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Quasi-Special Osserman Manifolds

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Abstract. In this paper we deal with a pseudo-Riemannian Osserman curvature tensor whose reduced Jacobi operator is diagonalizable with exactly two distinct eigenvalues. The main result gives new insight into the theory of the duality principle for pseudo-Riemannian Osserman manifolds. We concern with special Osserman curvature tensor and propose new ways to exclude some additional duality principle conditions from its definition.

1. Introduction

Let us start with the basic notation and terminology used throughout this work. Let R be an algebraic curvature tensor on a vector space $\mathcal{V}$ equipped with an indefinite metric g of the signature $(\nu, n-\nu)$. The sign $\varepsilon_X = g(X,X)$ denotes the norm of $X \in \mathcal{V}$, and it determines various types of vectors. We say that $X \in \mathcal{V}$ is timelike (if $\varepsilon_X < 0$), spacelike ($\varepsilon_X > 0$), null ($\varepsilon_X = 0$), nonnull ($\varepsilon_X \neq 0$), or unit ($\varepsilon_X \in \{-1, 1\}$). The curvature operator $\mathcal{R}$ is linked with R by equation $R(X,Y,Z,W) = g(\mathcal{R}(X,Y)Z, W)$. For the initial definitions and deeper explanations of this topic, the reader can consult Gilkey's book [11].

We need some natural operators associated with the curvature tensor. The polarized Jacobi operator $\mathcal{J}(X,Y) : \mathcal{V} \to \mathcal{V}$ is given by

$$\mathcal{J}(X,Y)(Z) = \frac{1}{2}\left(\mathcal{R}(Z,X)Y + \mathcal{R}(Z,Y)X\right),$$

for all $X, Y, Z \in \mathcal{V}$. Specifically, the Jacobi operator $\mathcal{J}_X : \mathcal{V} \to \mathcal{V}$ is defined by $\mathcal{J}_X = \mathcal{J}(X,X)$, which means that $\mathcal{J}_X(Z) = \mathcal{R}(Z,X)(X)$ holds for all $Z \in \mathcal{V}$. In the case of nonnull $X \in \mathcal{V}$, $\mathcal{J}_X$ preserves nondegenerate hyperspace $\{X\}^\perp = \{Y \in \mathcal{V} : X \perp Y\}$, and we have the reduced Jacobi operator $\tilde{\mathcal{J}}_X : \{X\}^\perp \to \{X\}^\perp$, given by $\tilde{\mathcal{J}}_X = \mathcal{J}_X|_{\{X\}^\perp}$.

We say that R is an Osserman curvature tensor if the characteristic polynomial of $\mathcal{J}_X$ is constant on both pseudo-spheres, in particular on the positive ($\varepsilon_X = 1$) and the negative ($\varepsilon_X = -1$) one. In a pseudo-Riemannian setting, Jordan normal form plays a crucial role, since characteristic polynomial does not determine the eigen-structure of a symmetric linear operator. We say that R is a Jordan Osserman curvature tensor if the Jordan normal form of $\mathcal{J}_X$ is constant on both pseudo-spheres.

In this text we study the Osserman curvature tensor R, whose Jacobi operator $\mathcal{J}_X$ is diagonalizable for all nonnull X, and we call such R - diagonalizable Osserman. Diagonalizability is a natural Riemannian-like

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condition, moreover, it is known that every Jordan Osserman curvature tensor of non-balanced signature ($n \neq 2\nu$) is necessarily diagonalizable [12].

In the Riemannian setting ($\nu = 0$), it is known that a local two-point homogeneous space (flat or locally rank one symmetric space) has a constant characteristic polynomial on the unit sphere bundle. Osserman wondered if the converse held [16], and this question has been called the Osserman conjecture by subsequent authors. During the solution of some particular cases of the conjecture, the implication

$$\mathcal{J}_X(Y) = \lambda Y \Rightarrow \mathcal{J}_Y(X) = \lambda X \tag{1}$$

appeared naturally, and if it holds, it can significantly simplify some calculations. The first results on this topic were published by Chi [8], who proved the conjecture in the cases of dimensions $n \neq 4k$, $k > 1$. In his paper he used the statement that (1) holds, if λ is an extremal (minimum or maximum) eigenvalue of the Jacobi operator. Rakić hoped that correctness of (1) could offer deeper understanding of the Osserman conjecture. He formulated the duality principle for Osserman manifolds and proved it in the Riemannian setting [17]. After that, the duality principle has been reproved by Gilkey [10], and it has become a beneficial tool for the conjecture solution. Moreover, the best results in this topic were achieved by Nikolayevsky [13–15], who used the duality principle [14] to prove the Osserman conjecture in all dimensions, except some possibilities in dimension $n = 16$.

The variant of the Osserman conjecture has appeared in a pseudo-Riemannian setting. For example, in the Lorentzian setting ($\nu = 1$), an Osserman manifold necessarily has a constant sectional curvature [5]. The observation of Osserman manifolds in the signature $(2, 2)$ become very popular, and it is worth noting results from [5], which are based on the discussion of possible Jordan normal forms of the Jacobi operator.

This is why we have started investigating the duality principle for Osserman curvature tensor in a pseudo-Riemannian setting. In a pseudo-Riemannian setting, the implication (1) looks inaccurate, and therefore we corrected it in the following way [1, 2, 4].

Definition 1 (Duality principle). *We say that the duality principle holds for curvature tensor R if for all mutually orthogonal units $X, Y \in \mathcal{V}$, and for all $\lambda \in \mathbb{R}$ holds*

$$\mathcal{J}_X(Y) = \varepsilon_X \lambda Y \Rightarrow \mathcal{J}_Y(X) = \varepsilon_Y \lambda X.$$

The duality principle for Osserman curvature tensor works for every known example, however we failed to prove it in general. In our previous work [1, 2, 4] we gave the affirmative answer only for the conditions of small index ($\nu \leq 1$) or low dimension ($n \leq 4$). In this text, we restrict our attention to small numbers of eigenvalues of the reduced Jacobi operator.

The simplest case is the diagonalizable Osserman curvature tensor whose reduced Jacobi operator has a single eigenvalue, and it has to be a real space form (constant sectional curvature) [9]. This is the reason why we devote our attention to the first nontrivial case, the diagonalizable Osserman curvature tensor whose reduced Jacobi operator has two distinct eigenvalues, and we call it a two-leaves Osserman (Definition 3). The duality principle holds for $n \leq 4$ [1–4], but every connected pointwise two-leaves Osserman manifold is a globally Osserman for $n > 4$ [2, 9]. This is why we put the problem into a pure algebraic concept with an algebraic Osserman curvature tensor instead of working with an Osserman manifold and associated tangent bundles. The more precise introduction and motivation of our specific problem will be given at the beginning of Section 4.

Our paper is organized as follows. Section 1 is devoted to the general introduction and motivation of the topic. Section 2 presents some original preliminaries, mostly concerning null vectors. In Section 3 we define two-leaves Osserman curvature tensor. We introduce valuable notations and give two important lemmas. Section 4 gives additional specific introduction of a special Osserman curvature tensor. We define a quasi-special Osserman curvature tensor, prove important lemmas, and try to exclude the specific special Osserman condition. At the end, we prove that every almost-special Osserman curvature tensor is a special Osserman.

