Operator Theory **69**, Issue 2, 2013, Pages 435-461.
- [15] D. Cichon, I.B. Jung and J. Stochel *Normality via local spectral radii* J. Operator Theory **61: 2** (2009), 253-278.
- [16] K. B. Laursen, M. M. Neumann *An introduction to local spectral theory*. Clarendon Press, Oxford, 2000.
- [17] M. Mbekhta *Sur la théorie spectrale locale et limite des nilpotents*. Proc. Amer. Math. Soc. **110** (1990), 621-31.
- [18] M. A. Naimark, *On the square of a closed symmetric operator* (Russian) Dokl. Akad. Nauk SSSR **26** (1940) 207–208.
- [19] P. R. Chernoff, *A semibounded closed symmetric operator whose square has trivial domain*, Proc. Amer. Math. Soc., **89/2** (1983) 289–290.
- [20] J. Danes *On local spectral radius*. Casopis Pest. Mat. **112**, (1987), 177-187.
- [21] F. Bagarello, C. Trapani, S. Triolo *Quasi *-algebras of measurable operators*, Studia Mathematica **172** (2006), no. 3, 289–305.
- [22] S. Triolo, *WQ*-Algebras of measurable operators*, Indian J. Pure Appl. Math. **43(6)**: , December 2012, 601-617.
- [23] P. Vrbová *On local spectral properties of operators in Banach spaces*. Czechoslovak Math. J. **23(98)** (1973a), 483-92.
- [24] C. La Russa; S. Triolo *RadonNikodym Theorem in Quasi *-Agebras*, J. Operator Theory **69:2** (2013), 423–433.