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New Bounds on the Rainbow Domination Subdivision Number

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Abstract. A *2-rainbow dominating function* (2RDF) of a graph G is a function f from the vertex set $V(G)$ to the set of all subsets of the set $\{1, 2\}$ such that for any vertex $v \in V(G)$ with $f(v) = \emptyset$ the condition $\bigcup_{u \in N(v)} f(u) = \{1, 2\}$ is fulfilled, where $N(v)$ is the open neighborhood of v. The *weight* of a 2RDF f is the value $\omega(f) = \sum_{v \in V} |f(v)|$. The *2-rainbow domination number* of a graph G, denoted by $\gamma_{r2}(G)$, is the minimum weight of a 2RDF of G. The *2-rainbow domination subdivision number* $\mathrm{sd}_{\gamma_{r2}}(G)$ is the minimum number of edges that must be subdivided (each edge in G can be subdivided at most once) in order to increase the 2-rainbow domination number. In this paper we prove that for every simple connected graph G of order $n \geq 3$,

$$\mathrm{sd}_{\gamma_{r2}}(G) \leq 3 + \min\{d_2(v) \mid v \in V \text{ and } d(v) \geq 2\}$$

where $d_2(v)$ is the number of vertices of G at distance 2 from v.

1. Introduction

In this paper, G is a simple graph with vertex set $V(G)$ and edge set $E(G)$ (briefly V and E). For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. Similarly, the *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \bigcup_{v \in S} N(v)$, and the *closed neighborhood* of S is the set $N[S] = N(S) \cup S$. The minimum and maximum degrees of G are respectively denoted by $\delta(G)$ and $\Delta(G)$ (briefly δ, Δ, when no ambiguity on the graph is possible). The distance between two vertices u and v is the length of a shortest path joining them. We denote by $N_2(v)$ the set of vertices at distance 2 from the vertex v and put $d_2(v) = |N_2(v)|$ and $\delta_2(G) = \min\{d_2(v) \mid v \in V(G)\}$. For a more thorough treatment of domination parameters and for terminology not presented here see [7, 10].

For a positive integer k, a *k-rainbow dominating function* (kRDF) of a graph G is a function f from the vertex set $V(G)$ to the set of all subsets of the set $\{1, 2, \ldots, k\}$ such that for any vertex $v \in V(G)$ with $f(v) = \emptyset$ the condition $\bigcup_{u \in N(v)} f(u) = \{1, 2, \ldots, k\}$ is fulfilled. The *weight* of a kRDF f is the value $\omega(f) = \sum_{v \in V} |f(v)|$. The *k-rainbow domination number* of a graph G, denoted by $\gamma_{rk}(G)$, is the minimum weight of a kRDF of G. A $\gamma_{rk}(G)$*-function* is a k-rainbow dominating function of G with weight $\gamma_{rk}(G)$. Note that $\gamma_{r1}(G)$ is the classical domination number $\gamma(G)$. The k-rainbow domination number was introduced by Brešar, Henning, and Rall [1] and has been studied by several authors (see for example [2–4, 8, 9, 11, 12]).

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The *2-rainbow domination subdivision number* $\mathrm{sd}_{\gamma_{r2}}(G)$ of a graph G is the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the 2-rainbow domination number of G. Since the rainbow domination subdivision number of the graph K_2 does not change when its only edge is subdivided, in the study of the rainbow domination subdivision number we must assume that one of the components of the graph has order at least 3. If $G_1, G_2, \ldots, G_s$ are the components of G, then $\gamma_{r2}(G) = \sum_{i=1}^{s} \gamma_{r2}(G_i)$ and if $G_1, G_2, \ldots, G_r$ are the components of G of order at least 3, then $\mathrm{sd}_{\gamma_{r2}}(G) = \min\{\mathrm{sd}_{\gamma_{r2}}(G_i) \mid 1 \le i \le s\}$. Hence, it is sufficient to study $\mathrm{sd}_{\gamma_{r2}}(G)$ for connected graphs. The rainbow domination subdivision number was introduced by Dehgardi, Sheikholeslami, and Volkmann [6] and has been studied in [5].

The parameter $\mathrm{sd}_{\gamma_{r2}}(G)$ can take large values [6]. Therefore an interesting problem is to find good upper bounds on $\mathrm{sd}_{\gamma_{r2}}(G)$ in terms of the order and possibly of other parameters of G. Some bounds are already known. For instance it has been proved that for any connected graph G of order n, $\mathrm{sd}_{\gamma_r 2}(G) \le n - \delta(G) + 1$ [5] and $\mathrm{sd}_{\gamma_{r2}}(G) \le n - \gamma_{r2}(G) + 3$ [6].

Our purpose in this paper is to prove that for every simple connected graph G of order $n \ge 3$,

$$\mathrm{sd}_{\gamma_{r2}}(G) \le 3 + \min\{d_2(v) \mid v \in V \text{ and } d(v) \ge 2\}$$

We make use of the following results in this paper. Their proofs can be found in [5, 6].

Theorem A. For any connected graph G with adjacent vertices u and v, each of degree at least two,

$$\mathrm{sd}_{\gamma_{r2}}(G) \le \deg(u) + \deg(v) - |N(u) \cap N(v)| - 1 = |N(u) \cup N(v)| - 1.$$

Theorem B. If G is a connected graph of order $n \ge 3$ with $\gamma_{r2}(G) = 2$, then $\mathrm{sd}_{\gamma_{r2}}(G) \le 2$.