2. Preliminaries

This section contains some small original results which we need later in the text. The lemmas are mostly related to null vectors, especially to totally isotropic spaces (Definition 2). The first lemma shows an interesting general property of a Jacobi operator.

Lemma 1. *For nonzero scalars $\alpha, \beta, \gamma, \delta$ holds*

$$\mathcal{J}_{\alpha X+\beta Y} = \frac{\alpha\beta}{\gamma\delta}\mathcal{J}_{\gamma X+\delta Y} + \frac{\alpha(\alpha\delta-\beta\gamma)}{\delta}\mathcal{J}_X + \frac{\beta(\beta\gamma-\alpha\delta)}{\gamma}\mathcal{J}_Y.$$

Proof. This is a simple calculation by definition.

$$\mathcal{J}_{\alpha X+\beta Y} = \mathcal{R}(\,\cdot\,, \alpha X+\beta Y)(\alpha X+\beta Y) = \alpha^2\mathcal{J}_X + \beta^2\mathcal{J}_Y + 2\alpha\beta\mathcal{J}(X,Y)$$

$$\mathcal{J}_{\gamma X+\delta Y} = \mathcal{R}(\,\cdot\,, \gamma X+\delta Y)(\gamma X+\delta Y) = \gamma^2\mathcal{J}_X + \delta^2\mathcal{J}_Y + 2\gamma\delta\mathcal{J}(X,Y)$$

Let us equate $2\mathcal{J}(X,Y)$ from the previous equations.

$$\frac{1}{\alpha\beta}\left(\mathcal{J}_{\alpha X+\beta Y} - \alpha^2\mathcal{J}_X - \beta^2\mathcal{J}_Y\right) = \frac{1}{\gamma\delta}\left(\mathcal{J}_{\gamma X+\delta Y} - \gamma^2\mathcal{J}_X - \delta^2\mathcal{J}_Y\right)$$

Thus

$$\mathcal{J}_{\alpha X+\beta Y} = \frac{\alpha\beta}{\gamma\delta}\mathcal{J}_{\gamma X+\delta Y} + \left(\alpha^2 - \frac{\alpha\beta\gamma}{\delta}\right)\mathcal{J}_X + \left(\beta^2 - \frac{\alpha\beta\delta}{\gamma}\right)\mathcal{J}_Y,$$

which proves the lemma. □

Definition 2 (Totally isotropic space). *A vector space is totally isotropic if it consists just of null vectors.*

Lemma 2. *Vectors from a totally isotropic space are mutually orthogonal.*

Proof. Let X and Y belong to a totally isotropic space $\mathcal{U}$. Then $X+Y \in \mathcal{U}$ gives $0 = \varepsilon_{X+Y} = \varepsilon_X + 2g(X,Y) + \varepsilon_Y = 2g(X,Y)$, and thus $X \perp Y$. □

Lemma 3. *If $\mathcal{V}$ is a vector space of the signature (p,q), then for every totally isotropic subspace $\mathcal{U} \leq \mathcal{V}$ holds $\dim\mathcal{U} \leq \min(p,q)$. Especially $\dim\mathcal{U} \leq \frac{1}{2}\dim\mathcal{V}$.*

Proof. We decompose $\mathcal{V} = \mathcal{V}^+ \oplus \mathcal{V}^-$ as an orthogonal sum, where $\mathcal{V}^+$ is a maximal spacelike subspace of dimension q and $\mathcal{V}^-$ is the complementary maximal timelike subspace of dimension p. Without loss of generality we can assume $p \leq q$. Totally isotropic $\mathcal{U} \leq \mathcal{V}$ with $\dim\mathcal{U} > p$, gives linearly independent vectors $V_1, \dots, V_p, V_{p+1} \in \mathcal{U}$. For all $i \in \{1, \dots, p+1\}$ we can decompose $V_i = P_i + Q_i$, with $P_i \in \mathcal{V}^+$ and $Q_i \in \mathcal{V}^-$. The dimension of $\mathcal{V}^-$ enables an existence of scalars $\alpha_1, \dots, \alpha_{p+1}$, which are not all zero, such that $\sum_{i=1}^{p+1}\alpha_i Q_i = 0$. Thus $\sum_{i=1}^{p+1}\alpha_i V_i = \sum_{i=1}^{p+1}\alpha_i P_i + \sum_{i=1}^{p+1}\alpha_i Q_i = \sum_{i=1}^{p+1}\alpha_i P_i$. The left side $\sum_{i=1}^{p+1}\alpha_i V_i \in \mathcal{U}$ has zero norm, while the right side $\sum_{i=1}^{p+1}\alpha_i P_i \in \mathcal{V}^+$ belongs to a definite subspace. Hence $\sum_{i=1}^{p+1}\alpha_i V_i = 0$, and therefore vectors V_i are not linearly independent. Especially $\dim\mathcal{U} \leq \min(p,q) \leq \frac{1}{2}(p+q) = \frac{1}{2}\dim\mathcal{V}$. □

Lemma 4. *Every null $N \neq 0$ from a nondegenerate space $\mathcal{V}$ can be decomposed as $N = S + T$, where $S, T \in \mathcal{V}$ and $\varepsilon_S = -\varepsilon_T > 0$.*

Proof. $N \in \mathcal{V} = \mathcal{V}^+ \oplus \mathcal{V}^-$, so $N = S + T$ with $S \in \mathcal{V}^+$ and $T \in \mathcal{V}^-$. $\mathcal{V}^+ \perp \mathcal{V}^-$ gives $S \perp T$, and therefore $0 = \varepsilon_N = g(S+T, S+T) = \varepsilon_S + \varepsilon_T$. Since $N \neq 0$, we conclude $S \neq 0$ and $\varepsilon_S \neq 0$, which finally implies $\varepsilon_S = -\varepsilon_T > 0$. □

A decomposition from Lemma 4 is not unique, even in the same plane $\mathrm{Span}\{S,T\}$. If we set $S_1 = \theta S + (1-\theta)T$ and $T_1 = (1-\theta)S + \theta T$ for some $\theta > \frac{1}{2}$, then $S_1, T_1 \in \mathcal{V}$ with $S_1 + T_1 = S + T = N$ and

$$g(S_1,T_1) = g(\theta S + (1-\theta)T, (1-\theta)S + \theta T) = \theta(1-\theta)(\varepsilon_S + \varepsilon_T) = 0.$$

Further we obtain $\varepsilon_{S_1} = \theta^2\varepsilon_S + (1-\theta)^2\varepsilon_T = (\theta^2 - (1-\theta)^2)\varepsilon_S = (2\theta-1)\varepsilon_S > 0$, while $S_1 \perp T_1$ gives $\varepsilon_{S_1} + \varepsilon_{T_1} = \varepsilon_{S_1+T_1} = \varepsilon_N = 0$ and $\varepsilon_{S_1} = -\varepsilon_{T_1} > 0$. Every $\theta > \frac{1}{2}$ makes a new decomposition, for example, $\theta = \frac{1+\varepsilon_S}{2\varepsilon_S} > \frac{1}{2}$ gives a natural decomposition

$$N = \left(\frac{\varepsilon_S+1}{2\varepsilon_S}S + \frac{\varepsilon_S-1}{2\varepsilon_S}T\right) + \left(\frac{\varepsilon_S-1}{2\varepsilon_S}S + \frac{\varepsilon_S+1}{2\varepsilon_S}T\right),$$

with $\varepsilon_{S_1} = -\varepsilon_{T_1} = 1$.

