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Existence of μ-pseudo Almost Automorphic Solutions to a Neutral Differential Equation by Interpolation Theory

Yong-Kui Chang[a], Xiao-Xia Luo[a]

[a]*Department of Mathematics, Lanzhou Jiaotong University, Lanzhou 730070, P. R. China*

Abstract. In this paper, we shall deal with μ-pseudo almost automorphic solutions to a neutral differential equation. To achieve this goal, we first prove a composition theorem for μ-pseudo almost automorphic functions under suitable conditions, and then apply it to investigate some existence results by the interpolation theory and fixed point methods.

1. Introduction

The concept of almost automorphy was first introduced in the literature by Bochner in [1], it is a natural generalization of almost periodicity in the sense of Bohr [2], for more details about this topic we refer to [3–8] and references therein. Since then, almost automorphy has become one of the most attractive topics in the qualitative theory of evolution equations, and there have been several interesting, natural and powerful generalizations of the classical almost automorphic functions. The concept of asymptotically almost automorphic functions was introduced by N'Guérékata in [9]. Liang, Xiao and Zhang in [10, 11] presented the concept of pseudo almost automorphy suggested by N'Guérékata in [4]. In [12], N'Guérékata and Pankov introduced another generalization of almost automorphic functions–Stepanov-like almost automorphic functions. Blot et al. introduced the notion of weighted pseudo almost automorphic functions with values in a Banach space in [13], which generalizes that of pseudo-almost automorphic functions. Zhang, Chang and N'Guérékata investigated some properties and new composition theorems of Stepanov-like weighted pseudo almost automorphic functions in [14] and investigated weighted pseudo almost automorphic solutions to some nonlinear equations with S^p-weighted pseudo almost automorphic coefficients in [15–17].

Recently, Blot, Cieutat and Ezzinbi in [18] applied the measure theory to define an ergodic function and they investigated many interesting properties of μ-pseudo almost automorphic functions. In this work, we first prove a composition theorem for μ-pseudo almost automorphic functions under suitable conditions, and then apply it to investigate the existence of μ-pseudo almost automorphic solutions to the following neutral differential equation:

$$\frac{d}{dt}[u(t) + f(t, u(t))] = Au(t) + g(t, u(t)), \qquad t \in \mathbb{R}, \tag{1}$$

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Email addresses: lzchangyk@163.com (Yong-Kui Chang), luoxx0931@foxmail.com (Xiao-Xia Luo)

where $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ is the generator of a hyperbolic analytic semigroup $T(t)_{t\geq 0}$, and $f : \mathbb{R}\times\mathbb{X} \to \mathbb{X}_\beta (0 < \alpha < \beta < 1)$, $g : \mathbb{R}\times\mathbb{X} \to \mathbb{X}$ are suitable continuous functions, $\mathbb{X}_\beta$ is a suitable interpolation space specified later. Our main results are based upon the interpolation theory developed in [19–21].

The rest of this paper is organized as follows. In Section 2, we present some basic definitions, lemmas, and preliminary results which will be used throughout this paper. In Section 3, we prove some existence results of μ-pseudo almost automorphic mild solutions to the neutral differential equation (1).

2. Preliminaries

This section is devoted to some preliminary results needed in the sequel. Throughout the paper, the notations $(\mathbb{X}, \|\cdot\|)$ and $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ are two Banach spaces and $BC(\mathbb{R},\mathbb{X})$ denotes the Banach space of all bounded continuous functions from $\mathbb{R}$ to $\mathbb{X}$, equipped with the supremum norm $\|f\|_\infty = \sup_{t\in\mathbb{R}} \|f(t)\|$. Let $\mathbb{X}_\alpha$ is an intermediate space between $D(A)$ and $\mathbb{X}$. $B(\mathbb{R},\mathbb{X}_\alpha)$ for $\alpha\in(0,1)$ stands for the Banach space of all bounded continuous functions $\varphi : \mathbb{R}\to\mathbb{X}_\alpha$ when equipped with the α-sup norm:

$$\|\varphi\|_{\alpha,\infty} := \sup_{t\in\mathbb{R}} \|\varphi(t)\|_\alpha$$

for $\varphi \in BC(\mathbb{R},\mathbb{X}_\alpha)$.

Throughout this work, we denote by $\mathcal{B}$ the Lebesgue σ-field of $\mathbb{R}$ and by $\mathcal{M}$ the set of all positive measures μ on $\mathcal{B}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a,b]) < +\infty$, for all $a, b \in \mathbb{R} (a < b)$.

Definition 2.1. *[1] A continuous function $f : \mathbb{R}\to\mathbb{X}$ is called almost automorphic if for every sequence of real numbers $(s_n)_{n\in\mathbb{N}}$ there exists a subsequence $(s'_n)_{n\in\mathbb{N}} \subset (s_n)_{n\in\mathbb{N}}$ such that*

$$\lim_{n,m\to\infty} \|f(t+s_n-s_m) - f(t)\| = 0.$$

Define

$$PAA_0(\mathbb{R},\mathbb{X}) = \left\{ \phi \in BC(\mathbb{R},\mathbb{X}) : \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \|\phi(\sigma)\| d\sigma = 0 \right\}.$$

In the same way, we define $PAA_0(\mathbb{R}\times\mathbb{X},\mathbb{X})$ as the collection of jointly continuous functions $f : \mathbb{R}\times\mathbb{X}\to\mathbb{X}$ which belong to $BC(\mathbb{R}\times\mathbb{X},\mathbb{X})$) and satisfy

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \|\phi(\sigma,x)\| d\sigma = 0$$

uniformly in compact subset of $\mathbb{X}$.