Theorem C. If G is a connected graph of order $n \ge 3$ with $\gamma_{r2}(G) = 3$, then $\mathrm{sd}_{\gamma_{r2}}(G) \le 3$.

Theorem D. Let G be a connected graph. If there is a path $v_3v_2v_1$ in G with $\deg(v_2) = 2$ and $\deg(v_1) = 1$, then G has a $\gamma_{r2}(G)$-function f such that $|f(v_1)| = 1$, $|f(v_3)| \ge 1$ and $f(v_1) \ne f(v_3)$.

Theorem E. For any connected graph G of order $n \ge 12$,

$$\mathrm{sd}_{\gamma_{r2}}(G) \le n - \delta(G) + 1.$$

Theorem F. Let G be a connected graph of order $n \ge 3$ with $\delta(G) = 1$. If v is a support vertex, then $\mathrm{sd}_{\gamma_{r2}}(G) \le \deg(v)$.

2. A new bound in terms of order and maximum degree

Lemma 2.1. Let G be a connected graph of order $n \ge 3$. If $v \in V(G)$ is a support vertex and has a neighbor u with $N(u) \setminus N[v] \ne \emptyset$, then $\mathrm{sd}_{\gamma_{r2}}(G) \le 2 + |N(u) - N[v]|$.

Proof. Assume $N(v) = \{u = v_1, v_2, \ldots, v_{\deg(v)}\}$ where $\deg(v_2) = 1$, and $N(u) \setminus N[v] = \{y_1, y_2, \ldots, y_k\}$. Let G_1 be the graph obtained from G by subdividing the edge vv_i with a vertex x_i for $i = 1, 2$, and the edge uy_j with a vertex z_j for $1 \le i \le k$. Let Z be the set of the $k + 2$ subdivision vertices and let f be a $\gamma_{r2}(G_1)$-function. Without loss of generality, we may assume $f(v_2) = \{1\}$, $f(x_2) = 0$ and $2 \in f(v)$ by Theorem D. Consider two cases.

Case 1. $f(v) = \{1, 2\}$.
Define $g : V(G) \to \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ by $g(v_2) = \emptyset$, $g(u) = f(u) \cup \cup_{z \in Z} f(z)$ and $g(x) = f(x)$ for each $x \in V(G) - \{v_2, u\}$. Clearly, g is a 2RDF of G of weight less than $\omega(f) = \gamma_{r2}(G_1)$ and hence $\mathrm{sd}_{\gamma_{r2}}(G) \le k+2 = 2+|N(u)-N[v]|$.

Case 2. $f(v) = \{2\}$.
To dominate x_1, we must have $f(x_1) \ne \emptyset$ or $f(u) \ne \emptyset$. Then the function g defined by $g(v) = \{1, 2\}$, $g(v_2) = g(u) = \emptyset$, $g(y_i) = f(y_i) \cup f(z_i)$ for $1 \le i \le k$ and $g(x) = f(x)$ otherwise, is a 2RDF of G of weight less than $\omega(f) = \gamma_{r2}(G_1)$ and hence $\mathrm{sd}_{\gamma_{r2}}(G) \le k + 2 = 2 + |N(u) - N[v]|$.

This completes the proof. □

Lemma 2.2. Let G be a connected graph of order $n \geq 3$. If $v \in V(G)$ is a support vertex, then

$$\mathrm{sd}_{\gamma_{r2}}(G) \leq 2 + d_2(v).$$

Proof. Assume that $N(v) = \{v_1, v_2, \ldots, v_{\deg(v)}\}$ where $\deg(v_1) = 1$. If v has a neighbor u such that $N(u) \setminus N[v] \neq \emptyset$, then it follows from Lemma 2.1 that $\mathrm{sd}_{\gamma_{r2}}(G) \leq 2 + |N(u) - N[v]| \leq 2 + d_2(v)$. Let $N(u) \subseteq N[v]$ for each $u \in N(v)$. If G is a star, then clearly $\mathrm{sd}_{\gamma_{r2}}(G) = 1$. Thus we may assume that $v_2v_3 \in E(G)$. Let G_1 be obtained from G by subdividing the edges vv_1 and v_2v_3, with vertices x, y, respectively. Let f be a $\gamma_{r2}(G_1)$-function. By Theorem D, we may assume $f(v_1) = \{1\}$ and $2 \in f(v)$.

To dominate y, we must have $|f(v_2)|+|f(y)|+|f(v_3)| \geq 1$. Define $g : V(G) \to \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ by $g(v) = \{1, 2\}$ and $g(x) = \emptyset$ for each $x \in V(G) - \{v\}$. It is easy to see that g is a 2RDF of G of weight 2 and hence $\mathrm{sd}_{\gamma_{r2}}(G) \leq 2$. This completes the proof. □

Lemma 2.3. Let G be a connected graph of order $n \geq 3$ and let G have a vertex $v \in V(G)$ which is contained in a triangle vuw such that $N(u) \cup N(w) \subseteq N[v]$. Then $\mathrm{sd}_{\gamma_{r2}}(G) \leq 3$.