New constructions are possible in other planes. If $\dim\mathcal{V} > 2$, there exists nonnull $W \in \mathcal{V}$, such that $W \perp \mathrm{Span}\{S,T\}$. We want to find α,β,γ, such that $S_1 = \alpha S + \beta T + \gamma W$ and $T_1 = (1-\alpha)S + (1-\beta)T - \gamma W$, which assure $S_1 + T_1 = S + T = N$. From the orthogonality $S_1 \perp T_1$ we have

$$0 = g(S_1,T_1) = \alpha(1-\alpha)\varepsilon_S + \beta(1-\beta)\varepsilon_T - \gamma^2\varepsilon_W = ((\alpha-\alpha^2) - (\beta-\beta^2))\varepsilon_S - \gamma^2\varepsilon_W,$$

and therefore

$$\gamma^2\varepsilon_W = (\alpha-\beta)(1-\alpha-\beta)\varepsilon_S. \tag{2}$$

The last condition $\varepsilon_{S_1} > 0$ gives

$$0 < \varepsilon_{S_1} = \alpha^2\varepsilon_S + \beta^2\varepsilon_T + \gamma^2\varepsilon_W = (\alpha^2-\beta^2)\varepsilon_S + (\alpha-\beta)(1-\alpha-\beta)\varepsilon_S = (\alpha-\beta)\varepsilon_S,$$

which is true for $\alpha > \beta$. In the case of spacelike W we choose $\alpha + \beta < 1$, while in the case of timelike W we choose $\alpha + \beta > 1$, which according to the equation (2) with $\alpha > \beta$ determines the final conditions for α and β. Finally, every α and β limited with the last two inequalities create a new decomposition, where $\gamma = \pm\sqrt{\frac{(\alpha-\beta)(1-\alpha-\beta)\varepsilon_S}{\varepsilon_W}}$.

Lemma 5. *Let $\mathcal{U} \leq \mathcal{V}$ be a nondegenerate subspace and $X \in \mathcal{V}$ be nonnull. Then every null $0 \neq N \in \mathcal{U}$ orthogonal to X can be decomposed as $N = S + T$, where $S, T \in \mathcal{U}$ and $\varepsilon_S = -\varepsilon_T > 0$, such that vectors $X - S$ and $X - T$ are not both null.*

Proof. We use Lemma 4 to set $N = S + T$, with $S,T \in \mathcal{U}$ and $\varepsilon_S = -\varepsilon_T > 0$. If $X - S$ and $X - T$ are both null, then $0 = \varepsilon_{X-S} = \varepsilon_X - 2g(X,S) + \varepsilon_S$ and $0 = \varepsilon_{X-T} = \varepsilon_X - 2g(X,T) + \varepsilon_T$. Thus

$$2g(X,S) = \varepsilon_X + \varepsilon_S, \quad 2g(X,T) = \varepsilon_X - \varepsilon_S. \tag{3}$$

Sum of the equations (3) is $2g(X,S) + 2g(X,T) = 2\varepsilon_X$, and therefore $0 = g(X,N) = \varepsilon_X \neq 0$, which completes the proof. □

Lemma 6. *Let $\mathcal{U}, \mathcal{W} \leq \mathcal{V}$, $\mathcal{U}$ is nondegenerate, and let $\Xi : \mathcal{U} \mapsto \mathcal{W}$ be a linear function such that $\Xi(X)$ is null for every nonnull $X \in \mathcal{U}$. Then $\Xi(\mathcal{U})$ is a totally isotropic space.*

Proof. It is needed to show that $\Xi(N)$ is null for every null $N \in \mathcal{U}$. Lemma 4 decomposes $N = S + T$, with $S, T \in \mathcal{U}$ and $\varepsilon_S = -\varepsilon_T > 0$, and therefore

$$\varepsilon_{\Xi(\alpha S+\beta T)} = \varepsilon_{\alpha\Xi(S)+\beta\Xi(T)} = \alpha^2\varepsilon_{\Xi(S)} + \beta^2\varepsilon_{\Xi(T)} + 2\alpha\beta g(\Xi(S),\Xi(T)). \tag{4}$$

Nonnull vectors S and T belong to $\mathcal{U}$, which implies $\varepsilon_{\Xi(S)} = 0$ and $\varepsilon_{\Xi(T)} = 0$. A simple calculation $\varepsilon_{2S+T} = 4\varepsilon_S + \varepsilon_T = 3\varepsilon_S > 0$ gives another nonnull $2S + T \in \mathcal{U}$, and therefore $\varepsilon_{\Xi(2S+T)} = 0$. The equation (4), for $\alpha = 2$ and $\beta = 1$, gives $g(\Xi(S),\Xi(T)) = 0$. Another look at the same equation, currently for $\alpha = 1$ and $\beta = 1$, brings $\varepsilon_{\Xi(S+T)} = 2g(\Xi(S),\Xi(T)) = 0$, and finally $\varepsilon_{\Xi(N)} = 0$. □

3. Two-leaves Osserman algebraic curvature tensors

In this section we introduce a concept of the two-leaves Osserman curvature tensor. Here, we establish the notation and the approach, which we use later in the text to solve some problems related to the duality principle. It is worth noting that through this section we do not differ eigenvalues of the reduced Jacobi operator, and therefore every statement has its adequate dual.

Definition 3 (Two-leaves Osserman). *Let R be an Osserman curvature tensor on a vector space $\mathcal{V}$ of the signature $(\nu, n-\nu)$, such that reduced Jacobi operator $\tilde{\mathcal{J}}_X$ is diagonalizable with exactly two distinct eigenvalues $\varepsilon_X\lambda$ and $\varepsilon_X\mu$ for every nonnull $X \in \mathcal{V}$. Then we say that R is the two-leaves Osserman.*

Diagonalizability and two eigenvalues ($\varepsilon_X\lambda$ and $\varepsilon_X\mu$) enable orthogonal decomposition of the vector space $\mathcal{V}$ for every nonnull X.

$$\mathcal{V} = \mathrm{Span}\{X\} \oplus \mathrm{Ker}(\tilde{\mathcal{J}}_X - \varepsilon_X\lambda \mathrm{Id}) \oplus \mathrm{Ker}(\tilde{\mathcal{J}}_X - \varepsilon_X\mu \mathrm{Id})$$

Let us introduce the following short notation for some important subspaces and their dimensions, which we use in the forthcoming text.

$$\begin{aligned} \mathcal{L}(X) &= \mathrm{Ker}(\tilde{\mathcal{J}}_X - \varepsilon_X\lambda \mathrm{Id}), & \dim \mathcal{L}(X) &= \tau \\ \mathcal{M}(X) &= \mathrm{Ker}(\tilde{\mathcal{J}}_X - \varepsilon_X\mu \mathrm{Id}), & \dim \mathcal{M}(X) &= \sigma \\ \mathcal{U}(X) &= \mathrm{Span}\{X\} \oplus \mathcal{L}(X), & \dim \mathcal{U}(X) &= \tau + 1 = n - \sigma \end{aligned}$$

Since each eigenspace of a self-adjoint diagonalizable linear operator is nondegenerate, all mentioned subspaces are nondegenerate. The previous decomposition can be written as

$$\mathcal{V} = \mathrm{Span}\{X\} \oplus \mathcal{L}(X) \oplus \mathcal{M}(X) = \mathcal{U}(X) \oplus \mathcal{M}(X),$$

and arbitrary $Y \in \mathcal{V}$ can be decomposed as

$$Y = \xi X + Y_L + Y_M,$$

where $Y_L \in \mathcal{L}(X)$ and $Y_M \in \mathcal{M}(X)$. The vectors X, Y_L, and Y_M are all mutually orthogonal, and we use the opportunity to denote relevant projections with

$$\Pi_{\mathcal{L}}(X, Y) = Y_L, \ \Pi_{\mathcal{M}}(X, Y) = Y_M, \ \Pi_{\mathcal{U}}(X, Y) = \xi X + Y_L.$$

The following lemma gives some kind of the linear extension along the line for $\mathcal{M}$ spaces using three points from that line. It is direct consequence of Lemma 1.