Definition 2.2. *[22, 23] A continuous function $f : \mathbb{R}\to\mathbb{X}$ (respectively $\mathbb{R}\times\mathbb{X}\to\mathbb{X}$) is called pseudo-almost automorphic if it can be decomposed as $f = g+\phi$, where $g \in AA(\mathbb{R},\mathbb{X})$(respectively $AA(\mathbb{R}\times\mathbb{X},\mathbb{X})$) and $\phi \in PAA_0(\mathbb{R},\mathbb{X})$(respectively $PAA_0(\mathbb{R}\times\mathbb{X},\mathbb{X})$). Denote by $PAA(\mathbb{R},\mathbb{X})$ (respectively $PAA(\mathbb{R}\times\mathbb{X},\mathbb{X})$) the set of all such functions.*

Definition 2.3. *[18] Let $\mu\in\mathcal{M}$. A bounded continuous function $f : \mathbb{R}\to\mathbb{X}$ is said to be μ-ergodic if*

$$\lim_{r\to+\infty} \frac{1}{\mu([-r,r])}\int_{[-r,r]} \|f(t)\| d\mu(t) = 0.$$

We denote the space of all such functions by $\varepsilon(\mathbb{R},\mathbb{X},\mu)$.

Definition 2.4. *[18] Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be μ-pseudo almost automorphic if f is written in the form $f = g + \phi$, where $g \in AA(\mathbb{R}, \mathbb{X})$ and $\phi \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$. We denote the space of all such functions by $PAA(\mathbb{R}, \mathbb{X}, \mu)$.*

Obviously, we have $AA(\mathbb{R}, \mathbb{X}) \subset PAA(\mathbb{R}, \mathbb{X}, \mu) \subset BC(\mathbb{R}, \mathbb{X})$.

Lemma 2.5. *[18, Proposition 2.13] Let $\mu \in \mathcal{M}$, then $(\varepsilon(\mathbb{R}, \mathbb{X}, \mu), \|\cdot\|_\infty)$ is a Banach space.*

Lemma 2.6. *[18, Theorem 4.1] Let $\mu \in \mathcal{M}$ and $f \in PAA(\mathbb{R}, \mathbb{X}, \mu)$ be such that $f = g + \phi$, where $g \in AA(\mathbb{R}, \mathbb{X})$ and $\phi \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$. If $PAA(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant, then $\{g(t) : t \in \mathbb{R}\} \subset \overline{\{f(t) : t \in \mathbb{R}\}}$, (the closure of the range of f).*

Lemma 2.7. *[18, Theorem 2.14] Let $\mu \in \mathcal{M}$ and I be the bounded interval (eventually $I = \emptyset$). Assume that $f \in BC(\mathbb{R}, \mathbb{X})$. Then the following assertions are equivalent.*
(i) $f \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$;
(ii) $\lim\limits_{r\to+\infty} \frac{1}{\mu([-r,r]\setminus I)} \int\limits_{[-r,r]\setminus I} \|f(t)\| d\mu(t) = 0$;
(iii) For any $\varepsilon > 0$, $\lim\limits_{r\to+\infty} \dfrac{\mu(\{t \in [-r, r] \setminus I : \|f(t)\| > \varepsilon\})}{\mu([-r, r] \setminus I)} = 0$.

Lemma 2.8. *[18, Theorem 4.7] Let $\mu \in \mathcal{M}$. Assume that $PAA(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant. Then the decomposition of a μ-pseudo almost automorphic function in the form $f = g + \phi$ where $g \in AA(\mathbb{R}, \mathbb{X})$ and $\phi \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$ is unique.*

Lemma 2.9. *[18, Theorem 4.9] Let $\mu \in \mathcal{M}$. Assume that $PAA(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant. Then $(PAA(\mathbb{R}, \mathbb{X}, \mu), \|\cdot\|_\infty)$ is a Banach space.*

Now, we introduce some notions and properties about hyperbolic semigroups and intermediate spaces.
Let $\mathbb{X}$ and $\mathbb{Z}$ be Banach spaces, with norms $\|\cdot\|$, $\|\cdot\|_{\mathbb{Z}}$ respectively, and suppose that $\mathbb{Z}$ is continuously embedded in $\mathbb{X}$, that is, $\mathbb{Z} \hookrightarrow \mathbb{X}$.

Definition 2.10. *[24, Definition 2.5] A Semigroup $(T(t))_{t\geq 0}$ on $\mathbb{X}$ is said to be hyperbolic if there is a projection P and constants $M, \delta > 0$ such that each $T(t)$ commutes with P, KerP is invariant with respect to $T(t)$, $T(t) : ImQ \to ImQ$ is invertible and for every $x \in \mathbb{X}$*

$$\|T(t)Px\| \leq Me^{-\delta t}\|x\|, \quad \text{for} \quad t \geq 0; \tag{2}$$

$$\|T(t)Qx\| \leq Me^{\delta t}\|x\|, \quad \text{for} \quad t \leq 0; \tag{3}$$

where $Q := I - P$ and, for $t < 0$, $T(t) = T(-t)^{-1}$.