Proof. Let G_1 be obtained from G by subdividing the edges vu, vw, uw with vertices x, y, z, respectively. Let f be a $\gamma_{r2}(G_1)$-function. We claim that $|f(v)| + |f(u)| + |f(w)| + |f(x)| + |f(y)| + |f(z)| \geq 3$. In the case, the function $g : V(G) \to \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ defined by $g(v) = \{1, 2\}$, $g(u) = g(w) = \emptyset$ and $g(l) = f(l)$ for each $l \in V(G) - \{v, u, w\}$, is a 2RDF of G of weight less than $\gamma_{r2}(G_1)$, as desired. If $f(v) = \{1, 2\}$, then to dominate z, we must have $|f(z)| + |f(u)| + |f(w)| \geq 1$. If $|f(v)| = 1$, then to dominate x, y, we must have $|f(x)| + |f(u)| \geq 1$ and $|f(y)| + |f(w)| \geq 1$, respectively, and we are done. Let $f(v) = \emptyset$. If $f(x) = \emptyset$ (the case $f(y) = \emptyset$ is similar) then we must have $f(u) = \{1, 2\}$ and $|f(y)| + |f(w)| \geq 1$, as desired. Let $|f(x)| \geq 1$ and $|f(y)| \geq 1$. To dominate z, we must have $|f(u)| + |f(z)| + |f(w)| \geq 1$ and the proof is complete. □

Lemma 2.4. Let G be a connected graph of order $n \geq 3$ and let G have a vertex $v \in V(G)$ which is contained in a triangle vuw such that $N(u) \subseteq N[v]$ and $N(w) \setminus N[v] \neq \emptyset$. Then

$$\mathrm{sd}_{\gamma_{r2}}(G) \leq 3 + |N(w) \setminus N[v]|.$$

Proof. Let $N(w) \setminus N[v] = \{w_1, w_2, \ldots, w_k\}$ and let G_1 be obtained from G by subdividing the edges vu, vw, uw with vertices x, y, z, respectively, and for each $1 \leq i \leq k$, the edge ww_i with the vertex z_i. Assume f is a $\gamma_{r2}(G_1)$-function. As in Lemma 2.3, $|f(v)| + |f(u)| + |f(w)| + |f(x)| + |f(y)| + |f(z)| \geq 3$. Define $g : V(G) \to \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ by $g(v) = \{1, 2\}, g(u) = g(w) = \emptyset, g(w_i) = f(w_i) \cup f(z_i)$ for $1 \leq i \leq k$ and $g(l) = f(l)$ for each $l \in V(G) - \{v, u, w, w_1, w_2, \ldots, w_k\}$. It is easy to see that g is a 2RDF of G of weight less than $\gamma_{r2}(G_1)$ and the proof is complete. □

Lemma 2.5. Let G be a connected graph of order $n \geq 3$ and v a vertex of degree at least 2 of G such that
(i) $N(y) \setminus N[v] \neq \emptyset$ for each $y \in N(v)$,
(ii) there exists a pair α, β of vertices in $N(v)$ such that $(N(\alpha) \cap N(\beta)) \setminus N[v] = \emptyset$.
Then $\mathrm{sd}_{\gamma_{r2}}(G) \leq 3 + |N_2(v)|$.

Proof. Let $N(v) = \{v_1, v_2, \ldots, v_{\deg(v)}\}$ and v_1, v_2 be any pair of adjacent vertices of $N(v)$ satisfying (ii) if such a pair exists. If not, then each pair of vertices of $N(v)$ satisfying (ii) is independent. Assume that $S = \{v_1, v_2, \ldots, v_k\}$ is a largest subset of $N(v)$ containing v_1, v_2 and such that every pair v_i, v_j, $1 \leq i \neq j \leq k$, of vertices satisfies (ii), and let $K = (\cup_{i=1}^{k} N(v_i)) \setminus N[v]$. Let $N(v_i) \setminus N[v] = \{v_{i1}, v_{i2}, \ldots, v_{i\ell_i}\}$ for $1 \leq i \leq k$. Let G_1 be obtained from G by subdividing the edges vv_1 and vv_2 with respectively x_1 and x_2, and for each $1 \leq i \leq k$, the edge v_iv_{ij}, $1 \leq j \leq \ell_i$, with v^{ij}. We put $T_i = \{v^{ij} \mid 1 \leq j \leq \ell_i\}$ and $T = \cup_{i=1}^{k} T_i$. If v_1 and v_2 are adjacent, we also subdivide the edge v_1v_2 with a vertex u. Let f be a $\gamma_{r2}(G_1)$-function. If $v_1v_2 \in E(G)$, then as in the proof of Lemma 2.4 we have $|f(v)|+|f(v_1)|+|f(v_2)|+|f(x_1)|+|f(x_2)|+|f(u)| \geq 3$. Define $g : V(G) \to \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ by $g(v) = \{1, 2\}, g(v_1) = g(v_2) = \emptyset, g(v_{ij}) = f(v_{ij}) \cup f(v^{ij})$ for each $1 \leq i \leq k$ and each $1 \leq j \leq \ell_i$ and $g(x) = f(x)$ otherwise. It is easy to see that g is a 2RDF of G of weight less than $\gamma_{r2}(G_1)$. Let $v_1v_2 \notin E(G)$. By the choice of v_1, v_2, we deduce that S is an independent set.