Lemma 7 (Three points lemma). *Let R be a two-leaves Osserman curvature tensor. If for nonnull vectors X, Y, and $\gamma X + \delta Y$ $(\gamma, \delta \neq 0)$ holds $Z \in \mathcal{M}(X) \cap \mathcal{M}(Y) \cap \mathcal{M}(\gamma X + \delta Y)$, then for every nonnull $\alpha X + \beta Y$ holds $Z \in \mathcal{M}(\alpha X + \beta Y)$. If for nonnull vectors X, Y, and $\gamma X + \delta Y$ $(\gamma, \delta \neq 0)$ holds $Z \in \mathcal{L}(X) \cap \mathcal{L}(Y) \cap \mathcal{L}(\gamma X + \delta Y)$, then for every nonnull $\alpha X + \beta Y$ holds $Z \in \mathcal{L}(\alpha X + \beta Y)$.*

Proof. The lemma obviously holds for $\alpha\beta = 0$, while otherwise $(\alpha, \beta \neq 0)$ we apply Lemma 1.

$$\mathcal{J}_{\alpha X+\beta Y}(Z) = \frac{\alpha\beta}{\gamma\delta}\mathcal{J}_{\gamma X+\delta Y}(Z) + \frac{\alpha(\alpha\delta - \beta\gamma)}{\delta}\mathcal{J}_X(Z) + \frac{\beta(\beta\gamma - \alpha\delta)}{\gamma}\mathcal{J}_Y(Z)$$

$$\begin{aligned} \mathcal{J}_{\alpha X+\beta Y}(Z) &= \frac{\alpha\beta}{\gamma\delta}\varepsilon_{\gamma X+\delta Y}\mu Z + \frac{\alpha(\alpha\delta - \beta\gamma)}{\delta}\varepsilon_X\mu Z + \frac{\beta(\beta\gamma - \alpha\delta)}{\gamma}\varepsilon_Y\mu Z \\ &= \left(\frac{\alpha\beta}{\gamma\delta}\left(\gamma^2\varepsilon_X + \delta^2\varepsilon_Y + 2\gamma\delta g(X, Y)\right) + \frac{\alpha(\alpha\delta - \beta\gamma)}{\delta}\varepsilon_X + \frac{\beta(\beta\gamma - \alpha\delta)}{\gamma}\varepsilon_Y\right)\mu Z \end{aligned}$$

$$\mathcal{J}_{\alpha X+\beta Y}(Z) = \left(\alpha^2\varepsilon_X + \beta^2\varepsilon_Y + 2\alpha\beta g(X,Y)\right)\mu Z$$

$$\mathcal{J}_{\alpha X+\beta Y}(Z) = \varepsilon_{\alpha X+\beta Y}\mu Z$$

The previous equation means $Z \in \mathcal{M}(\alpha X + \beta Y)$, which proves the first part of the lemma. The second part of the lemma is dual, because of the symmetry between μ and λ. □

Like the previous lemma, we can look for the linear extension along the line for $\mathcal{M}$ spaces using only two points from that line. The next lemma gives possible situations in that case.

Lemma 8 (Two points lemma). *Let R be a two-leaves Osserman curvature tensor, such that for nonnull A and B holds $Z \in \mathcal{M}(A) \cap \mathcal{M}(B)$. Then for every nonnull $\alpha A + \beta B$ holds $Z \in \mathcal{M}(\alpha A + \beta B)$, or there exists $N \neq 0$ such that for every nonnull $\alpha A + \beta B$ holds $N \in \mathcal{L}(\alpha A + \beta B)$.*

Proof. Let A, B, and $\alpha A+\beta B$ ($\alpha, \beta \neq 0$) be nonnull, such that $Z \in \mathcal{M}(A)$ and $Z \in \mathcal{M}(B)$. $Z \perp A$ and $Z \perp B$ imply $Z \perp \alpha A + \beta B$, and thus Z can be decomposed as $Z = L_{\alpha\beta} + M_{\alpha\beta}$, where $L_{\alpha\beta} = \Pi_{\mathcal{L}}(\alpha A + \beta B, Z) \in \mathcal{L}(\alpha A + \beta B)$ and $M_{\alpha\beta} = \Pi_{\mathcal{M}}(\alpha A + \beta B, Z) \in \mathcal{M}(\alpha A + \beta B)$. Let us start with calculations.

$$\mathcal{J}_{\alpha A+\beta B}(Z) = \mathcal{J}_{\alpha A+\beta B}(L_{\alpha\beta}) + \mathcal{J}_{\alpha A+\beta B}(M_{\alpha\beta})$$

$$\mathcal{R}(Z, \alpha A + \beta B)(\alpha A + \beta B) = \varepsilon_{\alpha A+\beta B}\lambda L_{\alpha\beta} + \varepsilon_{\alpha A+\beta B}\mu M_{\alpha\beta}$$

$$\alpha^2\mathcal{J}_A(Z) + \beta^2\mathcal{J}_B(Z) + 2\alpha\beta\mathcal{J}(A,B)(Z) \\ = \varepsilon_{\alpha A+\beta B}(\lambda-\mu)L_{\alpha\beta} + \varepsilon_{\alpha A+\beta B}\mu(L_{\alpha\beta} + M_{\alpha\beta})$$

$$\alpha^2\varepsilon_A\mu Z + \beta^2\varepsilon_B\mu Z + 2\alpha\beta\mathcal{J}(A,B)(Z) \\ = \varepsilon_{\alpha A+\beta B}(\lambda-\mu)L_{\alpha\beta} + \left(\alpha^2\varepsilon_A + \beta^2\varepsilon_B + 2\alpha\beta g(A,B)\right)\mu Z$$

$$2\alpha\beta\mathcal{J}(A,B)(Z) = \varepsilon_{\alpha A+\beta B}(\lambda-\mu)L_{\alpha\beta} + 2\alpha\beta g(A,B)\mu Z$$

$$\frac{1}{\alpha\beta}\varepsilon_{\alpha A+\beta B}(\lambda-\mu)L_{\alpha\beta} = 2\mathcal{J}(A,B)(Z) - 2g(A,B)\mu Z$$

The right hand side of the last equation, $N = 2\mathcal{J}(A,B)(Z) - 2g(A,B)\mu Z$, does not depend of the choice of α, β. Thus

$$\frac{1}{\alpha\beta}\varepsilon_{\alpha A+\beta B}(\lambda-\mu)L_{\alpha\beta} = N. \tag{5}$$

Let us suppose that there exist $\alpha, \beta \neq 0$, such that $L_{\alpha\beta} = 0$. This implies $Z = M_{\alpha\beta}$ and $Z \in \mathcal{M}(\alpha A + \beta B)$. We reach all conditions from Lemma 7, and therefore $Z \in \mathcal{M}(\alpha A + \beta B)$ for all nonnull $\alpha A + \beta B$. Otherwise, $L_{\alpha\beta} \neq 0$ for all $\alpha, \beta \neq 0$. Due to the equation (5) and the fact that $\frac{1}{\alpha\beta}\varepsilon_{\alpha A+\beta B}(\lambda-\mu) \neq 0$, vectors $L_{\alpha\beta}$ and N are collinear with $0 \neq N \in \mathcal{L}(\alpha A + \beta B)$. Finally, Lemma 7 enables appending $\alpha = 0$ and $\beta = 0$, which completes the proof. □

4. Quasi-special Osserman algebraic curvature tensors

Many known examples of the two-leaves Osserman curvature tensor have something in common, which motivate us to introduce some additional conditions.

