Filomat 28:2 (2014), 275–284
DOI 10.2298/FIL1402275V

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On coverings with special points and monodromy group a Weyl group of type B_d

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Abstract. In this paper we study Hurwitz spaces parameterizing coverings with special points and with monodromy group a Weyl group of type B_d. We prove that such spaces are irreducible if $k > 3d - 3$. Here, k denotes the number of local monodromies that are reflections relative to long roots.

1. Introduction

The study of irreducible components of Hurwitz spaces is a classic problem in algebraic geometry and it is valuable in many applications. The Lüroth - Clebsch - Hurwitz theorem states the irreducibility of the Hurwitz space of simple coverings of $\mathbb{P}^1$ with n branch points (see [11]). This result was used by Severi in order to prove the irreducibility of the moduli space of genus g curves (see [18]). Today, there were many generalizations of Lüroth - Clebsch - Hurwitz result. Let Y be a smooth, connected, projective complex curve of genus g. Specifically, the irreducibility of Hurwitz spaces of coverings of Y with monodromy group S_d and with an arbitrary number of special points has been studied both when $g = 0$ and when $g > 0$ (see [1, 9, 13–15, 19, 24, 27, 28]). We point out that, for example, Harris, Graber and Starr used the result of [9] in order to prove the existence of sections of one-parameter family of complex rationally connected varieties (see [10]). Hurwitz spaces of coverings whose monodromy group is a Weyl group different from S_d and their irreducible components were studied, for example, in [2, 20–23, 25, 26]. We note that coverings with monodromy group a Weyl group appear in the study of spectral curves, integrable systems and Prym - Tyurin varieties (see [6, 15, 16]). In fact, the Prym maps yield morphism from the Hurwitz spaces of coverings with monodromy group contained in a Weyl group to Siegel modular varieties which parameterize Abelian varieties. Thus, some property of these varieties can be studied by using these Hurwitz spaces.

In this paper we continue the investigation of the irreducibility of Hurwitz spaces that parameterize coverings with special fibers and with monodromy group a Weyl group of type B_d. In particular, we work with coverings that decompose into a sequence of type $X \xrightarrow{\pi} X' \xrightarrow{f} Y$ where π is a degree two covering with n_1 branch points and f is a degree d coverings with monodromy group S_d. Moreover, f has n_2 branch points, k of which are simple points and $n_2 - k$ of which are special points. Furthermore, $f(D_\pi) \cap D_f = \emptyset$ where D_π and D_f denote, respectively, the branch locus of π and f.

2010 *Mathematics Subject Classification*. Primary 14H30; Secondary 14H10
Keywords. Hurwitz spaces; special fibers; branched coverings; monodromy; braid moves.
Received: 10 August 2013; Accepted: 14 September 2013
Communicated by Vladimir Rakocevic
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We prove that, under the hypothesis $k > 3d - 3$, the corresponding Hurwitz spaces are irreducible both when $g = 0$ and when $g > 0$ (see Theorems 5.5 and 5.6). In this way, we generalize the results obtained for coverings as above but with one or two special fibers by the author in [20, 25]. Moreover, we extend the results obtained in the case in which the monodromy group is all S_d by Kulikov in [14] and by the author in [27] to coverings with monodromy group a Weyl group of type B_d.

Conventions and Notations Here, two sequences of coverings, $X_1 \xrightarrow{\pi_1} X_1' \xrightarrow{f_1} Y$ and $X_2 \xrightarrow{\pi_2} X_2' \xrightarrow{f_2} Y$, are *equivalent* if there exist two biholomorphic maps $p : X_1 \to X_2$ and $p' : X_1' \to X_2'$ such that $p' \circ \pi_1 = \pi_2 \circ p$ and $f_2 \circ p' = f_1$. We denote by $[f \circ \pi]$ the equivalence class containing $f \circ \pi$. Moreover, we denote by t^h the permutation $h^{-1}\, t\, h$ and we denote by $\langle t_1, \dots, t_l \rangle$ the subgroup of S_d generated by the permutations $t_1, \dots, t_l$.

2. Weyl groups of type B_d

Let $\{\varepsilon_1, \dots, \varepsilon_d\}$ be the standard base of $\boldsymbol{R}^d$ and let R be the root system $\{\pm\varepsilon_i,\ \pm\varepsilon_i \pm \varepsilon_j : 1 \leq i, j \leq d\}$. The Weyl group of type B_d is generated by the reflections s_{ε_i}, with $1 \leq i \leq d$, and by the reflections $s_{\varepsilon_i - \varepsilon_j}$, with $1 \leq i < j \leq d$ (see [4]). We denote this group by $W(B_d)$. We recall that the reflection $s_{\varepsilon_i - \varepsilon_j}$ exchanges ε_i with ε_j and $-\varepsilon_i$ with $-\varepsilon_j$, leaving fixed each ε_h with $h \neq i, j$. The reflection s_{ε_i} exchanges ε_i with $-\varepsilon_i$ and fixes all the ε_h with $h \neq i$. Hence, identifying $\{\pm\varepsilon_i : 1 \leq i \leq d\}$ with $\{\pm 1, \dots, \pm d\}$ by using the map $\pm\varepsilon_i \to \pm i$, we can define an injective homomorphism from $W(B_d)$ into S_{2d} such that

$$s_{\varepsilon_i - \varepsilon_j} \to (i\, j)(-i\ -j),\ \ s_{\varepsilon_i} \to (i\ -i),\ \ s_{\varepsilon_i + \varepsilon_j} = s_{\varepsilon_i}\, s_{\varepsilon_j}\, s_{\varepsilon_i - \varepsilon_j} \to (i\ -j)(-i\, j).$$