To dominate x_1, x_2, we must have $|f(v)| + |f(x_1)| + |f(x_2)| + |f(v_1)| + |f(v_2)| \geq 2$. If $|f(v)| + |f(x_1)| + |f(x_2)| + \sum_{i=1}^{k} |f(v_i)| \geq 3$, then the function $g : V(G) \to \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ defined by $g(v) = \{1,2\}, g(v_i) = \emptyset$ for $1 \leq i \leq k$, $g(v_{ij}) = f(v_{ij}) \cup f(v^{ij})$ for each $1 \leq i \leq k$ and each $1 \leq j \leq \ell_i$ and $g(x) = f(x)$ otherwise, is a 2RDF of G of weight less than $\gamma_{r2}(G_1)$. Thus we may assume that $|f(v)| + |f(x_1)| + |f(x_2)| + \sum_{i=1}^{k} |f(v_i)| = 2$. If $|f(v)| = 1$, then to dominate x_1, x_2 we must have $|f(x_1)| + |f(v_1)| \geq 1$ and $|f(x_2)| + |f(v_2)| \geq 1$ implying that $|f(v)| + |f(x_1)| + |f(x_2)| + \sum_{i=1}^{k} |f(v_i)| \geq 3$, a contradiction. Hence $f(v) = \emptyset$ or $f(v) = \{1,2\}$. We consider two cases.

Case 1. Assume that $f(v) = \{1,2\}$.
Thus $|f(x_1)| + |f(x_2)| + \sum_{i=1}^{k} |f(v_i)| = 0$. If $\sum_{j=1}^{\ell_i} |f(v^{ij})| \geq 3$ for some $1 \leq i \leq k$, say $i = 1$, then the function $g : V(G) \to \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ defined by $g(v_1) = \{1,2\}, g(v_{ij}) = f(v_{ij}) \cup f(v^{ij})$ for each $2 \leq i \leq k$ and each $1 \leq j \leq \ell_i$ and $g(x) = f(x)$ otherwise, is a 2RDF of G of weight less than $\gamma_{r2}(G_1)$. Suppose that $\sum_{j=1}^{\ell_i} |f(v^{ij})| \leq 2$ for each i. If $f(v^{ij}) = \{1,2\}$ for some $1 \leq i \leq k$ and some $1 \leq j \leq \ell_i$, say $i = j = 1$, then define g by $g(v_{11}) = \{1\}, g(v_{ij}) = f(v_{ij}) \cup f(v^{ij})$ when $ij \neq 11$ and $g(x) = f(x)$ otherwise. Clearly g is a 2RDF of G of weight less than $\gamma_{r2}(G_1)$. Thus we may assume $|f(v^{ij})| \leq 1$ for each i and each j. Note that $f(v_{ij}) = \{1,2\}$ when $f(v^{ij}) = \emptyset$. Define $g : V(G) \to \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ by $g(v) = \{1\}, g(v_{ij}) = \{2\} \cup f(v_{ij})$ when $|f(v_{ij})| = 1$ and $g(x) = f(x)$ otherwise. It is easy to see that g is a 2RDF of G of weight less than $\gamma_{r2}(G_1)$.

Case 2. Assume that $f(v) = \emptyset$.
Since $|f(v)|+|f(x_1)|+|f(x_2)|+\sum_{i=1}^{k} |f(v_i)| = 2$, we deduce that that $|f(x_1)| = |f(x_2)| = 1$. Then $f(v_i) = \emptyset$ for each $i \in \{1, 2, \ldots, k\}$. If $f(x_1) = f(x_2)$, then the function $g : V(G) \to \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ defined by $g(v) = f(x_1), g(v_{ij}) = f(v_{ij}) \cup f(v^{ij})$ for each $2 \leq i \leq \deg(v)$ and each $1 \leq j \leq i\ell_i$ and $g(x) = f(x)$ otherwise, is a 2RDF of G of weight less than $\gamma_{r2}(G_1)$. Let $f(x_1) \neq f(x_2)$. As in Case 1, we may assume that $|f(v^{ij})| \leq 1$ for each i and each j. Then the function $g : V(G) \to \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ by $g(v) = \{1\}, g(v_{ij}) = \{2\} \cup f(v_{ij})$ when $|f(v_{ij})| = 1$ and $g(x) = f(x)$ otherwise is a 2RDF of G of weight less than $\gamma_{r2}(G_1)$.

In all cases we defined a 2RDF of G of weight less than $\gamma_{r2}(G_1)$. Since G_1 was obtained by inserting at most $3 + |T| \leq 3 + |N_2(v)|$ vertices, $\mathrm{sd}_{\gamma_{r2}}(G) \leq 3 + |N_2(v)|$. □

Lemma 2.6. Let G be a connected graph of order $n \geq 3$ and v a vertex of degree at least 2 of G such that
(i) $N(y) \setminus N[v] \neq \emptyset$ for each $y \in N(v)$,
(ii) for every pair of vertices α, β in $N(v)$, $(N(\alpha) \cap N(\beta)) \setminus N[v] \neq \emptyset$.
Then $\mathrm{sd}_{\gamma_{r2}}(G) \leq 3 + |N_2(v)|$.