Definition 2.11. *[25] A linear operator $A : D(A) \subset \mathbb{X} \to \mathbb{X}$(not necessarily densely defined) is said to be sectorial if the following hold: There exist constants $\omega \in \mathbb{R}$, $\theta \in (\frac{\pi}{2}, \pi)$, and $M > 0$ such that*

$$\rho(A) \subset S_{\theta,\omega} := \{\lambda \in \mathbb{C} : \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta\},$$

$$\|R(\lambda, A)\| \leq \frac{M}{|\lambda - \omega|}, \quad \lambda \in S_{\theta,\omega}.$$

Remark 2.12. *[24, Remark 2.6] The existence of a hyperbolic semigroup on a Banach space $\mathbb{X}$ give us a nice algebraic information about this vectorial space. In fact, let $(T(t))_{t\geq 0}$ be a hyperbolic semigroup on $\mathbb{X}$. Then there are $(T(t))_{t\geq 0}$-invariant closed subspaces $\mathbb{X}_s$ and $\mathbb{X}_u$ such that $\mathbb{X} = \mathbb{X}_s \bigoplus \mathbb{X}_u$. Furthermore, the restricted semigroups $(T_s(t))_{t\geq 0}$ on $\mathbb{X}_s$ and $(T_u(t))_{t\geq 0}$ on $\mathbb{X}_u$ have the following properties:*
(i)The semigroup $(T_s(t))_{t\geq 0}$ is uniformly exponentially stable on $\mathbb{X}_s$.
(ii)The operators $T_u(t)$ are invertible on $\mathbb{X}_u$, and $(T_u(t)^{-1})_{t\geq 0}$ is uniformly exponentially stable on $\mathbb{X}_u$.

Definition 2.13. *[24, Definition 2.7] Let $0 \le \alpha \le 1$. A Banach space $\mathbb{Y}$ such that $\mathbb{Z} \hookrightarrow \mathbb{Y} \hookrightarrow \mathbb{X}$ is said to the class J_α between $\mathbb{X}$ and $\mathbb{Z}$ if there is a constant $c > 0$ such that*

$$\|x\|_{\mathbb{Y}} \le c\|x\|^{1-\alpha}\|x\|_{\mathbb{Z}}^{\alpha} \quad (x \in \mathbb{Z}).$$

In this case we write $\mathbb{Y} \in J_\alpha((X), \mathbb{Z})$.

Definition 2.14. *[24, Definition 2.8] Let $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ be a sectorial operator. A Banach space $(\mathbb{X}_\alpha, \|\cdot\|_\alpha)$, $\alpha \in (0,1)$, is said to be an intermediate space between $\mathbb{X}$ and $D(A)$ if $\mathbb{X}_\alpha \in J_\alpha(\mathbb{X}, D(A))$.*

Examples of intermediate spaces between $\mathbb{X}$ and $D(A)$ are the domains of the fractional powers $D(-A^\alpha)$ and the interpolation spaces $D_A(\alpha, \infty)$, defined as follows

$$D_A(\alpha,\infty) = \{x \in \mathbb{X} : [x]_\alpha = \sup_{0<t\le 1} \|t^{1-\alpha}AT(t)x\| < +(\infty)\}$$

$$\|x\|_{D_A(\alpha,\infty)} = \|x\| + [x]_\alpha.$$

Lemma 2.15. *[24, Lemma 2.10] Let $(T(t))_{t\ge 0}$ be a hyperbolic analytic semigroup on $\mathbb{X}$ with generator A. For $\alpha \in (0,1)$, let $(\mathbb{X}_\alpha, \|\cdot\|_\alpha)$ be intermediate spaces between $\mathbb{X}$ and $D(A)$. Then there are positive constants $C(\alpha)$, $M(\alpha)$, δ and γ such that*

$$\|T(t)Px\|_\alpha \le M(\alpha)t^{-\alpha}e^{-\gamma t}\|x\|, \quad (t > 0), \tag{4}$$

and

$$\|T(t)Qx\|_\alpha \le C(\alpha)e^{\delta t}\|x\|, \quad (t \le 0). \tag{5}$$

Lemma 2.16. *[26] Let $0 < \alpha, \beta < 1$. Then*

$$\|AT(t)Px\|_\alpha \le ct^{\beta-\alpha-1}e^{-\gamma t}\|x\|_\beta, \quad \text{for} \quad t > 0, \tag{6}$$

$$\|AT(t)Qx\|_\alpha \le ce^{\delta t}\|x\|_\beta, \quad \text{for} \quad t \le 0. \tag{7}$$

For the problem (1), we list the following assumptions:
(H1) If $0 \le \alpha < \beta < 1$, then we let k_1 be the bound of the embedding $\mathbb{X}_\alpha \hookrightarrow \mathbb{X}$, that is

$$\|u\| \le k_1\|u\|_\alpha \quad \text{for} \quad u \in \mathbb{X}_\alpha.$$

(H2) Let $0 \le \alpha < \beta < 1$ and the function $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}_\beta$ belongs to $PAA(\mathbb{R}, \mathbb{X}_\beta, \mu)$ while $g : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ belongs to $PAA(\mathbb{R}, \mathbb{X}, \mu)$. Moreover, the functions f, g are uniformly Lipschitz with respect to the second argument in the following sense: there exist $K > 0$ such that

$$\|f(t,u) - f(t,v)\|_\beta \le K\|u - v\|$$

and

$$\|g(t,u) - g(t,v)\| \le K\|u - v\|$$

for all $u, v \in \mathbb{X}$ and $t \in \mathbb{R}$.