Let $(\mathbf{Z}_2)^d$ be the set of the functions from $\{1, \dots, d\}$ into $\mathbf{Z}_2$ equipped with the sum operation. Let us denote by Ψ the homomorphism from S_d in $Aut((\mathbf{Z}_2)^d)$ which assigns to $t \in S_d$ the element $\Psi(t) \in Aut((\mathbf{Z}_2)^d)$ where

$$[\Psi(t)\, a]\, (j) := a\, (j^t) \quad \text{for each} \quad a \in (\mathbf{Z}_2)^d.$$

Let $(\mathbf{Z}_2)^d \times^s S_d$ be the semidirect product of $(\mathbf{Z}_2)^d$ and S_d through the homomorphism Ψ. Given $(a';\, t_1), (a'';\, t_2) \in (\mathbf{Z}_2)^d \times^s S_d$, we put

$$(a';\, t_1) \cdot (a'';\, t_2) := (a' + \Psi(t_1)\, a''; t_1\, t_2).$$

Moreover, we use $\bar{1}_j$ by denote the function of $(\mathbf{Z}_2)^d$ defined as

$$\bar{1}_j(j) = \bar{1} \quad \text{and} \quad \bar{1}_j(h) = \bar{0} \quad \text{for each} \quad h \neq j$$

and we use z_{ij} to denote the function of $(\mathbf{Z}_2)^d$ defined as

$$z_{ij}(i) = z_{ij}(j) = z \quad \text{and} \quad z_{ij}(h) = \bar{0} \quad \text{for each} \quad h \neq i, j \quad \text{and } z \in \mathbf{Z}_2.$$

We notice that the homomorphism from $W(B_d)$ into $(\mathbf{Z}_2)^d \times^s S_d$ defined by

$$s_{\varepsilon_i - \varepsilon_j} \to (0; (i\, j)),\ \ s_{\varepsilon_i} \to (\bar{1}_i; id),\ \ s_{\varepsilon_i + \varepsilon_j} \to (\bar{1}_{ij}; (i\, j))$$

is an isomorphism. In what follows, we will use this isomorphism in order to identify $W(B_d)$ by $(\mathbf{Z}_2)^d \times^s S_d$.

Definition 2.1. *Let h be a positive integer. Let $(c;\, \xi)$ be an element of $W(B_d)$ satisfying the following: ξ is a h-cycle of S_d and c is a function that sends to $\bar{0}$ all the indexes fixed by ξ. We call an such element* positive h-cycle *if c is either zero or a function which sends to $\bar{1}$ an even number of indexes. We call it* negative h-cycle *if it is not positive.*

We recall that two cycles $(c;\, \xi)$ and $(c';\, \xi')$ in $W(B_d)$ are disjoint if ξ and ξ' are disjoint. Furthermore, all the elements in $W(B_d)$ can be expressed as a product of disjoint positive and negative cycles. The lengths of such disjoint cycles together with their signs determine the signed cycle type of the elements of $W(B_d)$. Two elements of $W(B_d)$ are conjugate if and only if they have the same signed cycle type (see [5]).

3. Hurwitz spaces of type $H^{W(B_d)}_{d,n_1,k,q_1\underline{e}^1,\dots,q_r\underline{e}^r}(Y)$

Let X, X' and Y be smooth, connected, projective complex curves and let g be the genus of Y. Let d, n_1, n_2, k be integers such that $d \geq 3, n_1 > 0$ and $n_2 > k > 0$. In this paper we are interested in degree $2d$ coverings that decompose in a sequence of coverings, $X \xrightarrow{\pi} X' \xrightarrow{f} Y$, satisfying the followings:

π is a degree 2 covering with n_1 branch points and f is a degree d coverings, with monodromy group S_d and with n_2 branch points, k of which are simple points and $n_2 - k$ of which are special points. Moreover, $f(D_\pi) \cap D_f = \emptyset$ where D_π and D_f denote, respectively, the branch locus of π and f.

Let b_0 be a point of Y and let D be a finite subset of Y such that $b_0 \in Y - D$. By Riemann's existence theorem (see [8], Proposition 1.2) there is a natural one-to-one correspondence between:

− *the set of equivalence classes of degree* $2d$ *branched coverings of* Y *with branch locus* D

and

− *the set of equivalence classes of homomorphisms* $m : \pi_1(Y - D, b_0) \to S_{2d}$ *whose images are transitive subgroups of* S_{2d}*, where two homomorphisms* m *and* m' *are equivalent if there exists* $h \in S_{2d}$ *such that* $m'([\gamma]) = h^{-1}m([\gamma])h$ *for each* $[\gamma] \in \pi_1(Y - D, b_0)$.

From now on, we will denote by D and by m, respectively, the branch locus and the monodromy homomorphism of $f \circ \pi$.

Let $\underline{e}^1, \dots, \underline{e}^r$ be partitions of d such that $\underline{e}^i = (e^i_1, \dots, e^i_{s_i})$ and $e^i_1 \geq \dots \geq e^i_{s_i}$. Let $q_1, \dots, q_r$ be positive integers such that $q_1 + \dots + q_r = n_2 - k$. Let us denote by $H^{W(B_d)}_{d,n_1,k,q_1\underline{e}^1,\dots,q_r\underline{e}^r}(Y)$ the Hurwitz space of equivalence classes of sequences of coverings, $f \circ \pi$, defined as above such that q_i among the special points of f have local monodromy whose cycle type is given by the partition $\underline{e}^i$, for $i = 1, \dots, r$.