Proof. Let $N(v) = \{v_1, v_2, \ldots, v_k\}$ and $M = N(v_1) \setminus N[v] = \{w_1, w_2, \ldots, w_p\}$. By the hypothesis, each $y \in N(v) \setminus \{v_1\}$ has a neighbor in M. Let T be a largest subset of $N(v) \setminus \{v_1\}$ such that for each subset $T_1 \subseteq T$, $|N(T_1) \setminus (N[v] \cup M)| \geq |T_1|$. By the definition of T, $|N_2(v)| \geq |M| + |T|$ and for every vertex $u \in U = N(v) \setminus (T \cup \{v_1\})$, $N(u) \setminus N[v] \subseteq M \cup N(T)$. Moreover, M dominates $N(v)$ by (ii). If $|U| \leq 1$, then by Theorem A we have

$$\begin{aligned} \mathrm{sd}_{\gamma_{r2}}(G) &\leq |N(v) \cup N(v_1)| - 1 \\ &= |T| + |U| + 1 + |M| + 1 - 1 \\ &\leq |U| + 1 + |N_2(v)| \\ &\leq |N_2(v)| + 2, \end{aligned}$$

as desired. Let $|U| \geq 2$. Suppose that $T = \emptyset$ or, without loss of generality, $T = \{v_2, v_3, \ldots, v_s\}$. Let G_1 be obtained from G by subdividing the $|M| + |T| + 3$ edges $v_1 w_j$ with vertex y_i for $1 \leq j \leq p$ and vv_i with vertex x_i for $1 \leq i \leq s + 2$ ($1 \leq i \leq 3$ when $T = \emptyset$). Let f be a $\gamma_{r2}(G_1)$-function. If $|f(v)| + |f(v_1)| + \sum_{i=1}^{s+2} |f(x_i)| \geq 3$, then define $g : V(G) \to \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ by $g(v) = \{1,2\}, g(w_j) = f(w_j) \cup f(y_j)$ for $1 \leq j \leq p$ and $g(x) = f(x)$ otherwise. It is easy to see that g is a 2RDF of G of weight less than $\gamma_{r2}(G_1)$. Assume that

$$|f(v)| + |f(v_1)| + \sum_{i=1}^{s+2} |f(x_i)| \leq 2. \tag{1}$$

Consider three cases.

Case 1. Assume that $f(v) = \{1, 2\}$.
By (1), $f(v_1) = \emptyset$. Hence, if $f(y_j) = \emptyset$ for some j, then $f(w_j) = \{1, 2\}$. Define $g : V(G) \to \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ by $g(v) = \{1\}, g(w_j) = \{2\} \cup f(w_j)$ when $|f(y_j)| \geq 1$ and $g(x) = f(x)$ otherwise. Since each v_i has a neighbor in $\{w_1, w_2, , \ldots, w_p\}$, we deduced that g is a 2RDF of G of weight less than $\gamma_{r2}(G_1)$.

Case 2. Assume that $f(v) = \emptyset$.
To dominate x_1, we must have $|f(x_1)| + |f(v_1)| \geq 1$. If $|f(x_1)| + |f(v_1)| \geq 2$, then by (1), we have $f(x_{s+1}) = f(x_{s+2}) = \emptyset$ and hence $f(v_{s+1}) = f(v_{s+2}) = \{1, 2\}$. Define $g : V(G) \to \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ by $g(v) = g(v_1) = \{1, 2\}, g(v_{s+1}) = g(v_{s+2}) = \emptyset$, and $g(x) = f(x)$ otherwise. Since for every vertex $u \in U = N(v) \setminus (T \cup \{v_1\})$, $N(u) \setminus N[v] \subseteq M \cup N(T)$, g is a 2RDF of G of weight less than $\gamma_{r2}(G_1)$. Let $|f(x_1)| + |f(v_1)| = 1$. By (1), we may assume that $f(x_{s+1}) = \emptyset$ and so $f(v_{s+1}) = \{1, 2\}$. To dominate x_{s+2}, we must have $|f(x_{s+2})| + |f(v_{s+2})| \geq 1$. Define $g : V(G) \to \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ by $g(v) = \{1, 2\}, g(v_1) = f(v_1) \cup f(x_1), g(v_{s+1}) = g(v_{s+2}) = \emptyset, g(w_j) = f(w_j) \cup f(y_j)$ for $1 \leq j \leq p$, and $g(x) = f(x)$ otherwise. Since for every vertex $u \in U = N(v) \setminus (T \cup \{v_1\})$, $N(u) \setminus N[v] \subseteq M \cup N(T)$, g is a 2RDF of G of weight less than $\gamma_{r2}(G_1)$.