Definition 4 (Quasi-special Osserman). *We say that R is a quasi-special Osserman if it is the two-leaves Osserman and for all nonnull $X, Y \in \mathcal{V}$ holds*

$$Y \in \mathcal{U}(X) \Rightarrow \mathcal{U}(X) = \mathcal{U}(Y). \tag{6}$$

Definition 5 (Special Osserman). *We say that R is special Osserman if it is quasi-special Osserman and for all nonnull* $X, Z \in \mathcal{V}$ *holds*

$$Z \in \mathcal{M}(X) \Rightarrow X \in \mathcal{M}(Z). \tag{7}$$

García-Río and Vázquez-Lorenzo proposed the concept of special Osserman manifolds [7, 9]. The original definition in our terminology can be recognized as Definition 5, which means two-leaves Osserman with additional conditions (6) and (7). They managed to give the complete classification of special Osserman manifolds [7, 9]. Let us recall this results.

Theorem 1 (García-Río, Vázquez-Lorenzo). *The complete and simply connected special Osserman manifold is isometric to one of the following:*
1) an indefinite complex space form of signature $(2k, 2r)$, $k, r \geq 0$,
2) an indefinite quaternionic space form of signature $(4k, 4r)$, $k, r \geq 0$,
3) a paracomplex space form of signature (k, k)
4) a paraquaternionic space form of signature $(2k, 2k)$, *or*
5) a Cayley plane over the octaves with definite or indefinite metric tensor, or a Cayley plane over the anti-octaves with indefinite metric tensor of signature $(8, 8)$.

At the first sight it seems that we have too many conditions in the definition of a special Osserman curvature tensor. However, this work is the first serious attempt to exclude the condition (7). If we look closely, we can link that condition with the duality principle. In fact, the additional condition which made quasi-special Osserman to be special Osserman, is the duality principle for the value μ. Let us note that eigenspaces $\mathcal{L}(X)$ and $\mathcal{M}(X)$ play different roles now, and the duality principle for the value λ is already included in the condition (6). The aim of our work is to examine if quasi-special Osserman curvature tensors are necessarily special Osserman. Can we prove that strong quasi-special Osserman condition implies the duality principle? Unfortunately, this article does not give the final answer to this question, but it contains good grounds for the future investigations.

Let us remark that special Osserman curvature tensors are not the only examples of two-leaves Osserman. For example, let $\mathcal{V}$ be a vector space furnished with a metric g and a quaternionic structure, where $\{J_1, J_2, J_3 = J_1 J_2\}$ is a canonical basis of that quaternionic structure. Let us define a curvature operator by $\mathcal{R} = \mathcal{R}^{J_1} + \mathcal{R}^{J_2}$, where

$$\mathcal{R}^J(X, Y)Z = g(JX, Z)JY - g(JY, Z)JX + 2g(JX, Y)JZ$$

presents one of the first examples of an Osserman curvature operator. The associated curvature tensor of such defined $\mathcal{R}$ is diagonalizable Osserman, whose reduced Jacobi operator has exactly two distinct eigenvalues: $\lambda = -3$ (with multiplicity $\tau = 2$) and $\mu = 0$. Therefore it is a two-leaves Osserman curvature tensor. However, since for every unit X holds

$$\mathcal{U}(X) = \operatorname{Span}\{X, J_1X, J_2X\} \neq \operatorname{Span}\{X, J_1X, J_3X\} = \mathcal{U}(J_1X),$$

it is not quasi-special Osserman.

The following lemma has a profound effect on our theory of quasi-special Osserman curvature tensors.

Lemma 9 (Weak duality lemma). *Let R be a quasi-special Osserman curvature tensor and let* $A, B \in \mathcal{V}$ *be nonnull. Then*

$$\frac{\varepsilon_{\Pi_{\mathcal{L}}(B,A)}}{\varepsilon_A} = \frac{\varepsilon_{\Pi_{\mathcal{L}}(A,B)}}{\varepsilon_B}, \quad \frac{\varepsilon_{\Pi_{\mathcal{M}}(B,A)}}{\varepsilon_A} = \frac{\varepsilon_{\Pi_{\mathcal{M}}(A,B)}}{\varepsilon_B}, \quad \frac{\varepsilon_{\Pi_{\mathcal{U}}(B,A)}}{\varepsilon_A} = \frac{\varepsilon_{\Pi_{\mathcal{U}}(A,B)}}{\varepsilon_B}.$$

Proof. Let $A = \xi_1 B + A_L + A_M$ and $B = \xi_2 A + B_L + B_M$ be decompositions with $A_L = \Pi_{\mathcal{L}}(B, A) \in \mathcal{L}(B)$, $A_M = \Pi_{\mathcal{M}}(B, A) \in \mathcal{M}(B)$, $B_L = \Pi_{\mathcal{L}}(A, B) \in \mathcal{L}(A)$, and $B_M = \Pi_{\mathcal{M}}(A, B) \in \mathcal{M}(A)$. Our statement is the consequence of the symmetric property of a Jacobi operator.

$$g(\mathcal{J}_A(B), B) = R(B, A, A, B) = R(A, B, B, A) = g(\mathcal{J}_B(A), A)$$

$$g(\mathcal{J}_A(\xi_2 A) + \mathcal{J}_A(B_L) + \mathcal{J}_A(B_M), B) = g(\mathcal{J}_B(\xi_1 B) + \mathcal{J}_B(A_L) + \mathcal{J}_B(A_M), A)$$
$$g(\varepsilon_A \lambda B_L + \varepsilon_A \mu B_M, \xi_2 A + B_L + B_M) = g(\varepsilon_B \lambda A_L + \varepsilon_B \mu A_M, \xi_1 B + A_L + A_M)$$
$$g(\varepsilon_A \lambda B_L, B_L) + g(\varepsilon_A \mu B_M, B_M) = g(\varepsilon_B \lambda A_L, A_L) + g(\varepsilon_B \mu A_M, A_M)$$
$$\varepsilon_A \varepsilon_{B_L} \lambda + \varepsilon_A \varepsilon_{B_M} \mu = \varepsilon_B \varepsilon_{A_L} \lambda + \varepsilon_B \varepsilon_{A_M} \mu$$
$$(\varepsilon_A \varepsilon_{B_L} - \varepsilon_B \varepsilon_{A_L})\lambda = (\varepsilon_B \varepsilon_{A_M} - \varepsilon_A \varepsilon_{B_M})\mu \tag{8}$$