Definition 3.1. *Let G be an arbitrary group. An ordered sequence*

$$(t_1, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g) := (\underline{t}, \underline{\lambda}, \underline{\mu})$$

of elements in G is a Hurwitz system *if* $t_i \neq id$ *for each* $i \in \{1, \dots, n\}$ *and* $t_1 \cdots t_n = [\lambda_1, \mu_1] \cdots [\lambda_g, \mu_g]$. *The subgroup of G generated by* t_i, λ_s, μ_s *with* $i = 1, \dots, n$ *and* $s = 1, \dots, g$ *is called the* monodromy group *of the Hurwitz system. Two Hurwitz systems* $(\underline{t}, \underline{\lambda}, \underline{\mu})$ *and* $(\underline{t}', \underline{\lambda}', \underline{\mu}')$ *with elements in G are equivalent if there exsists* $h \in G$ *such that* $t'_i = h^{-1}t_ih$, $\lambda'_s = h^{-1}\lambda_sh$ *and* $\mu'_s = h^{-1}\mu_sh$ *for each* $i = 1, \dots, n$ *and* $s = 1, \dots, g$.

Remark 3.2. *We notice that an order sequence* $(t_1, \dots, t_n)$ *of elements in G, with* $t_i \neq id$ *for each i, is a Hurwitz system if* $t_1 \cdots t_n = id$.

Let $(\gamma_1, \dots, \gamma_{n_1+n_2}, \alpha_1, \beta_1, \dots, \alpha_g, \beta_g)$ be a standard generating system for $\pi_1(Y - D, b_0)$. The images via m of $\gamma_1, \dots, \gamma_{n_1+n_2}, \alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ determine an equivalence class of Hurwitz systems

$$[t_1, \dots, t_{n_1+n_2}; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g]$$

with monodromy group $W(B_d)$ satisfying the following conditions: k among the t_h are elements of type $(z_{ij}; (i, j))$, n_1 are elements of type $(\bar{1}_l; id)$ and q_i, with $i = 1, \dots, r$, are product of s_i positive disjoint cycles whose lengths are given by the elements of the partition $\underline{e}^i$. Let us denote by $A^o_{k,n_1,q_1\underline{e}^1,\dots,q_r\underline{e}^r,g}$ the set of all equivalence classes of Hurwitz systems as above.

We notice that by Riemann's existence theorem, we can identify the set of equivalence classes $[f \circ \pi] \in H^{W(B_d)}_{d,n_1,k,q_1\underline{e}^1,\dots,q_r\underline{e}^r}(Y)$ such that $f \circ \pi$ has branch locus D with the set $A^o_{k,n_1,q_1\underline{e}^1,\dots,q_r\underline{e}^r,g}$.

4. Braid moves

Let n be a positive integer. Let $Y^{(n)}$ be the n-fold symmetric product of Y and Δ be the codimension 1 locus of $Y^{(n)}$ consisting of non simple divisors. In this paper we are interested in the way in which the generators of the braid group $\pi_1(Y^{(n)} - \Delta, D)$ act on Hurwitz systems. So, we recall that such group is generated by the elementary braids σ_i with $i = 1, \ldots, n-1$ and by the braids ρ_{js}, τ_{js} with $1 \leq j \leq n$ and $1 \leq s \leq g$ (see [3, 7, 17]). Here, we denote by σ_i', $\sigma_i'' = (\sigma_i')^{-1}$ the pair of moves associated to σ_i. We call σ_i', σ_i'' elementary moves. The moves σ_i', σ_i'' fix all the λ_s, all the μ_s and all the t_h with $h \neq i, i+1$. They transform (t_i, t_{i+1}) into

$$(t_i t_{i+1} t_i^{-1}, t_i) \quad \text{and} \quad (t_{i+1}, t_{i+1}^{-1} t_i t_{i+1}),$$

respectively (see [11]). We denote by ρ_{js}', $\rho_{js}'' = (\rho_{js}')^{-1}$ and by τ_{js}', $\tau_{js}'' = (\tau_{js}')^{-1}$, respectively, the pair of moves associated to ρ_{js} and τ_{js}. We use the following result.

Proposition 4.1 ([12], Theorem 1. 8). *Let $(t_1, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g)$ be a Hurwitz system. Let $u_0 = 1$ and let $u_s = [\lambda_1, \mu_1] \cdots [\lambda_s, \mu_s]$ for $s = 1, \ldots, g$. The following formulae hold:*

$$(t_1, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g) \to (t_1', \ldots, t_n'; \lambda_1', \mu_1', \ldots, \lambda_g', \mu_g')$$

- *For ρ_{is}' where $1 \leq i \leq n$, $1 \leq s \leq g$*

  $t_j' = t_j$ for each $j \neq i$, $\lambda_l' = \lambda_l$ for each l, $\mu_l' = \mu_l$ for each $l \neq s$ and

$$(t_i, \mu_s) \to (t_i', \mu_s') = (a_1^{-1} t_i a_1,\ b_1^{-1} t_i^{-1} b_1\, \mu_s)$$

  where $a_1 = (t_1 \cdots t_{i-1})^{-1} u_{s-1} \lambda_s (u_s^{-1} u_g)(t_{i+1} \cdots t_n)^{-1}$ and $b_1 = (t_1 \cdots t_{i-1})^{-1} u_{s-1} \lambda_s$.

- *For τ_{is}' where $1 \leq i \leq n$, $1 \leq s \leq g$*

  $t_j' = t_j$ for each $j \neq i$, $\lambda_l' = \lambda_l$ for each $l \neq s$, $\mu_l' = \mu_l$ for each l and

$$(t_i, \lambda_s) \to (t_i', \lambda_s') = (c_1^{-1} t_i c_1,\ d_1^{-1} t_i d_1\, \lambda_s)$$

  where $c_1 = t_{i+1} \cdots t_n (u_s^{-1} u_g)^{-1} \mu_s (u_{s-1})^{-1} t_1 \cdots t_{i-1}$ and $d_1 = t_{i+1} \cdots t_n (u_s^{-1} u_g)^{-1} \mu_s$.