Case 3. Assume that $|f(v)| = 1$.
To dominate x_1, we must have $|f(x_1)| + f(v_1)| \geq 1$. It follows from (1) that $f(x_{s+1}) = f(x_{s+2}) = \emptyset$ and hence $f(v_{s+1}) = f(v_{s+2}) = \{1, 2\}$. Define $g : V(G) \to \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ by $g(v) = \{1, 2\}, g(v_1) = f(v_1) \cup f(x_1), g(v_{s+1}) = g(v_{s+2}) = \emptyset, g(w_j) = f(w_j) \cup f(y_j)$ for $1 \leq j \leq p$, and $g(x) = f(x)$ otherwise. Clearly, g is a 2RDF of G of weight less than $\gamma_{r2}(G_1)$ and the proof is complete. □

Theorem 2.7. Let G be a connected graph of order $n \geq 3$. Then

$$\mathrm{sd}_{\gamma_{r2}}(G) \leq 3 + \min\{d_2(v) \mid v \in V \text{ and } d(v) \geq 2\}.$$

Proof. If G is a star $K_{1,n-1}$ then $\mathrm{sd}_{\gamma_{r2}}(G) = 1$. Otherwise, let v be a vertex of degree at least 2 of G such that $d_2(v)$ is minimum. The result is a consequence of Lemmas 2.1 and 2.2 if v is a support vertex, of Lemmas 2.3 and 2.4 if some neighbor u of v different from a leaf satisfies $N(u) \subseteq N[v]$, and of Lemmas 2.5 and 2.6 if $N(y) \setminus N[v] \neq \emptyset$ for every $y \in N(v)$. □

Corollary 2.8. Let G be a connected graph of minimum degree at least 2. Then $\mathrm{sd}_{\gamma_{r2}}(G) \leq \delta_2(G) + 3$.

For a vertex v of degree Δ, $|N_2(v)| \leq n - \Delta - 1$. Therefore the following improvement of the bound in Theorem E is an immediate corollary of Theorem 2.7 for non-regular graphs.

Corollary 2.9. Let G be a connected graph of order $n \geq 3$. Then $\mathrm{sd}_{\gamma_{r2}}(G) \leq n - \Delta + 2$.

3. An upper bound in terms of rainbow domination number

In this section we present an upper bound on $\mathrm{sd}_{\gamma_{r2}}(G)$ in terms of the rainbow domination number of G.

Lemma 3.1. Let G be a connected graph of order $n \geq 3$ with $\delta(G) \geq 2$ and let G have a vertex $v \in V(G)$ which has a neighbor u such that $N(u) \subseteq N[v]$. Then subdividing the edges at v strictly increase the rainbow domination number.

Proof. Let $N(v) = \{v_1, v_2 \ldots, v_k\}$ where $u = v_k$ and let G' be the graph obtained from G by subdividing the edge vv_i with subdivision vertex x_i for $i = 1, 2, \ldots, k$. Assume that f is a $\gamma_{r2}(G')$-function. If $|f(v)| + \sum_{i=1}^{k} |f(x_i)| \geq 3$, then the function $g : V \to \mathcal{P}(\{1, 2\})$ defined by $g(v) = \{1, 2\}$ and $g(z) = f(z)$ for $z \in V \setminus \{v\}$, is a 2RDF of G of weight less than $\gamma_{r2}(G')$. Let

$$|f(v)| + \sum_{i=1}^{k} |f(x_i)| \leq 2. \tag{2}$$

We consider three cases.

Case 1. Assume that $f(v) = \{1, 2\}$.
Then $f(x_i) = \emptyset$ for each i, by (2). Define $g : V(G) \to \mathcal{P}(\{1, 2\})$ by $g(v) = \{1\}$ and $g(z) = f(z)$ for each $z \in V \setminus \{v\}$. It is easy to see that g is a 2RDF of G of weight less than $\gamma_{r2}(G')$.

Case 2. Assume that $|f(v)| = 1$.
We may assume, without loss of generality, that $f(v) = \{1\}$. If $f(x_i) \neq \emptyset$ for exactly one $i = i_0$, then the function $g : V \to \mathcal{P}(\{1, 2\})$ by $g(v) = f(x_{i_0})$ and $g(z) = f(z)$ for each $z \in V \setminus \{v\}$ is a 2RDF of G of weight less than $\gamma_{r2}(G')$. Let $f(x_i) = \emptyset$ for each i. It follows that $2 \in f(v_i)$ for each i. Since $\delta(G) \geq 2$ and $N(u) \subseteq N[v]$, the function $g : V \to \mathcal{P}(\{1, 2\})$ by $g(u) = \emptyset$ and $g(z) = f(z)$ for each $z \in V \setminus \{u\}$ is a 2RDF of G of weight less than $\gamma_{r2}(G')$.

Case 3. Assume that $f(v) = \emptyset$.
Then $\bigcup_{i=1}^{k} f(x_i) = \{1, 2\}$. If $f(x_i) = \{1, 2\}$ for some i, then the function $g : V(G) \to \mathcal{P}(\{1, 2\})$ by $g(v_i) = f(v_i) \cup \{1\}$ and $g(z) = f(z)$ for each $z \in V \setminus \{v_i\}$ is a 2RDF of G of weight less than $\gamma_{r2}(G')$. Let $f(x_i) = \{1\}, f(x_j) = \{2\}$ for some $i \neq j$. If $f(v_i) \neq \emptyset$ (the case $f(v_j) \neq \emptyset$ is similar), then the function $g : V(G) \to \mathcal{P}(\{1, 2\})$ by $g(v) = f(x_j)$ and $g(z) = f(z)$ for each $z \in V \setminus \{v\}$ is a 2RDF of G of weight less than $\gamma_{r2}(G')$. Thus we may assume that $f(v_i) = \emptyset$ and $f(v_j) = \emptyset$.