3. Main results

In this section, we first prove a composition theorem for μ-pseudo almost automorphic functions under suitable conditions, and then apply it to establish some existence results for the problem (1).

Theorem 3.1. *Let $\mu \in \mathcal{M}$ and $f = g + h \in PAA(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \mu)$. Assume that*
(H3) $f(t, x)$ is uniformly continuous on any bounded subset $K \subset \mathbb{X}$ uniformly in $t \in \mathbb{R}$.
(H4) $g(t, x)$ is uniformly continuous on any bounded subset $K \subset \mathbb{X}$ uniformly in $t \in \mathbb{R}$.
Then the function defined by $F(\cdot) := f(\cdot, \phi(\cdot)) \in PAA(\mathbb{R}, \mathbb{X}, \mu)$ if $\phi \in PAA(\mathbb{R}, \mathbb{X}, \mu)$.

Proof. Let $f = g + h$ with $g \in AA(\mathbb{R} \times \mathbb{X}, \mathbb{X})$, $h \in \varepsilon(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \mu)$, and $\phi = u + v$, with $u \in AA(\mathbb{R}, \mathbb{X})$, and $v \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$.
Now we define

$$\begin{aligned} F(t) &= g(t, u(t)) + f(t, \phi(t)) - g(t, u(t)) \\ &= g(t, u(t)) + f(t, \phi(t)) - f(t, u(t)) + h(t, u(t)). \end{aligned}$$

Let us rewrite

$$G(t) = g(t, u(t)), \Phi(t) = f(t, \phi(t)) - f(t, u(t)), H(t) = h(t, u(t)).$$

Thus, we have $F(t) = G(t) + \Phi(t) + H(t)$. In view of [11, Lemma 2.2], $G(t) \in AA(\mathbb{R}, \mathbb{X})$. Next we prove that $\Phi(t) \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$. Clearly , $\Phi(t) \in BC(\mathbb{R}, \mathbb{X})$. For Φ to be in $\varepsilon(\mathbb{R}, \mathbb{X}, \mu)$, it is enough to show that

$$\lim_{r\to\infty} \frac{1}{\mu([-r, r])} \int_{[-r,r]} \|\Phi(t)\| d\mu(t) = 0.$$

By Lemma 2.6, $u(\mathbb{R}) \subset \overline{\phi(\mathbb{R})}$ which is a bounded set. From assumption (H3) with $K = \overline{\phi(\mathbb{R})}$, we conclude that for each $\varepsilon > 0$, there exists a constant $\delta > 0$ such that for all $t \in \mathbb{R}$,

$$\|\phi - u\| \leq \delta \Rightarrow \|f(t, \phi(t)) - f(t, u(t))\| \leq \varepsilon.$$

Denote by the following set $A_{r,\varepsilon} = \{t \in [-r, r] : \|f(t)\| > \varepsilon\}$. Thus we obtain

$$\begin{aligned} A_{r,\varepsilon}(\Phi) &= A_{r,\varepsilon}(f(t, \phi(t)) - f(t, u(t))) \subset A_{r,\delta}(\phi(t) - u(t)) \\ &= A_{r,\delta}(v). \end{aligned}$$

Therefore the following inequality holds

$$\frac{\mu(\{t \in [-r, r] : \|f(t, \phi(t)) - f(t, u(t))\| > \varepsilon\})}{\mu([-r, r])} \leq \frac{\mu(\{t \in [-r, r] : \|\phi(t) - u(t)\| > \delta\})}{\mu([-r, r])}.$$

Since $\phi(t) = u(t) + v(t)$ and $v \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$, Lemma 2.7 yields that for the above-mentioned δ we have

$$\lim_{r\to\infty} \frac{\mu(\{t \in [-r, r] : \|\phi(t) - u(t)\| > \delta\})}{\mu([-r, r])} = 0,$$

and then we obtain

$$\lim_{r\to\infty} \frac{\mu(\{t \in [-r, r] : \|f(t, \phi(t)) - f(t, u(t))\| > \varepsilon\})}{\mu([-r, r])} = 0. \tag{8}$$

From Lemma 2.7 and relation (8), we draw a conclusion that $\Phi(t) \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$.

Finally, it is only to show that $H(t) = h(t, u(t)) \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$. We have the set $u([-r, r])$ is compact since u is continuous on $\mathbb{R}$ as an almost automorphic function. So, the function g belongs to $AA(\mathbb{R} \times \mathbb{X}, \mathbb{X})$, and g is uniformly continuous on $[-r, r] \times u([-r, r])$. Then it follows from (H3) that $h(t, x)$ is uniformly continuous with $x \in u([-r,r])$ uniformly in $t \in [-r,r]$. Thus for any $\varepsilon > 0$, there exists a constant $\delta > 0$ such that for $x_1,\ x_2 \in u([-r,r])$ with $\|x_1 - x_2\| < \delta$ we have

$$\|h(t,x_1) - h(t,x_2)\| < \frac{\varepsilon}{2},\ \forall t \in [-r,r]. \tag{9}$$

On the other hand, since the set $u([-r,r])$ is compact, there exist finite balls O_k with $\beta_k \in u([-r,r])$, $k = 1,\cdots,m$, and radius δ given above, such that $u([-r,r]) \subset \cup_{k=1}^m O_k$. Then the sets $U_k := \{t \in [-r,r] : u(t) \in O_k\}$, $k = 1,\cdots,m$ are open in $[-r,r]$ and $[-r,r] = \cup_{k=1}^m U_k$.