- *For ρ_{is}'' where $1 \leq i \leq n$, $1 \leq s \leq g$*

  $t_j' = t_j$ for each $j \neq i$, $\lambda_l' = \lambda_l$ for each l, $\mu_l' = \mu_l$ for each $l \neq s$ and

$$(t_i, \mu_s) \to (t_i', \mu_s') = (a_2^{-1} t_i a_2,\ b_2^{-1} t_i b_2\, \mu_s)$$

  where $a_2 = t_{i+1} \cdots t_n (u_s^{-1} u_g)^{-1} \lambda_s^{-1} (u_{s-1})^{-1} t_1 \cdots t_{i-1}$ and $b_2 = t_{i+1} \cdots t_n (u_s^{-1} u_g)^{-1}$.

- *For τ_{is}'' where $1 \leq i \leq n$, $1 \leq s \leq g$*

  $t_j' = t_j$ for each $j \neq i$, $\lambda_l' = \lambda_l$ for each $l \neq s$, $\mu_l' = \mu_l$ for each l and

$$(t_i, \lambda_s) \to (t_i', \lambda_s') = (c_2^{-1} t_i c_2,\ d_2^{-1} t_i^{-1} d_2\, \lambda_s)$$

  where $c_2 = (t_1 \cdots t_{i-1})^{-1} u_{s-1} \mu_s^{-1} (u_s^{-1} u_g)(t_{i+1} \cdots t_n)^{-1}$ and $d_2 = (t_1 \cdots t_{i-1})^{-1} u_{s-1}$.

Remark 4.2. *The moves ρ_{is}', ρ_{is}'', τ_{is}' and τ_{is}'' transform t_i into an element belonging to the same conjugacy class. Furthermore, we notice that when $\lambda_1 = \cdots = \lambda_s = \mu_1 = \cdots = \mu_{s-1} = id$, the braid move ρ_{1s}' transforms*

$$\mu_s \quad \text{into} \quad t_1^{-1} \mu_s.$$

Analogously when $\lambda_1 = \cdots = \lambda_{s-1} = \mu_1 = \cdots = \mu_{s-1} = id$, the braid move τ_{1s}'' transforms

$$\lambda_s \quad \text{into} \quad t_1^{-1} \lambda_s.$$

Definition 4.3. *Two Hurwitz systems are said* braid equivalent *if one is obtained from the other by using a finite sequence of braid moves σ_i', ρ_{js}', τ_{js}', σ_i'', ρ_{js}'', τ_{js}'' where $1 \leq i \leq n-1$, $1 \leq j \leq n$ and $1 \leq s \leq g$. Two ordered sequences of permutations $(t_1, \ldots, t_l)$ and $(t_1', \ldots, t_l')$ are said* braid equivalent *if $(t_1', \ldots, t_l')$ is obtained from $(t_1, \ldots, t_l)$ by using a finite sequence of braid moves of type σ_i', σ_i''. We denote the braid equivalence by $\sim$.*

5. Irreducibility of $H^{W(B_d)}_{d,n_1,k,q_1\underline{e}^1,\dots,q_r\underline{e}^r}(Y)$

In what follows, we write $|\underline{e}^1|$ to denote $\sum_{j=1}^{s_1}(e^1_j - 1)$. Moreover, we associate to the partition $\underline{e}^i$ the following element in S_d having cycle type given by $\underline{e}^i$

$$\epsilon_i := (1, 2, \dots, e^i_1)(e^i_1 + 1, \dots, e^i_1 + e^i_2) \cdots (\sum_{j=1}^{s_i - 1} e^i_{j+1} \dots d).$$

Let ϵ be the following permutation of S_d

$$(\epsilon_1 \cdots \epsilon_1 \epsilon_2 \cdots \epsilon_2 \cdots \epsilon_r \cdots \epsilon_r)^{-1}$$

where ϵ_i, with $i = 1, \dots, r$, appears q_i times. Let $\xi_1, \dots, \xi_q$ be disjoint cycles of lengths $h_1, \dots, h_q$, with $h_1 \geq h_2 \geq \cdots \geq h_q$, such that $\epsilon = \xi_1 \cdots \xi_q$. Let $\xi_j = (l^j_1 \dots l^j_{h_j})$ where $l^j_1 < l^j_b$ for each $b = 2, \dots, h_j$. In the sequel, we denote by Z_j the sequence of transpositions $((l^j_1, l^j_2), (l^j_1, l^j_3), \dots, (l^j_1, l^j_{h_j}))$ and by Z the concatenation $Z_1, Z_2, \dots, Z_q$.

For a convenience of the reader we recall the following results.

Lemma 5.1 ([19], Proposition 3). *Let $(t_1, t_2, \dots, t_l)$ be a sequence of permutations in S_d such that t_1 has cycle type $\underline{e}^1$ and $t_2, \dots, t_l$ are transpositions.*

If $l - 1 + |\underline{e}^1| \geq 2d$ then $(t_1, t_2, \dots, t_l)$ is braid equivalent to

$$(t'_1, t'_2, \dots, t'_{l-2}, t'_{l-1}, t'_l)$$

where t'_1 has cycle type $\underline{e}^1$, $t'_2, \dots, t'_l$ are transpositions, $t'_{l-1} = t'_l$ and

$$\langle t'_1, t'_2, \dots, t'_{l-2} \rangle = \langle t'_1, \dots, t'_{l-2}, t'_{l-1}, t'_l \rangle.$$