If $k \notin \{i, j\}$, then $f(x_k) = \emptyset$ and we must have $f(u) = \{1, 2\}$. It is easy to see that the function $g : V(G) \to \mathcal{P}(\{1, 2\})$ defined by $g(v) = \{1, 2\}, g(u) = \emptyset$ and $g(z) = f(z)$ for each $z \in V \setminus \{u, v\}$ is a 2RDF of G of weight less than $\gamma_{r2}(G')$. Let $k \in \{i, j\}$. Assume, without loss of generality, that $i = k$. To dominate u, u must have a neighbor $v_s \in N(v)$ for which $2 \in f(v_s)$ since $f(u) = \emptyset$. It follows that $s \neq j$ because $f(v_j) = \emptyset$. Since $f(v) = f(x_s) = \emptyset$, to dominate x_s we must have $f(v_s) = \{1, 2\}$. Then the function $g : V(G) \to \mathcal{P}(\{1, 2\})$ by $g(v) = \{2\}$ and $g(z) = f(z)$ for each $z \in V \setminus \{v\}$ is a 2RDF of G of weight less than $\gamma_{r2}(G')$. This completes the proof. □

Theorem F and Lemma 3.1 imply the next result immediately.

Corollary 3.2. Let G be a connected graph of order $n \geq 3$. If G contains two adjacent vertices u and v such that $N(u) \subseteq N[v]$, then

$$\mathrm{sd}_{\gamma_{r2}}(G) \leq d(v) \leq \Delta.$$

Theorem 3.3. Let G be a connected graph of order $n \geq 3$ with $\delta(G) \geq 2$. If for each vertex $v \in V(G)$, subdividing the edges at v don't increase the rainbow domination number, then

$$\mathrm{sd}_{\gamma_{r2}}(G) \leq \gamma_{r2}(G).$$

Proof. If $\gamma_{r2}(G) = 2, 3$, then the result follows by Theorems B and C. Assume now that $\gamma_{r2}(G) \geq 4$. Let u be a vertex of degree δ and let $uv \in E(G)$. Since subdividing the edges at u don't increase the rainbow domination number, it follows from Lemma 3.1 that $N(v) \nsubseteq N[u]$. Similarly, we have $N(u) \nsubseteq N[v]$. Let $N(v) = \{v_1, v_2 \ldots, v_k\}$ where $u = v_k$ and let $N(u) - N[v] = \{u_1, u_2, \ldots, u_s\}$. Dehgardi et al. [6] proved that subdividing the edge vv_i for $i = 1, 2, \ldots, k$, and the edge uu_j for $j = 1, 2, \ldots, s$ increase the rainbow domination number (Theorem A). Let T be a maximal subset of $\{uu_1, uu_2, \ldots, uu_s\}$ such that subdividing the edges in T and the edge vv_i for $i = 1, 2, \ldots, k$, does not increases the rainbow domination number. Then

$$\mathrm{sd}_{\gamma_{r2}}(G) \leq k + |T| + 1. \tag{3}$$

Since $\deg(u) \leq \deg(v)$, we observe that $|T| \leq k - 2$ and hence

$$\mathrm{sd}_{\gamma_{r2}}(G) \leq 2k - 1. \tag{4}$$

Without loss of generality, assume that $T = \{uu_1, uu_2, \ldots, uu_r\}$ when $T \neq \emptyset$. Let G' be the graph obtained from G by subdividing the edge uu_i with subdivision vertex x_i for $i = 1, 2, \ldots, r$, and the edge vv_j with subdivision vertex y_j for $j = 1, 2, \ldots, k$. Let f be a $\gamma_{r2}(G')$-function. Then $\omega(f) = \gamma_{r2}(G') = \gamma_{r2}(G)$. If $f(v) = \{1, 2\}$, then the function $g : V(G) \to \mathcal{P}(\{1, 2\})$ defined by $g(v) = \{1\}, g(v_j) = f(v_j) \cup f(y_j)$ for each $1 \leq j \leq k$, $g(u_i) = f(u_i) \cup f(x_i)$ for each $1 \leq j \leq r$, and $g(z) = f(z)$ for each $z \in V \setminus \{v, v_1, \ldots, v_k, u_1, \ldots, u_r\}$ is a 2RDF of G of weight less than

$\gamma_{r2}(G')$, a contradiction. Thus $|f(v)| \leq 1$. If $|f(v)| + \sum_{j=1}^{k} |f(y_j)| \geq 3$, then the function $g : V(G) \to \mathcal{P}(\{1,2\})$ defined by $g(v) = \{1,2\}, g(u_i) = f(u_i) \cup f(x_i)$ for each $1 \leq j \leq r$, and $g(z) = f(z)$ for each $z \in V \setminus \{v, u_1, \ldots, u_r\}$ is a 2RDF of G of weight less than $\gamma_{r2}(G')$, a contradiction. Thus we may assume

$$|f(v)| + \sum_{j=1}^{k} |f(y_j)| \leq 2. \tag{5}$$

We consider two cases.

Case 1. Assume that $|f(v)| = 1$.