Define V_k by

$$V_1 = U_1,\ V_k = U_k - \cup_{i=1}^{k-1} U_i,\ 2 \le k \le m.$$

Then it is obvious that $V_i \cap V_j = \emptyset$, if $i \neq j,\ 1 \le i,\ j \le m$. So we get

$$\begin{aligned}
\Lambda :&= \{t \in [-r,r] : \|H(t)\| \ge \varepsilon\} = \{t \in [-r,r] : \|h(t,u(t))\| \ge \varepsilon\} \\
&\subset \cup_{k=1}^m \{t \in V_k : \|h(t,u(t)) - h(t,\beta_k)\| + \|h(t,\beta_k)\| \ge \varepsilon\} \\
&\subset \cup_{k=1}^m \left(\left\{t \in V_k : \|h(t,u(t)) - h(t,\beta_k)\| \ge \frac{\varepsilon}{2}\right\} \cup \left\{t \in V_k : \|h(t,\beta_k)\| \ge \frac{\varepsilon}{2}\right\}\right).
\end{aligned}$$

It follows from relation (9) that

$$\left\{t \in V_k : \|h(t,u(t)) - h(t,\beta_k)\| \ge \frac{\varepsilon}{2}\right\} = \emptyset,\ \ k = 1,\ldots,m.$$

Thus, if we set $A_{r,\frac{\varepsilon}{2}}(h_k) := A_{r,\frac{\varepsilon}{2}}(h(t,\beta_k))$, then $A_{r,\varepsilon}(H) \subset U_{k=1}^m A_{r,\frac{\varepsilon}{2}}(h_k)$ and

$$\frac{1}{\mu([-r,r])}\int_{[-r,r]} \|H(t)\| d\mu(t) \le \sum_{k=1}^m \frac{1}{\mu([-r,r])}\int_{[-r,r]} \|h_k(t)\| d\mu(t).$$

And since $h \in \varepsilon(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \mu)$, we have

$$\lim_{r\to\infty} \frac{1}{\mu([-r,r])}\int_{[-r,r]} \|h_k(t)\| d\mu(t) = 0,\ k = 1,\cdots,m.$$

It follows that $\lim_{r\to\infty} \frac{1}{\mu([-r,r])}\int_{[-r,r]} \|H(t)\| d\mu(t) = 0$. According to Lemma 2.7, we deduce that $H(t) = h(t,u(t)) \in \varepsilon(\mathbb{R},\mathbb{X},\mu)$. This completes the proof. $\square$

Throughout the rest of this paper we suppose that there exists two real numbers α, β such that $0 < \alpha < \beta < 1$ with

$$2\beta > \alpha + 1.$$

Moreover, we denote by Γ_1, Γ_2, Γ_3, and Γ_4 the nonlinear integral operators defined by

$$(\Gamma_1(u)(t)) := \int_{-\infty}^{t} AT(t-s)Pf(s,u(s))ds,$$

$$(\Gamma_2(u)(t)) := \int_{t}^{\infty} AT(t-s)Qf(s,u(s))ds,$$

$$(\Gamma_3(u)(t)) := \int_{-\infty}^{t} T(t-s)Pg(s,u(s))ds,$$

$$(\Gamma_4(u)(t)) := \int_{t}^{\infty} T(t-s)Qg(s,u(s))ds.$$

Lemma 3.2. *Let $\mu \in \mathcal{M}$, let $u \in PAA(\mathbb{R}, \mathbb{X}_\alpha, \mu)$. Under assumptions (H1)-(H2), the integral operators Γ_3 and Γ_4 defined above map $PAA(\mathbb{R}, \mathbb{X}_\alpha, \mu)$ into itself.*

Proof. Let $u \in PAA(\mathbb{R}, \mathbb{X}_\alpha, \mu)$. Setting $h(t) = g(t, u(t))$ and by Theorem 3.1, it follows that $h \in PAA(\mathbb{R}, \mathbb{X}, \mu)$ for each $u \in PAA(\mathbb{R}, \mathbb{X}_\alpha, \mu)$. Now write $h = \phi + \zeta$ where $\phi \in AA(\mathbb{R}, \mathbb{X})$ and $\zeta \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$. Thus $\Gamma_3 u$ can be rewritten as

$$(\Gamma_3(u)(t)) := \int_{-\infty}^{t} T(t-s)P\phi(s)ds + \int_{-\infty}^{t} T(t-s)P\zeta(s)ds.$$

Set $\Phi(t) = \int_{-\infty}^{t} T(t-s)P\phi(s)ds$ and $\Psi(t) = \int_{-\infty}^{t} T(t-s)P\zeta(s)ds$ for each $t \in \mathbb{R}$.
Now, we shall show that $\Phi \in AA(\mathbb{R}, \mathbb{X}_\alpha)$. Let us take a sequence $(s'_n)_{n\in\mathbb{N}}$, since $\phi \in AA(\mathbb{R}, \mathbb{X})$, there is a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n,m\to\infty} \|\phi(t + s_n - s_m) - \phi(t)\| = 0. \tag{10}$$

Furthermore,

$$\begin{aligned} \Phi(t + s_n - s_m) - \Phi(t) &= \int_{-\infty}^{t+s_n-s_m} T(t + s_n - s_m - s)P\phi(s)ds - \int_{-\infty}^{t} T(t-s)P\phi(s)ds \\ &= \int_{-\infty}^{0} T(-s)P[\phi(s + t + s_n - s_m) - \phi(s+t)]ds. \end{aligned}$$