Lemma 5.2 ([12], Main Lemma 2.1). *Let $(t_1, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g)$ be a Hurwitz system of permutations in S_d. Suppose that $t_i t_{i+1} = id$. Let H be the subgroup of S_d generated by $\{t_1, \dots, t_{i-1}, t_{i+2}, \dots, t_n, \lambda_1, \mu_1, \dots, \lambda_g, \mu_g\}$. Then for every $h \in H$ the given Hurwitz system is braid equivalent to*

$$(t_1, \dots, t_{i-1}, t^h_i, t^h_{i+1}, t_{i+2}, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g).$$

Proposition 5.3 ([14], Theorem 2. 3). *Let $[t_1, \dots, t_{n_2}]$ be an equivalence class of Hurwitz systems of permutations in S_d, with monodromy group S_d, satisfying the followings: k among the t_j are transpositions and q_i among the t_j are permutations whose cycle type is given by the partition $\underline{e}^i$ of d, for $i = 1, \dots, r$. If $k > 3d - 3$, $[t_1, \dots, t_{n_2}]$ is braid equivalent to the class $[\tilde{t}_1, \dots, \tilde{t}_{n_2}]$ where*

$$\tilde{t}_1 = \ldots = \tilde{t}_{q_1} = \epsilon_1, \;\; \tilde{t}_{q_j+1} = \ldots = \tilde{t}_{q_{j+1}} = \epsilon_{j+1}$$

with $j = 1, \dots, r - 1$. Moreover the sequence $(\tilde{t}_{n_2-k+1}, \dots, \tilde{t}_{n_2})$ is equal to

$$(Z, (1, 2), \dots, (1, 2), (2, 3), (2, 3), \dots, (d - 1, d), (d - 1, d))$$

where $(1, 2)$ appears an even number of times.

Now, by using Proposition 5.3, we show that any two class in $A^o_{k,n_1,q_1\underline{e}^1,\dots,q_r\underline{e}^r,0}$ are braid equivalent.

Proposition 5.4. *If $k > 3d - 3$, each equivalence class in $A^o_{k,n_1,q_1\underline{e}^1,\dots,q_r\underline{e}^r,0}$ is braid equivalent to a class of the form:*

$$[(0; \tilde{t}_1), \dots, (0; \tilde{t}_{n_2}), (\bar{1}_1; id), \dots, (\bar{1}_1; id)]$$

where $(\tilde{t}_1, \dots, \tilde{t}_{n_2})$ is the sequence in Proposition 5.3.

Proof. Let $[\,\underline{t}\,] \in A^{o}_{k,n_1,q_1\underline{e}^1,\dots,q_r\underline{e}^r,0}$. We act with elementary moves of type σ'_i and move on the right all the elements $(\bar{1}_*; id)$. In this way, we have that $[\,\underline{t}\,]$ is braid equivalent to a class of the form

$$[(*;t'_1),\dots,(*;t'_{n_2}),(\bar{1}_*;id),\dots,(\bar{1}_*;id)].$$

We notice that the equivalence class $[t'_1,\dots,t'_{n_2}]$ satisfies all the hypothesis in Proposition 5.3, so it is braid equivalent to the class $[\tilde{t}_1,\dots,\tilde{t}_{n_2}]$. Then, $[\underline{t}]$ is braid equivalent to a class of type

$$[(*;\tilde{t}_1),\dots,(*;\tilde{t}_{n_2}),(\bar{1}_*;id),\dots,(\bar{1}_*;id)].$$

Step 1**.** We show that $[\underline{t}]$ is braid equivalent to a class of the form

$$[(*;\tilde{t}_1),\dots,(*;\tilde{t}_{n_2}),(\bar{1}_1;id),\dots,(\bar{1}_1;id)].$$

Let i and j be two arbitrary indexes in $\{1,\dots,d\}$ such that $i<j$ and $j\neq i+1$. We notice that the sequence $((*;(i,i+1)),(*;(i,i+1)),\dots,(*;(j-1,j)),(*;(j-1,j))$ is braid equivalent to the sequence

$$((*;(i,j)),(*;(i,j)),(*;(i,i+1)),(*;(i,i+1)),\dots,(*;(j-2,j-1)),(*;(j-2,j-1))).$$

In fact, if the elements of the pair $((*;(j-1,j)),(*;(j-1,j))$ are at the places $h,h+1$ and the elements of the pair $((*;(i,i+1)),(*;(i,i+1))$ are at the places $l,l+1$, in order to obtain the claim we can act with the sequence of moves

$$\sigma'_{h-1},\sigma'_h,\sigma''_{h-2},\sigma''_{h-1},\sigma'_{h-3},\sigma'_{h-2},\sigma''_{h-4},\sigma''_{h-3},\dots,\sigma'_{l+1},\sigma'_{l+2},\sigma''_l,\sigma''_{l+1}.$$

Now, we can bring the elements of the pair $((*;(i,j)),(*;(i,j)))$ to the places n_2-1 and n_2 by using the sequence of moves $\sigma'_{l+1},\sigma'_l,\sigma'_{l+2},\sigma'_{l+1},\dots,\sigma'_{n_2-1},\sigma'_{n_2-2}$.