The initial decompositions for A and B provide the following scalar products: $\varepsilon_A = \xi_1^2 \varepsilon_B + \varepsilon_{A_L} + \varepsilon_{A_M}$, $\varepsilon_B = \xi_2^2 \varepsilon_A + \varepsilon_{B_L} + \varepsilon_{B_M}$, $g(A, B) = \xi_1 \varepsilon_B$, and $g(B, A) = \xi_2 \varepsilon_A$. Hence $\varepsilon_A \varepsilon_B = \xi_1^2 \varepsilon_B^2 + \varepsilon_B \varepsilon_{A_L} + \varepsilon_B \varepsilon_{A_M}$ and $\varepsilon_B \varepsilon_A = \xi_2^2 \varepsilon_A^2 + \varepsilon_A \varepsilon_{B_L} + \varepsilon_A \varepsilon_{B_M}$. Because of $\xi_1^2 \varepsilon_B^2 = (g(A, B))^2 = \xi_2^2 \varepsilon_A^2$, the previous equations give

$$\varepsilon_B \varepsilon_{A_L} + \varepsilon_B \varepsilon_{A_M} = \varepsilon_A \varepsilon_B - (g(A, B))^2 = \varepsilon_A \varepsilon_{B_L} + \varepsilon_A \varepsilon_{B_M},$$

and therefore

$$\varepsilon_A \varepsilon_{B_L} - \varepsilon_B \varepsilon_{A_L} = \varepsilon_B \varepsilon_{A_M} - \varepsilon_A \varepsilon_{B_M}. \tag{9}$$

Finally, using the fact $\lambda \neq \mu$, equations (8) and (9) imply

$$\varepsilon_A \varepsilon_{B_L} - \varepsilon_B \varepsilon_{A_L} = 0 = \varepsilon_B \varepsilon_{A_M} - \varepsilon_A \varepsilon_{B_M}.$$

Thus $\frac{\varepsilon_{A_L}}{\varepsilon_A} = \frac{\varepsilon_{B_L}}{\varepsilon_B}$ and $\frac{\varepsilon_{A_M}}{\varepsilon_A} = \frac{\varepsilon_{B_M}}{\varepsilon_B}$, which are equations $\frac{\varepsilon_{\Pi_{\mathcal{L}}(B,A)}}{\varepsilon_A} = \frac{\varepsilon_{\Pi_{\mathcal{L}}(A,B)}}{\varepsilon_B}$ and $\frac{\varepsilon_{\Pi_{\mathcal{M}}(B,A)}}{\varepsilon_A} = \frac{\varepsilon_{\Pi_{\mathcal{M}}(A,B)}}{\varepsilon_B}$. At the end

$$\frac{\varepsilon_{\Pi_{\mathcal{U}}(B,A)}}{\varepsilon_A} = \frac{\varepsilon_A - \varepsilon_{\Pi_{\mathcal{M}}(B,A)}}{\varepsilon_A} = \frac{\varepsilon_B - \varepsilon_{\Pi_{\mathcal{M}}(A,B)}}{\varepsilon_B} = \frac{\varepsilon_{\Pi_{\mathcal{U}}(A,B)}}{\varepsilon_B},$$

which completes the proof. □

Lemma 9 has important consequences, especially in special cases. In the case of $B \in \mathcal{L}(A)$, holds $\Pi_{\mathcal{M}}(A, B) = 0$, and therefore $\varepsilon_{\Pi_{\mathcal{M}}(B,A)} = \varepsilon_{A_M} = 0$. In the case of $B \in \mathcal{M}(A)$, holds $\Pi_{\mathcal{L}}(A, B) = 0$, and therefore $\varepsilon_{\Pi_{\mathcal{L}}(B,A)} = \varepsilon_{A_L} = 0$. The duality principle for $B \in \mathcal{L}(A)$ gives $A_M = 0$, instead of $\varepsilon_{A_M} = 0$, which justifies the name "Weak duality lemma".

One of the key questions is a possible existence of a nontrivial intersection of $\mathcal{U}(A)$ and $\mathcal{U}(B)$. The next lemma gives some useful limits in that direction.

Lemma 10. *Let R be a quasi-special Osserman curvature tensor, such that $\mathcal{U}(A) \cap \mathcal{U}(B) \neq 0$ holds for some nonnull $A, B \in \mathcal{V}$. Then $\varepsilon_{\Pi_{\mathcal{M}}(B,A)} = 0$.*

Proof. The lemma condition enables existence of N with $0 \neq N \in \mathcal{U}(A) \cap \mathcal{U}(B)$. N is null, otherwise $A \in \mathcal{U}(A) = \mathcal{U}(N) = \mathcal{U}(B)$, and $\Pi_{\mathcal{M}}(B, A) = 0$. Lemma 4 decomposes $N \in \mathcal{U}(B)$ with $N = S + T$, where $S, T \in \mathcal{U}(B)$ and $\varepsilon_S = -\varepsilon_T > 0$. Let us apply Lemma 9 twice, firstly on the pair S, A and then on the pair T, A.

$$\frac{\varepsilon_{\Pi_{\mathcal{M}}(A,S)}}{\varepsilon_S} = \frac{\varepsilon_{\Pi_{\mathcal{M}}(S,A)}}{\varepsilon_A}, \quad \frac{\varepsilon_{\Pi_{\mathcal{M}}(A,T)}}{\varepsilon_T} = \frac{\varepsilon_{\Pi_{\mathcal{M}}(T,A)}}{\varepsilon_A}.$$

$S, T \in \mathcal{U}(B)$ gives $\mathcal{U}(S) = \mathcal{U}(B) = \mathcal{U}(T)$, and therefore $\mathcal{M}(S) = \mathcal{M}(B) = \mathcal{M}(T) = \mathcal{U}(B)^\perp$. Hence $\Pi_{\mathcal{M}}(S, A) = \Pi_{\mathcal{M}}(B, A) = \Pi_{\mathcal{M}}(T, A)$, which implies

$$\frac{\varepsilon_{\Pi_{\mathcal{M}}(A,S)}}{\varepsilon_S} = \frac{\varepsilon_{\Pi_{\mathcal{M}}(B,A)}}{\varepsilon_A} = \frac{\varepsilon_{\Pi_{\mathcal{M}}(A,T)}}{\varepsilon_T}.$$

$S = \Pi_{\mathcal{U}}(A, S) + \Pi_{\mathcal{M}}(A, S)$, so $T = N - S = (N - \Pi_{\mathcal{U}}(A, S)) + (-\Pi_{\mathcal{M}}(A, S))$, where $N - \Pi_{\mathcal{U}}(A, S) \in \mathcal{U}(A)$ and $-\Pi_{\mathcal{M}}(A, S) \in \mathcal{M}(A)$. This is why $\Pi_{\mathcal{M}}(A, T) = -\Pi_{\mathcal{M}}(A, S)$, and since the sign does not effect the norm, $\varepsilon_{\Pi_{\mathcal{M}}(A,T)} = \varepsilon_{\Pi_{\mathcal{M}}(A,S)}$. Thus arise

$$\frac{\varepsilon_{\Pi_{\mathcal{M}}(A,S)}}{\varepsilon_S} = \frac{\varepsilon_{\Pi_{\mathcal{M}}(A,S)}}{\varepsilon_T},$$

therefore $\varepsilon_S = -\varepsilon_T$ implies $\varepsilon_{\Pi_{\mathcal{M}}(A,S)} = 0$, and finally $\varepsilon_{\Pi_{\mathcal{M}}(B,A)} = 0$ holds. □

We proved important lemmas and everything is prepared for the final result. Let R be a quasi-special Osserman curvature tensor and let A and B be arbitrary nonnull such that $B \in \mathcal{M}(A)$. The basic idea for completing the proof is to show $\mathcal{U}(B) \leq \mathcal{M}(A)$, because it implies $A \perp \mathcal{M}(A) \geq \mathcal{U}(B)$, with the final $A \in \mathcal{U}(B)^\perp = \mathcal{M}(B)$ and R has to be a special Osserman.