Then, we obtain

$$\|\Phi(t + s_n - s_m) - \Phi(t)\|_\alpha \le \int_{-\infty}^{0} \|T(-s)P[\phi(s + t + s_n - s_m) - \phi(s+t)]\|_\alpha ds.$$

Hence, by (4) we deduce

$$\|\Phi(t + s_n - s_m) - \Phi(t)\|_\alpha \le \int_{-\infty}^{0} M(\alpha)s^{-\alpha}e^{-\gamma s}\|\phi(s + t + s_n - s_m) - \phi(s+t)\|ds.$$

The result follows by (10) and the Lebesgue's dominated convergence theorem.

Finally, it is only to show that $\Psi(t) \in \varepsilon(\mathbb{R}, \mathbb{X}_\alpha, \mu)$. We have

$$\begin{aligned}
\frac{1}{\mu([-r,r])}\int_{[-r,r]} \|\Psi(t)\|_\alpha d\mu(t) &= \frac{1}{\mu([-r,r])}\int_{[-r,r]} \left\| \int_{-\infty}^{t} T(t-s)P\zeta(s)ds \right\|_\alpha d\mu(t) \\
&\leq \frac{1}{\mu([-r,r])}\int_{[-r,r]} \int_{-\infty}^{t} \|T(t-s)P\zeta(s)\| ds d\mu(t) \\
&\leq \frac{1}{\mu([-r,r])}\int_{[-r,r]} \int_{-\infty}^{t} M(\alpha)(t-s)^{-\alpha}e^{-\gamma(t-s)}\|\zeta(s)\| ds d\mu(t) \\
&\leq M(\alpha)\int_{0}^{\infty} s^{-\alpha}e^{-\gamma s}\left(\frac{1}{\mu([-r,r])}\int_{[-r,r]} \|\zeta(t-s)\| d\mu(t)\right) ds.
\end{aligned}$$

By the fact that the space $\varepsilon(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant, it follows that $t \mapsto \zeta(t-s)$ belongs to $\varepsilon(\mathbb{R}, \mathbb{X}, \mu)$ for each $s \in \mathbb{R}$ and hence

$$\lim_{r\to\infty} \frac{1}{\mu([-r,r])}\int_{[-r,r]} \|\zeta(t-s)\| d\mu(t) = 0.$$

One completes the proof by using the well-known Lebesgue dominated convergence theorem and the fact $\lim_{r\to\infty} M(\alpha)\int_{0}^{\infty} s^{-\alpha}e^{-\gamma s}\left(\frac{1}{\mu([-r,r])}\int_{[-r,r]} \|\zeta(t-s)\| d\mu(t)\right) ds = 0$. The proof is now completed. □

The proof for $\Gamma_4 u$ is similar to that $\Gamma_3 u$. However one makes use of (5) rather that (4).

Lemma 3.3. *Let $\mu \in \mathcal{M}$, and let $u \in PAA(\mathbb{R}, \mathbb{X}, \mu)$. Under assumptions (H1)-(H2), the integral operators Γ_1 and Γ_2 defined above map $PAA(\mathbb{R}, \mathbb{X}_\alpha, \mu)$ into itself.*

Proof. Let $u \in PAA(\mathbb{R}, \mathbb{X}_\alpha, \mu)$. Setting $h(t) = f(t, u(t))$ and in view of Theorem 3.1, it follows that $h \in PAA(\mathbb{R}, \mathbb{X}_\beta, \mu)$ whenever $u \in PAA(\mathbb{R}, \mathbb{X}_\alpha, \mu)$. In particular,

$$\|h\|_{\infty,\beta} = \sup_{t\in\mathbb{R}} \|f(t, u(t))\|_\beta < \infty.$$

Now write $h = \phi + \psi$, where $\phi \in AA(\mathbb{R}, \mathbb{X}_\beta)$, $\psi \in \varepsilon(\mathbb{R}, \mathbb{X}_\beta, \mu)$, that is, $\Gamma_1 h = \Xi\phi + \Xi\psi$ where

$$\Xi\phi(t) := \int_{-\infty}^{t} AT(t-s)P\phi(s)ds,$$

$$\Xi\psi(t) := \int_{-\infty}^{t} AT(t-s)P\psi(s)ds.$$

First, we need to prove that $\Xi\phi(t) \in AA(\mathbb{R}, \mathbb{X}_\alpha)$. Let us take a sequence $(s'_n)_{n\in\mathbb{N}}$ in $t \in \mathbb{R}$, since $\phi(t) \in AA(\mathbb{R}, \mathbb{X}_\beta)$, there is a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n,m\to\infty} \|\phi(t + s_n - s_m) - \phi(t)\|_\beta = 0. \tag{11}$$

Furthermore, since

$$\begin{aligned}\Xi\phi(t+s_n-s_m)-\Xi\phi(t) &= \int_{-\infty}^{t+s_n-s_m} AT(t+s_n-s_m-s)P\phi(s)ds - \int_{-\infty}^{t} AT(t-s)P\phi(s)ds \\ &= \int_{-\infty}^{0} AT(-s)P[\phi(s+t+s_n-s_m)-\phi(s+t)]ds.\end{aligned}$$