Assume, without loss of generality, that $f(v) = \{1\}$. If $|f(v)| + \sum_{j=1}^{k} |f(y_j)| = 2$, then $|f(y_i)| = 1$ for exactly one $i = i_0$. Then the function $g : V(G) \to \mathcal{P}(\{1,2\})$ by $g(v) = f(y_{i_0}), g(u_j) = f(u_j) \cup f(x_j)$ for each $1 \leq j \leq r$, and $g(z) = f(z)$ for each $z \in V \setminus \{v, u_1, \ldots, u_r\}$ is a 2RDF of G of weight less than $\gamma_{r2}(G')$ which is a contradiction. Thus we may assume that $f(y_j) = \emptyset$ for each $1 \leq j \leq k$. This implies that $2 \in f(v_j)$ for each j. If $1 \in \cup_{j=1}^{k} f(v_j)$, then the function $g : V(G) \to \mathcal{P}(\{1,2\})$ by $g(v) = \emptyset, g(u_i) = f(u_i) \cup f(x_i)$ for each $1 \leq j \leq r$, and $g(z) = f(z)$ for each $z \in V \setminus \{v, u_1, \ldots, u_r\}$ is a 2RDF of G of weight less than $\gamma_{r2}(G')$ which is a contradiction. Thus $f(v_j) = \{2\}$ for each j. If $\sum_{i=1}^{r} |f(x_i)| \geq 1$, then the function $g : V(G) \to \mathcal{P}(\{1,2\})$ by $g(v) = \emptyset, g(u) = \{1,2\}$, and $g(z) = f(z)$ for each $z \in V \setminus \{v, u\}$ is a 2RDF of G of weight less than $\gamma_{r2}(G')$ which is a contradiction. Hence $f(x_i) = \emptyset$ for each i which implies that $1 \in f(u_i)$ for $1 \leq i \leq r$. Thus $\omega(f) \geq k + 1 + r$. It follows from (3) that $\mathrm{sd}_{\gamma_{r2}}(G) \leq \gamma_{r2}(G)$.

Case 2. Assume that $f(v) = \emptyset$.

Then $\cup_{j=1}^{k} f(y_j) = \{1,2\}$. It follows from (5) that $\sum_{j=1}^{k} |f(y_j)| = 2$. First let $f(y_j) = \{1,2\}$ for some j, say $j = 1$. Then by (5) we have $f(y_j) = \emptyset$ and hence $f(v_j) = \{1,2\}$ for $2 \leq j \leq k$. Then $\gamma_{r2}(G) = \omega(f) \geq 2k > \mathrm{sd}_{\gamma_{r2}}(G)$ by (4).

Now let $|f(y_{j_1})| = |f(y_{j_2})| = 1$. Then $f(y_j) = \emptyset$ and so $f(v_j) = \{1,2\}$ for each $j \in \{1,2,\ldots,k\} - \{j_1, j_2\}$. If $f(v_{j_1}) \neq \emptyset$ (the case $f(v_{j_2}) \neq \emptyset$ is similar) or if $N(v_{j_1}) \cap (N(v) - \{v_{j_1}, v_{j_2}\}) \neq \emptyset$ (the case $N(v_{j_2}) \cap (N(v) - \{v_{j_1}, v_{j_2}\}) \neq \emptyset$ is similar), then the function $g : V \to \mathcal{P}(\{1,2\})$ by $g(v) = f(v_{j_2}), g(u_i) = f(u_i) \cup f(x_i)$ for $1 \leq i \leq r$, and $g(z) = f(z)$ for each $z \in V \setminus \{v_1, u_1, \ldots, u_r\}$ is a 2RDF of G of weight less than $\gamma_{r2}(G')$ which is a contradiction. Suppose that $f(v_{j_1}) = f(v_{j_2}) = \emptyset$, $N(v_{j_1}) \cap (N(v) - \{v_{j_1}, v_{j_2}\}) = \emptyset$ and $N(v_{j_2}) \cap (N(v) - \{v_{j_1}, v_{j_2}\}) = \emptyset$. To dominate v_{j_1}, v_{j_1} must have a neighbor w for which $\{1,2\} \setminus f(y_{j_1}) \subseteq f(w)$. This implies that

$$\gamma_{r2}(G) = \omega(f) \geq |f(y_{j_1})| + |f(y_{j_2})| + \sum_{j=1}^{k} |f(v_j)| + |f(w)| \geq 3 + 2(k-2) = 2k - 1$$

and hence $\mathrm{sd}_{\gamma_{r2}}(G) \leq \gamma_{r2}(G)$ by (4).

This complete the proof. □

Theorem F and Theorem 3.3 lead to the following general result.

Corollary 3.4. If G is a connected graph of order $n \geq 3$, then

$$\mathrm{sd}_{\gamma_{r2}}(G) \leq \max\{\gamma_{r2}(G), \Delta(G)\}.$$

To conclude the paper, let us mention the following conjecture which was established in some classes of graphs.

Conjecture 3.5. For any connected graph G of order $n \geq 3$, $\mathrm{sd}_{\gamma_{r2}}(G) \leq \gamma_{r2}(G)$.

Since $\gamma_{r2}(G) \leq n - \Delta(G) + 1$ if G is connected of order at least 3, this conjecture, if true, would imply $\mathrm{sd}_{\gamma_{r2}}(G) \leq n - \Delta(G) + 1$ improving Corollary 2.9. Thus we post the following problem.

Problem 3.6. Is it true that for any connected graph G of order $n \geq 3$,

$$\mathrm{sd}_{\gamma_{r2}}(G) \leq n - \Delta + 1.$$

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