This ensures that acting by suitable elementary moves on the sequence $((*;(1,2)),(*;(1,2)),\dots,(*;(d-1,d)),(*;(d-1,d)))$ we can replace it with

$$((*;(1,2)),(*;(1,2)),\dots,(*;(u-2,u-1)),(*;(u-2,u-1)),(*;(u,u+1)),$$
$$(*;(u,u+1)),\dots,(*;(d-1,d)),(*;(d-1,d)),(*;(1,u)),(*;(1,u)))$$

where u is an arbitrary index in $\{1,\dots,d\}$. Let $(\bar{1}_v;id)$ be the element that occupies the place n_2+1. Then, we choose $u=v$ and we act with σ'_{n_2} in order to replace $(\bar{1}_v;id)$ with $(\bar{1}_1;id)$. Now, we move this element to the last place. If the elements $(*;(u,u+1)),(*;(u,u+1))$ are in the places $h+2,h+3$, then we use the moves

$$\sigma'_{n_2-2},\sigma'_{n_2-1},\sigma'_{n_2-3},\sigma'_{n_2-2},\dots,\sigma'_{n_2-2(d-1)+2},\sigma'_{n_2-2(d-1)+3},\sigma'_{n_2-2(d-1)+3},\sigma'_{n_2-2(d-1)+2},$$
$$\sigma''_{n_2-2(d-1)+4},\sigma''_{n_2-2(d-1)+3},\sigma'_{n_2-2(d-1)+5},\sigma'_{n_2-2(d-1)+4},\dots,\sigma''_{h-1},\sigma''_h,\sigma'_h,\sigma'_{h-1}$$

in order to obtain again a sequence of the type

$$((*;(1,2)),(*;(1,2)),\dots,(*;(u-1,u)),(*;(u-1,u)),\dots,(*;(d-1,d)),(*;(d-1,d))).$$

Now, we can proceed as above for all the elements of type $(\bar{1}_*;id)$. In this way, we obtain the claim.

Step 2. By Step 1, $[\underline{t}]$ is braid equivalent to a class of the form

$$[(b_1;\tilde{t}_1),\dots,(*;\tilde{t}_{n_2}),(\bar{1}_1;id),(\bar{1}_1;id),\dots,(\bar{1}_1;id)].$$

Now we claim that $[\underline{t}]$ is braid equivalent to a class of type

$$[(0;\tilde{t}_1),\dots,(0;\tilde{t}_{n_2-2(d-1)}),(*;\tilde{t}_{n_2-2(d-1)+1}),\dots,(*;\tilde{t}_{n_2}),(\bar{1}_1;id),\dots,(\bar{1}_1;id)].$$

Let $i_1,\ i_2,\dots,i_l$ be the indexes which b_1 sends to $\bar{1}$. We suppose that $i_1<i_2<\cdots<i_{l-1}<i_l$. We notice that, by Step 1, we can assume that the element at the place n_2+1 is $(\bar{1}_{i_l};id)$. In fact, in order to obtain the claim it

is sufficient to choose $u = i_l$. By using elementary moves of type σ''_j, we move $(\bar{1}_{i_l}; id)$ to the place 2. We act two times with the moves σ''_1 and so we replace the pair $((b_1;\tilde{t}_1),(\bar{1}_{i_l};id))$ with $((\hat{b}_1;\tilde{t}_1),(\bar{1}_{i_l+1};id))$ where $\hat{b}_1$ is a function that sends to $\bar{1}$ the indexes $i_1, i_2, \dots, i_{l-1}, i_l - 1$. Here, $i_l - 1$ and $i_l + 1$ are, respectively, the index that precede and the index that follow i_l in $\tilde{t}_1$. Now, we move the element $(\bar{1}_{i_{l+1}}; id)$ to the place $n_2 + 1$. By Step 1, we can replace $(\bar{1}_{i_{l+1}}; id)$ with $(\bar{1}_1; id)$.

Since b_1 is a function which sends to $\bar{1}$ an even number of indexes (see Definition 2.1), acting as above, after a finite number of steps, we can replace the element $(\hat{b}_1;\tilde{t}_1)$ with $(0;\tilde{t}_1)$.

We can proceed as done for $(b_1;\tilde{t}_1)$, also for all the elements of type $(*;\tilde{t}_j)$ with $j = 2,\dots,n_2-2(d-1)$. In this way, we obtain the claim.

Step 3 . By Step 2, $[\underline{t}]$ is braid equivalent to the class

$$[(0;\tilde{t}_1),\dots,(0;\tilde{t}_{n_2-2(d-1)}),(*;(1,2)),(*;(1,2)),\dots,(*;(d-1,d)),(*;(d-1,d)),(\bar{1}_1;id),\dots,(\bar{1}_1;id)].$$

Since n_1 is even, one has

$$(0;\tilde{t}_1)\cdots(0;\tilde{t}_{n_2-2(d-1)})(*;(1,2))(*;(1,2))\cdots(*;(d-1,d))(*;(d-1,d)) = (0;id).$$

From this it follows that the sequence $((*;(1,2)),(*;(1,2)),\dots,(*;(d-1,d)),(*;(d-1,d)))$ is equal to either

$$((0;(1,2)),(0;(1,2)),\dots,(0;(d-1,d)),(0;(d-1,d)))$$

or

$$((\bar{1}_{12};(1,2)),(\bar{1}_{12};(1,2)),\dots,(\bar{1}_{d-1d};(d-1,d)),(\bar{1}_{d-1d};(d-1,d))).$$

In the first case, we have the claim. So, we analyze the second case. We use the moves $\sigma'_{n_2},\sigma'_{n_2-1},\dots,\sigma'_{n_2-2(d-1)+3}$ in order to shift one element of type $(\bar{1}_1;id)$ to the right of the pair $((\bar{1}_{12};(1,2)),(\bar{1}_{12};(1,2)))$. We use the moves $\sigma''_{n_2-2(d-1)+2},\sigma''_{n_2-2(d-1)+2},\sigma''_{n_2-2(d-1)+1}$ in order to replace the sequence $((\bar{1}_{12};(1,2)),(\bar{1}_{12};(1,2)),(\bar{1}_1;id))$ with $((0;(1,2)),(\bar{1}_{12};(1,2)),(\bar{1}_2;id))$.