Let us start with the important subspace $\mathcal{P}$ defined by

$$\mathcal{P} = \mathcal{M}(A) \cap \mathcal{M}(B).$$

Its dimension can be evaluated by $\mathcal{V} \geq \mathcal{M}(A) + \mathcal{M}(B)$.

$$\dim \mathcal{V} \geq \dim(\mathcal{M}(A) + \mathcal{M}(B)) = \dim \mathcal{M}(A) + \dim \mathcal{M}(B) - \dim \mathcal{P}.$$

Thus $\dim \mathcal{P} \geq n - 2(\tau + 1)$, and therefore $\dim \mathcal{P}^\perp = n - \dim \mathcal{P} \leq 2(\tau + 1)$. $\mathcal{U}(A) \perp \mathcal{M}(A) \supset \mathcal{P}$ and $\mathcal{U}(B) \perp \mathcal{M}(B) \supset \mathcal{P}$ give $\mathcal{U}(A) + \mathcal{U}(B) \perp \mathcal{P}$, and thus $\mathcal{U}(A) + \mathcal{U}(B) \leq \mathcal{P}^\perp$.

On the other hand, the initial condition $B \in \mathcal{M}(A)$ means $\Pi_{\mathcal{M}}(A, B) = B$. Lemma 9 gives

$$\varepsilon_{\Pi_{\mathcal{M}}(B,A)} = \frac{\varepsilon_A}{\varepsilon_B} \cdot \varepsilon_{\Pi_{\mathcal{M}}(A,B)} = \frac{\varepsilon_A}{\varepsilon_B} \cdot \varepsilon_B = \varepsilon_A \neq 0,$$

and therefore by Lemma 10

$$\mathcal{U}(A) \cap \mathcal{U}(B) = 0.$$

This is why $\mathcal{U}(A) + \mathcal{U}(B)$ is a direct sum and accordingly $\dim(\mathcal{U}(A) + \mathcal{U}(B)) = 2(\tau + 1)$. Comparing this with the previous results ($\mathcal{U}(A) + \mathcal{U}(B) \leq \mathcal{P}^\perp$ with $\dim \mathcal{P}^\perp \leq 2(\tau + 1)$), we conclude $\dim \mathcal{P}^\perp = 2(\tau + 1)$ and finally

$$\mathcal{P}^\perp = \mathcal{U}(A) \oplus \mathcal{U}(B).$$

Every $X \in \mathcal{U}(B)$ can be decomposed as $X = \Pi_{\mathcal{U}}(A, X) + \Pi_{\mathcal{M}}(A, X)$. In the case of nonnull X we have $\mathcal{U}(X) = \mathcal{U}(B)$, and applying Lemma 9 twice (on the pair A, X and on the pair A, B) gives

$$\frac{\varepsilon_{\Pi_{\mathcal{U}}(A,X)}}{\varepsilon_X} = \frac{\varepsilon_{\Pi_{\mathcal{U}}(X,A)}}{\varepsilon_A} = \frac{\varepsilon_{\Pi_{\mathcal{U}}(B,A)}}{\varepsilon_A} = \frac{\varepsilon_{\Pi_{\mathcal{U}}(A,B)}}{\varepsilon_B} = 0,$$

since $B \in \mathcal{M}(A)$ implies $\Pi_{\mathcal{U}}(A, B) = 0$. Thus $\varepsilon_{\Pi_{\mathcal{U}}(A,X)} = 0$ holds for every nonnull $X \in \mathcal{U}(B)$. The projection $\Xi : \mathcal{U}(B) \mapsto \mathcal{U}(A)$ given by $\Xi(Y) = \Pi_{\mathcal{U}}(A, Y)$ is linear, and therefore Lemma 6 extends $\varepsilon_{\Pi_{\mathcal{U}}(A,X)} = 0$ for every $X \in \mathcal{U}(B)$, which can be written as

$$X = \Pi_{\mathcal{U}}(A, X) + \Pi_{\mathcal{M}}(A, X), \ \varepsilon_{\Pi_{\mathcal{U}}(A,X)} = 0. \tag{10}$$

Let us define new important subspaces $\mathcal{S}$ and $\mathcal{F}$ with

$$\mathcal{S} := \{\Pi_{\mathcal{U}}(A, X) : X \in \mathcal{U}(B)\}, \ \mathcal{F} := \mathcal{M}(A) \cap \mathcal{U}(B).$$

A space $\mathcal{S}$ is a subspace of $\mathcal{V}$ as a projection. Moreover, by (10), it is a totally isotropic subspace of $\mathcal{U}(A)$, and according to Lemma 3 $\dim \mathcal{S} \leq \frac{\tau+1}{2}$ holds. It is not difficult to notice that $\dim \mathcal{S} + \dim \mathcal{F} = \dim \mathcal{U}(B) = \tau + 1$, which implies $\dim \mathcal{F} \geq \frac{\tau+1}{2}$. Of course, we need $\dim \mathcal{F} = \tau + 1$ to show $\mathcal{U}(B) \leq \mathcal{M}(A)$ and solve our problem. This is why we assume existence of some $D \in \mathcal{U}(B)$, such that $\Pi_{\mathcal{U}}(A, D) \neq 0$. It is easy to find nonnull D with the same properties, because if D is null we choose $D' = D + \theta B \in \mathcal{U}(B)$ for some θ with $\theta \neq 0$ and $\theta \neq \frac{-2g(D,B)}{\varepsilon_B}$. Thus

$$\varepsilon_{D'} = \varepsilon_D + 2\theta g(D, B) + \theta^2 \varepsilon_B = \theta(2g(D, B) + \theta\varepsilon_B) \neq 0,$$

and $D' \in \mathcal{U}(B)$ is nonnull with $\Pi_{\mathcal{U}}(A, D') = \Pi_{\mathcal{U}}(A, D + \theta B) = \Pi_{\mathcal{U}}(A, D) \neq 0$.

After the previous argument we have nonnull $D = E + C \in \mathcal{U}(B)$, with $E = \Pi_{\mathcal{U}}(A, D) \neq 0$ and $C = \Pi_{\mathcal{M}}(A, D)$. Since $\mathcal{S}$ is totally isotropic, $E \in \mathcal{S}$ is null, and therefore $\varepsilon_C = \varepsilon_D$ and C is nonnull. Both E and C are orthogonal to E, and therefore $E \perp E + C = D$. Lemma 5 applied on null $E \in \mathcal{U}(A)$ regards nonnull $D \perp E$ gives a decomposition $E = S + T$, with $S, T \in \mathcal{U}(A)$ and $\varepsilon_S = -\varepsilon_T > 0$, where either $D - S$ or $D - T$ is nonnull. Without loss of generality we can assume that $D - S$ is nonnull.