Then, we obtain

$$\|\Xi\phi(t+s_n-s_m)-\Xi\phi(t)\|_\alpha \le \int_{-\infty}^{0} \|AT(-s)P[\phi(s+t+s_n-s_m)-\phi(s+t)]\|_\alpha ds.$$

Hence, by (6) we deduce

$$\|\Xi\phi(t+s_n-s_m)-\Xi\phi(t)\|_\alpha \le \int_{-\infty}^{0} cs^{\beta-\alpha-1}e^{-\gamma s}\|\phi(s+t+s_n-s_m)-\phi(s+t)\|_\beta ds.$$

The result follows by (11) and the Lebesgue's dominated theorem.
Finally, it is only to show that $\Xi\psi(t) \in \varepsilon(\mathbb{R}, \mathbb{X}_\alpha, \mu)$. We have

$$\begin{aligned}\frac{1}{\mu([-r,r])}\int_{[-r,r]} \|\Xi\psi(t)\|_\alpha d\mu(t) &= \frac{1}{\mu([-r,r])}\int_{[-r,r]} \left\|\int_{-\infty}^{t} AT(t-s)P\psi(s)ds\right\|_\alpha d\mu(t) \\ \le\ & \frac{1}{\mu([-r,r])}\int_{[-r,r]}\int_{-\infty}^{t} \|AT(t-s)P\psi(s)\|_\alpha ds d\mu(t) \\ \le\ & \frac{1}{\mu([-r,r])}\int_{[-r,r]}\int_{-\infty}^{t} c(t-s)^{\beta-\alpha-1}e^{-\gamma(t-s)}\|\psi(s)\|_\beta ds d\mu(t) \\ \le\ & c\int_{0}^{\infty} s^{\beta-\alpha-1}e^{-\gamma s}\left(\frac{1}{\mu([-r,r])}\int_{[-r,r]} \|\psi(t-s)\|_\beta d\mu(t)\right)ds.\end{aligned}$$

Now, $\lim_{r\to\infty}\dfrac{1}{\mu([-r,r])}\int_{[-r,r]} \|\psi(t-s)\|_\beta d\mu(t) = 0$ as $s \to \psi(t-s) \in \varepsilon(\mathbb{R}, \mathbb{X}_\beta, \mu)$ for every $s \in \mathbb{R}$. One completes the proof by using the Lebesgue dominated convergence theorem. □

The proof for $\Gamma_2 u$ is similar to that of $\Gamma_1 u$ except that one makes use of (7) instead of (6).

The rest of this section is devoted to the existence of μ-pseudo almost automorphic solutions to the (1).

Definition 3.4. *Let $\alpha \in (0,1)$. A bounded continuous function $u : \mathbb{R} \to \mathbb{X}_\alpha$ is said to be a mild solution to (1) provide that the function $s \to AT(t-s)Pf(s,u(s))$ is integrable on $(-\infty,t)$, $s \to AT(t-s)Qf(s,u(s))$ is integrable on (t,∞) and*

$$\begin{aligned}u(t) &= -f(t,u(t)) - \int_{-\infty}^{t} AT(t-s)Pf(s,u(s))ds + \int_{t}^{\infty} AT(t-s)Qf(s,u(s))ds \\ &+ \int_{-\infty}^{t} T(t-s)Pg(s,u(s))ds - \int_{t}^{\infty} T(t-s)Qg(s,u(s))ds\end{aligned}$$

for each $t \in \mathbb{R}$.

Theorem 3.5. *Let* $\mu \in \mathcal{M}$. *Under assumptions (H1)-(H2), the neutral differential equation (1) admits a unique* μ*-pseudo almost aitomorphic mild solution whenever* K *is small enough.*

Proof. Consider the operator $\Lambda : PAA(\mathbb{R}, \mathbb{X}_\alpha, \mu) \to PAA(\mathbb{R}, \mathbb{X}_\alpha, \mu)$ such that

$$\begin{aligned}\Lambda u(t) : &= -f(t, u(t)) - \int_{-\infty}^{t} AT(t-s)Pf(s, u(s))ds + \int_{t}^{\infty} AT(t-s)Qf(s, u(s))ds \\ &+ \int_{-\infty}^{t} T(t-s)Pg(s, u(s))ds - \int_{t}^{\infty} T(t-s)Qg(s, u(s))ds.\end{aligned}$$

As we have previously seen, for every $u \in PAA(\mathbb{R}, \mathbb{X}_\alpha, \mu)$, $f(\cdot, u(\cdot)) \in PAA(\mathbb{R}, \mathbb{X}_\alpha, \mu)$. In view of Lemmas 3.2 and 3.3, it follows that Λ maps $PAA(\mathbb{R}, \mathbb{X}_\alpha, \mu)$ into itself. To complete the proof one has to show that Λ has a unique fixed-point.
Let $v, w \in PAA(\mathbb{R}, \mathbb{X}_\alpha, \mu)$

$$\begin{aligned}&\|\Gamma_1(v)(t) - \Gamma_1(w)(t)\|_\alpha \\ \leq\ & \int_{-\infty}^{t} \|AT(t-s)P[f(s, v(s)) - f(s, w(s))]\|_\alpha ds \\ \leq\ & \int_{-\infty}^{t} c(t-s)^{\beta-\alpha-1} e^{-\gamma(t-s)} \|f(s, v(s)) - f(s, w(s))\|_\beta ds \\ \leq\ & k_1 c K \int_{-\infty}^{t} (t-s)^{\beta-\alpha-1} e^{-\gamma(t-s)} \|v(s) - w(s)\|_\alpha ds \\ \leq\ & c k_1 K \gamma^{\alpha-\beta} \Gamma(\beta-\alpha) \|v - w\|_{\alpha,\infty}.\end{aligned}$$