By using the moves

$$\sigma'_{n_2-2(d-1)+3},\sigma''_{n_2-2(d-1)+4},\sigma'_{n_2-2(d-1)+5},\sigma''_{n_2-2(d-1)+6},\dots,\sigma'_{n_2},\sigma''_{n_2+1}$$

we replace

$$((\bar{1}_2;id),(\bar{1}_{23};(2,3)),(\bar{1}_{23};(2,3)),\dots,(\bar{1}_{d-1d};(d-1,d)),(\bar{1}_{d-1d};(d-1,d)))$$

with

$$((0;(2,3)),(\bar{1}_{23};(2,3)),\dots,(0;(d-1,d)),(\bar{1}_{d-1d};(d-1,d)),(\bar{1}_d;id)).$$

Now, we apply the sequence of moves $\sigma''_{n_2},\sigma'_{n_2-1},\dots,\sigma''_{n_2-2(d-1)+4},\sigma'_{n_2-2(d-1)+3},\sigma''_{n_2-2(d-1)+2},\sigma''_{n_2-2(d-1)+2}$. In this way, we have that the above sequence is braid equivalent to

$$((0;(1,2)),(\bar{1}_1;id),(0;(2,3)),(0;(2,3)),\dots,(0;(d-1,d)),(0;(d-1,d))).$$

We obtain the claim by using the moves

$$\sigma''_{n_2-2(d-1)+3},\sigma''_{n_2-2(d-1)+4},\dots,\sigma''_{n_2-1},\sigma''_{n_2}.$$

□

The purpose of this paper is to show that the space $H^{W(B_d)}_{d,n_1,k,q_1\underline{e}^1,\dots,q_r\underline{e}^r}(Y)$ is irreducible. We notice that such space is smooth. So, if we prove that it is connected then we also prove that it is irreducible. Let

$$\delta : H^{W(B_d)}_{d,n_1,k,q_1\underline{e}^1,\dots,q_r\underline{e}^r}(Y) \to Y^{(n_1+n_2)} - \Delta$$

be the map which assigns to each equivalence class $[f \circ \pi]$ the branch locus of $f \circ \pi$. The topology defined on $H^{W(B_d)}_{d,n_1,k,q_1\underline{e}^1,\dots,q_r\underline{e}^r}(Y)$ is such that δ is a topological covering map (see [8]). Therefore the braid group $\pi_1(Y^{(n_1+n_2)} - \Delta, D)$ acts on $A^o_{k,n_1,q_1\underline{e}^1,\dots,q_r\underline{e}^r,g}$. The orbits of this action are in one-to-one correspondence with the connected components of $H^{W(B_d)}_{d,n_1,k,q_1\underline{e}^1,\dots,q_r\underline{e}^r}(Y)$. So, if we prove that $\pi_1(Y^{(n_1+n_2)} - \Delta, D)$ acts transitively on $A^o_{k,n_1,q_1\underline{e}^1,\dots,q_r\underline{e}^r,g}$ then we also prove that $H^{W(B_d)}_{d,n_1,k,q_1\underline{e}^1,\dots,q_r\underline{e}^r}(Y)$ is connected. We notice that, in order to check the transitivity of this action, it is sufficient to prove that any class in $A^o_{k,n_1,q_1\underline{e}^1,\dots,q_r\underline{e}^r,g}$ is braid equivalent to a given normal form. Hence, an immediate consequence of the previous proposition is the following theorem.

Theorem 5.5. *If $k > 3d - 3$, then the Hurwitz space $H^{W(B_d)}_{d,n_1,k,q_1\underline{e}^1,\dots,q_r\underline{e}^r}(\boldsymbol{P}^1)$ is irreducible.*

From Prposition 5.4, it follows also the following result.

Theorem 5.6. *Let Y be a smooth, connected, projective complex curve of genus ≥ 1. If $k > 3d - 3$, then the Hurwitz space $H^{W(B_d)}_{d,n_1,k,q_1\underline{e}^1,\dots,q_r\underline{e}^r}(Y)$ is irreducible.*

Proof. In order to obtain the claim it is sufficient to prove that each equivalence class in $A^o_{k,n_1,q_1\underline{e}^1,\dots,q_r\underline{e}^r,g}$ is braid equivalente to a class of the form $[\underline{t}'; (0; id), (0; id), \dots, (0; id), (0; id)]$. In fact, $[\underline{t}']$ belongs to $A^o_{k,n_1,q_1\underline{e}^1,\dots,q_r\underline{e}^r,0}$ and so the theorem follows by Proposition 5.4. Let $[\underline{t}; \underline{\lambda}, \underline{\mu}] \in A^o_{k,n_1,q_1\underline{e}^1,\dots,q_r\underline{e}^r,g}$.

Step 1. At first, we show that $[\underline{t}; \underline{\lambda}, \underline{\mu}]$ is braid equivalent to a class of type $[\dots, (\bar{1}_i; id), \dots; \underline{\lambda}, \underline{\mu}]$ where i is an arbitrary index in $\{1, \dots, d\}$.

Using suitable elementary moves σ'_i, we shift on the right the elements of the form $(\bar{1}_*; id)$. We act with elementary moves σ''_i in order to bring to the first place an element of type $(*; \eta)$, where η is a permutation with cycle type given by the partition $\underline{e}^1$ of d. Now, we move to the places $2, \dots, k+1$ the elements of type $(z_{ij}; (i, j))$. In this way, we have that our class is braid equivalent to

$$[\bar{t}_1, \dots, \bar{t}_{n_2}, (\bar{1}_h; id), \dots, (\bar{1}_*; id); \lambda_1, \mu_1, \dots, \lambda_g, \mu_g]$$

where $\bar{t}_i = (*; t'_i)$, $\lambda_k = (*; \lambda'_k)$, $\mu_k = (*; \mu'_k)$, $t'_1 = \eta$ and $t'_2, \dots, t'_{k+1}$ are transpositions.