Because of $\mathcal{P} \subset \mathcal{M}(B) = \mathcal{M}(D)$, $\mathcal{P} \subset \mathcal{M}(A) = \mathcal{M}(S)$, and $\mathcal{U}(D) \cap \mathcal{U}(S) = \mathcal{U}(B) \cap \mathcal{U}(A) = 0$, Lemma 8 gives $\mathcal{P} \subset \mathcal{M}(D - S)$. The fact $\Pi_{\mathcal{M}}(T, D - S) = \Pi_{\mathcal{M}}(A, D - S) = \Pi_{\mathcal{M}}(A, D) = C$ gives $\varepsilon_{\Pi_{\mathcal{M}}(T, D-S)} = \varepsilon_C \neq 0$, and therefore Lemma 10 implies $\mathcal{U}(D - S) \cap \mathcal{U}(T) = 0$. We have $\mathcal{P} \subset \mathcal{M}(D - S)$ and $\mathcal{P} \subset \mathcal{M}(T)$, so another application of Lemma 8 gives $\mathcal{P} \subset \mathcal{M}(D - S - T) = \mathcal{M}(C)$.

The facts $\mathcal{P} \subset \mathcal{M}(C)$ and $\mathcal{U}(A) \cap \mathcal{U}(C) = 0$ ($C \in \mathcal{M}(A)$ and Lemma 10), after a similar technique, establish $\mathcal{P}^\perp = \mathcal{U}(A) \oplus \mathcal{U}(C)$.

Subspaces $\mathcal{S}$ and $\mathcal{M}(A)$ are both orthogonal to $\mathcal{S}$, so $\mathcal{S} \perp \mathcal{S} + \mathcal{M}(A) \geq \mathcal{U}(B)$, and therefore $\mathcal{S} \perp \mathcal{U}(B)$. It gives $\mathcal{S} \leq \mathcal{U}(B)^\perp = \mathcal{M}(B)$, and finally $\mathcal{S} \leq \mathcal{M}(B) \cap \mathcal{P}^\perp$. Since $E \in \mathcal{S}$, the decomposition $C = D - E$ has $D \in \mathcal{U}(B)$ and $-E \in \mathcal{M}(B)$. Consequently $\Pi_{\mathcal{M}}(B, C) = \Pi_{\mathcal{M}}(B, D - E) = -E \in \mathcal{S}$, and therefore

$$\varepsilon_{\Pi_{\mathcal{M}}(B,C)} = 0. \tag{11}$$

The equation (11), using Weak duality lemma, guarantees null $\Pi_{\mathcal{M}}(X, Y)$ and null $\Pi_{\mathcal{M}}(Y, X)$ for all nonnull $X \in \mathcal{U}(B)$ and $Y \in \mathcal{U}(C)$.

$$\frac{\varepsilon_{\Pi_{\mathcal{M}}(X,Y)}}{\varepsilon_Y} = \frac{\varepsilon_{\Pi_{\mathcal{M}}(Y,X)}}{\varepsilon_X} = \frac{\varepsilon_{\Pi_{\mathcal{M}}(C,X)}}{\varepsilon_X} = \frac{\varepsilon_{\Pi_{\mathcal{M}}(X,C)}}{\varepsilon_C} = \frac{\varepsilon_{\Pi_{\mathcal{M}}(B,C)}}{\varepsilon_C} = 0.$$

Let us define subspaces $\mathcal{H}$ and $\mathcal{Z}$ by

$$\mathcal{H} := \{\Pi_{\mathcal{M}}(B, X) : X \in \mathcal{U}(C)\}, \ \mathcal{Z} := \mathcal{U}(B) \cap \mathcal{U}(C).$$

The previous consideration gives null $\Pi_{\mathcal{M}}(B, X)$ for all nonnull $X \in \mathcal{U}(C)$. Lemma 6 can easily extend $\varepsilon_{\Pi_{\mathcal{M}}(B,X)} = 0$ for all $X \in \mathcal{U}(C)$, and therefore $\mathcal{H}$ is totally isotropic. The condition (6) with $\mathcal{U}(B) \neq \mathcal{U}(C)$ implies that $\mathcal{Z}$ is totally isotropic, too.

We need to prove that $\mathcal{P}^\perp$ is nondegenerate. Let $(B_0, B_1, ..., B_\tau)$ be a pseudo-orthonormal basis of the nondegenerate space $\mathcal{U}(B)$. Let us decompose

$$B_i = A_i + D_i, \ A_i = \Pi_{\mathcal{U}}(A, B_i) \in \mathcal{S}, \ D_i = \Pi_{\mathcal{M}}(A, B_i) \in \mathcal{M}(A),$$

for every $i \in \{0, 1, ..., \tau\}$. Hence

$$g(B_i, B_j) = g(A_i + D_i, A_j + D_j) = g(A_i, A_j) + g(D_i, D_j) = g(D_i, D_j),$$

and therefore $(D_0, D_1, ..., D_\tau)$ is a pseudo-orthonormal basis of the space $\mathcal{M}(A) \cap \mathcal{P}^\perp$. This is why $\mathcal{P}^\perp = \mathcal{U}(A) \oplus (\mathcal{M}(A) \cap \mathcal{P}^\perp)$ is nondegenerate as an orthogonal sum of nondegenerate spaces. $\mathcal{M}(B) \cap \mathcal{P}^\perp$ is a nondegenerate space of dimension $\tau + 1$ and $\mathcal{H}$ is its subspace. This is why Lemma 3 gives $\dim \mathcal{Z} \leq \frac{\tau+1}{2}$ and $\dim \mathcal{H} \leq \frac{\tau+1}{2}$. Once again, it is not difficult to see that $\dim \mathcal{Z} + \dim \mathcal{H} = \dim \mathcal{U}(C) = \tau + 1$, which gives the only possible case

$$\dim \mathcal{Z} = \frac{\tau + 1}{2} = \dim \mathcal{H}. \tag{12}$$

Let us stop here and introduce the following definition in order to explain some results.

Definition 6 (Almost-special Osserman). *We say that R is almost-special Osserman if it is two-leaves Osserman and for all nonnull $X, Y \in \mathcal{V}$ holds*

$$\mathcal{U}(X) \cap \mathcal{U}(Y) \neq 0 \Rightarrow \mathcal{U}(X) = \mathcal{U}(Y).$$

It is easy to see that special Osserman curvature tensor is almost-special Osserman, and that almost-special Osserman is quasi-special Osserman. Almost-special Osserman curvature tensors allowed only trivial intersections of $\mathcal{U}$ spaces, and consequently $\dim \mathcal{Z} = 0$. This contradicts the equation (12), and the following theorem holds.

Theorem 2. *Almost-special Osserman curvature tensor is special Osserman.*

Let us give the final remark. The duality principle for an Osserman curvature tensor is a very complex problem. The present author is not aware of any counterexample, although we failed to prove the duality principle, even under very strong additional conditions. It seems that the solution of our quasi-special Osserman problem is not far away, because of the equation (12), which looks peculiar. For example, it immediately solves the problem in the case of even τ. This is why we hope for an affirmative answer of the problem in our future investigations.

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