Now

$$\begin{aligned}&\|\Gamma_2(v)(t) - \Gamma_2(w)(t)\|_\alpha \\ \leq\ & \int_{t}^{\infty} \|AT(t-s)Q[f(s, v(s)) - f(s, w(s))]\|_\alpha ds \\ \leq\ & \int_{t}^{\infty} c e^{\delta(t-s)} \|f(s, v(s)) - f(s, w(s))\|_\beta ds \\ \leq\ & k_1 c K \int_{t}^{\infty} e^{\delta(t-s)} \|v(s) - w(s)\|_\alpha ds \\ \leq\ & c k_1 K \delta^{-1} \|v - w\|_{\alpha,\infty}.\end{aligned}$$

Now for Γ_3 and Γ_4, we have the following approximations:

$$\begin{aligned}
&\|\Gamma_3(v)(t) - \Gamma_3(w)(t)\|_\alpha \\
\leq\ & \int_{-\infty}^{t} \|T(t-s)P[g(s,v(s)) - g(s,w(s))]\|_\alpha ds \\
\leq\ & \int_{-\infty}^{t} M(\alpha)(t-s)^{-\alpha}e^{-\gamma(t-s)}\|g(s,v(s)) - g(s,w(s))\|ds \\
\leq\ & k_1 M(\alpha)K\int_{-\infty}^{t} (t-s)^{-\alpha}e^{-\gamma(t-s)}\|v(s) - w(s)\|_\alpha ds \\
\leq\ & k_1 M(\alpha)K\gamma^{\alpha-1}\Gamma(1-\alpha)\|v-w\|_{\alpha,\infty}.
\end{aligned}$$

and

$$\begin{aligned}
&\|\Gamma_4(v)(t) - \Gamma_4(w)(t)\|_\alpha \\
\leq\ & \int_{t}^{\infty} \|T(t-s)Q[g(s,v(s)) - g(s,w(s))]\|_\alpha ds \\
\leq\ & \int_{t}^{\infty} C(\alpha)e^{\delta(t-s)}\|g(s,v(s)) - g(s,w(s))\|ds \\
\leq\ & k_1 C(\alpha)K\int_{t}^{\infty} e^{\delta(t-s)}\|v(s) - w(s)\|_\alpha ds \\
\leq\ & k_1 C(\alpha)K\delta^{-1}\|v-w\|_{\alpha,\infty}.
\end{aligned}$$

Combining previous inequalities it follows that

$$\|\Lambda v - \Lambda w\|_{\alpha,\infty} \leq K\Theta\|v-w\|_{\alpha,\infty},$$

where

$$\Theta := ck_1\gamma^{\alpha-\beta}\Gamma(\beta-\alpha) + ck_1\delta^{-1} + k_1M(\alpha)\gamma^{\alpha-1}\Gamma(1-\alpha) + k_1C(\alpha)\delta^{-1}.$$

Therefore, if K is small enough, that is, $K < \Theta^{-1}$, then the $Eq.$(1) has unique solution, which obviously is its only μ-pseudo almost automorphic mild solution. □

From [24], we have the following results.

Remark 3.6. *Throughout the rest of the paper, we consider a locally bounded function $\mathcal{L} : \mathbb{X}_\alpha \times \mathbb{X}_\alpha \to [0,\infty)$ such that for every $r \geq 0$ there is a constant $k(r) \geq 0$ such that $\mathcal{L}(x,y) \leq k(r)$, for all $x, y \in \mathbb{X}_\alpha$ with $\|x\|_\alpha \leq r$ and $\|y\|_\alpha \leq r$.*

Corollary 3.7. *Let $\mu \in \mathcal{M}$. Let also $f = g + h \in PAA(\mathbb{R}, \mathbb{X}_\alpha, \mu)$, assume that there is a locally bounded function $\mathcal{L} : \mathbb{X}_\alpha \times \mathbb{X}_\alpha \to [0,\infty)$ such that for every $x, y \in \mathbb{X}_\alpha$ we have*

$$\|f(t,x) - f(t,y)\| \leq \mathcal{L}(x,y)(1 + \|x\|_\alpha^{l-1} + \|y\|_\alpha^{l-1})\|x-y\|_\alpha, \quad (t \in \mathbb{R}),$$

$$\|g(t,x) - g(t,y)\| \leq \mathcal{L}(x,y)(1 + \|x\|_\alpha^{l-1} + \|y\|_\alpha^{l-1})\|x-y\|_\alpha, \quad (t \in \mathbb{R}).$$

where $l \geq 1$. If there is $R \geq 0$ such that

$$\Theta = K(R)\left(1 + c\gamma^{\alpha-\beta}\Gamma(\beta-\alpha) + c\delta^{-1} + C(\alpha)\delta^{-1} + M(\alpha)\gamma^{\alpha-1}\Gamma(1-\alpha)\right) < 1,$$

where $K(R) := k(R)(1 + 2R^{l-1})$, with $k(R)$ as in Remark 3.6, and $M(\alpha)$ and $C(\alpha)$ are the constants given in Lemma 2.15. Then (1) has a unique μ-pseudo almost automorphic mild solution.

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