We observe that the condition $k > 3d - 3$ ensures that $k + |\underline{e}^1| \geq 2d$. So, by Lemma 5.1, we have that the sequence of permutations $(\eta, t'_2, \dots, t'_{k+1})$ is braid equivalent to a sequence of type $(\eta', t''_2, \dots, t''_{k+1})$ where η' has cycle type $\underline{e}^1$, $t''_2, \dots, t''_{k+1}$ are transpositions, $t''_k = t''_{k+1}$ and

$$\langle \eta', t''_2, \dots, t''_{k+1} \rangle = \langle \eta', t''_2, \dots, t''_{k-1} \rangle.$$

Now, we notice that $(\eta', t''_2, \dots, t''_{k+1}, \dots, t'_{n_2}; \lambda'_1, \mu'_1, \dots, \lambda'_g, \mu'_g)$ is the Hurwitz system of a degree d branched covering of Y with monodromy group S_d. So, by Lemma 5.2, it is braid equivalent to a system of type $(\dots, \nu, \nu \dots; \lambda'_1, \mu'_1, \dots, \lambda'_g, \mu'_g)$ where ν is an arbitrary transposition of S_d. From this, it follows that our class is braid equivalent to a class of type

$$[\dots, (*; \nu), (*; \nu), \dots, (\bar{1}_h; id), \dots; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g].$$

Now, in order to obtain the claim it is sufficient to choose $\nu = (i, h)$, to move one element of type $(*; \nu)$ to the place n_2 and to act with σ'_{n_2}.

Step 2. Now, we claim that $[\underline{t}; \underline{\lambda}, \underline{\mu}]$ is braid equivalent to a class of type $[\underline{t}'; (0; id), (0; id), \dots, (0; id), (0; id)]$.

Acting by suitable elementary moves σ'_j we have that our class is braid equivalent to

$$[\hat{t}_1, \dots, \hat{t}_{n_2}, (\bar{1}_*; id), \dots, (\bar{1}_*; id); \lambda_1, \mu_1, \dots, \lambda_g, \mu_g]$$

where $\hat{t}_i = (*; \dot{t}_i)$, $\lambda_k = (*; \lambda'_k)$ and $\mu_k = (*; \mu'_k)$.

We notice that $(\dot{t}_1, \dots, \dot{t}_{n_2}; \lambda'_1, \mu'_1, \dots, \lambda'_g, \mu'_g)$ is the Hurwitz system of a degree $d \geq 3$ covering of Y, with monodromy group S_d and with n_2 branch points, k of which are simple points and $n_2 - k$ of which are special points. Moreover, q_i among of the special points have local monodromies with cycle type given by the partition $\underline{e}^i$ of d. Since, under the condition $k > 3d - 3$, the Hurwitz space parameterizing coverings as above is irreducible (see [27], Theorem 2), the Hurwitz system $(\dot{t}_1, \dots, \dot{t}_{n_2}; \lambda'_1, \mu'_1, \dots, \lambda'_g, \mu'_g)$ is braid equivalent to a system of type

$$(\ddot{t}_1, \dots, \ddot{t}_{n_2}; id, id, \dots, id, id).$$

Hence, $[\underline{t}; \underline{\lambda}, \underline{\mu}]$ is braid equivalent to a class of the form

$$[\bar{t}_1, \dots, \bar{t}_{n_2}, (\bar{1}_*; id), \dots; (a_1; id), (b_1; id), \dots, (a_g; id), (b_g; id)].$$

We notice that if $a_s = 0$ and $b_v = 0$ for each $1 \leq s, v \leq g$ we have the claim. So, let $a_1 \neq 0$ and i be one of the indexes that a_1 sends to $\bar{1}$.

By Step 1, $[\bar{t}_1, \dots, \bar{t}_{n_2}, (\bar{1}_*; id), \dots; (a_1; id), (b_1; id), \dots, (a_g; id), (b_g; id)]$ is braid equivalent to the class

$$[\dots, (\bar{1}_i; id), \dots; (a_1; id), (b_1; id), \dots, (a_g; id), (b_g; id)].$$

Acting with elementary moves σ''_j we bring to the first place the element $(\bar{1}_i; id)$ and then we use the move τ''_{11} to replace $(a_1; id)$ with $(\bar{1}_i; id)\,(a_1; id)$ where $\bar{1}_i + a_1$ is a function that sends i to $\bar{0}$.

So reasoning for all the indexes that a_1 sends to $\bar{1}$, after a finite number of steps, we obtain that our class is braid equivalent to

$$[\dots; (0; id), (b_1; id), \dots, (a_g; id), (b_g; id)].$$

If $a_1 = 0$, $b_1 \neq 0$ and b_1 sends i to $\bar{1}$, we again use elementary moves of type σ''_j to shift $(\bar{1}_i; id)$ to the first place. We act by the braid move ρ'_{11} and so we transform $(b_1; id)$ into $(\bar{1}_i; id)\,(b_1; id)$ where the function $\bar{1}_i + b_1$ sends i to $\bar{0}$. Following this line for all the indexes that b_1 sent to $\bar{1}$, we can replace our class with

$$[\dots; (0; id), (0; id), \dots, (a_g; id), (b_g; id)].$$

We notice that if $a_s \neq 0$ and $a_l = b_l = 0$, for each $l \leq s - 1$, in order to obtain the claim one can reason in the same way but this time applying the braid move τ'_{1s}. Analogously if $b_s \neq 0$, $a_l = b_l = 0$, for each $l \leq s - 1$, and $a_s = 0$ one can apply the braid move ρ'_{1k} to transform $(b_k; id)$ into $(0; id)$. □

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