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Some Fixed Point Results for (α, β)-(ψ, ϕ)-Contractive Mappings

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Abstract. In this paper, we introduce the notions of cyclic (α, β)-admissible mappings, (α, β)-(ψ, ϕ)-contractive and weak $\alpha - \beta - \psi-$ rational contraction mappings via cyclic (α, β)-admissible mappings. We prove some new fixed point results for such mappings in the setting of complete metric spaces. The obtained results generalize, unify and modify some recent theorems in the literature. Some examples and an application to integral equations are given here to illustrate the usability of the obtained results.

1. Introduction

In the fixed point theory of continuous mappings, a well-known theorem of Banach [3] states that if (X, d) is a complete metric space and if f is a self-mapping on X which satisfies the inequality $d(fx, fy) \leq kd(x, y)$ for some $k \in [0, 1)$ and all $x, y \in X$, then f has a unique fixed point z and the sequence of successive approximations $\{f^n x\}$ converges to z for all $x \in X$. On the other hand, the condition $d(fx, fy) < d(x, y)$ does not insure that f has a fixed point. In the last decades, the Banach's theorem [3] has been extensively studied and generalized on many settings, see for example [1] and [4]-[35].

In 2008, Dutta and Choudhury [12] proved the following theorem:

Theorem 1.1. *Let (X, d) be a complete metric space and $f : X \to X$ be a mapping such that*

$$\psi(d(fx, fy)) \leq \psi(d(x, y)) - \phi(d(x, y)),$$

for all $x, y \in X$, where $\psi, \phi : [0, +\infty) \to [0, +\infty)$ are continuous, non-decreasing and $\psi(t) = \phi(t) = 0$ if and only if $t = 0$. Then, f has a unique fixed point $x^* \in X$.

Notice that above Theorem remains true if the hypothesis on ϕ is replaced by "*ϕ is lower semi-continuous and $\phi(t) = 0$ if and only if $t = 0$*" (see e.g. [1, 11]).

Recently, Samet et al. [31] introduced the concept of α-ψ-contractive type mappings and established various fixed point theorems for such mappings in complete metric spaces. Very recently, Salimi et al. [30]

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modified the concept of α-ψ-contractive type mappings and established some new fixed point theorems for certain contractive conditions (Also see [13, 17, 22] and references therein)

Motivated by [30], we introduce the notions of (α,β)-(ψ,ϕ)-contractive and weak $\alpha-\beta-\psi$- rational contraction mappings via cyclic (α,β)-admissible mappings and establish some results of fixed point for this class of mappings in the setting of metric spaces. The obtained results generalize, unify and modify some recent theorems in the literature. Some examples and an application to integral equations are given here to illustrate the usability of the obtained results.

2. Main Results

We denote by Ψ the set of continuous and increasing functions $\psi:[0,+\infty)\to[0,+\infty)$ and denote by Φ the set of lower semicontinuous functions $\phi:[0,+\infty)\to[0,+\infty)$ such that $\phi(t)=0$ iff $t=0$.

Let X be a nonempty set and let $f:X\to X$ be an arbitrary mapping. We say that $x\in X$ is a fixed point for f, if $x=fx$. We denote by $Fix(f)$ the set of all fixed points of f.

Definition 2.1. *Let $f:X\to X$ be a mapping and $\alpha,\beta:X\to\mathbb{R}^+$ be two functions. We say that f is a cyclic (α,β)-admissible mapping if*

(i) $\alpha(x)\geq 1$ for some $x\in X$ implies $\beta(fx)\geq 1$,

(ii) $\beta(x)\geq 1$ for some $x\in X$ implies $\alpha(fx)\geq 1$.

Example 2.2. *Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $fx=-(x+x^3)$. Suppose that $\alpha,\beta:\mathbb{R}\to\mathbb{R}^+$ are given by $\alpha(x)=e^x$ for all $x\in\mathbb{R}$ and $\beta(y)=e^{-y}$ for all $y\in\mathbb{R}$. Then f is a cyclic (α,β)-admissible mapping.*

Indeed, if $\alpha(x)=e^x\geq 1$, then $x\geq 0$ which implies $-fx\geq 0$. Therefore, $\beta(fx)=e^{-fx}\geq 1$. Also, if $\beta(y)=e^{-y}\geq 1$, then $y\leq 0$ which implies $fy\geq 0$. So, $\alpha(fy)=e^{fy}\geq 1$.

Definition 2.3. *Let (X,d) be a metric space and $f:X\to X$ be a cyclic (α,β)-admissible mapping. We say that f is a (α,β)-(ψ,ϕ)-contractive mapping if*

$$\alpha(x)\beta(y)\geq 1\Longrightarrow\psi(d(fx,fy))\leq\psi(d(x,y))-\phi(d(x,y)),\tag{1}$$

for $x,y\in X$, where $\psi\in\Psi$ and $\phi\in\Phi$.

Now we are ready to prove our first theorem.

Theorem 2.4. *Let (X,d) be a complete metric space and $f:X\to X$ be a (α,β)-(ψ,ϕ)-admissible mapping. Suppose that the following conditions hold:*

(a) there exists $x_0\in X$ such that $\alpha(x_0)\geq 1$ and $\beta(x_0)\geq 1$;

(b) f is continuous, or

(c) if $\{x_n\}$ is a sequence in X such that $x_n\to x$ and $\beta(x_n)\geq 1$ for all n, then $\beta(x)\geq 1$;

then f has a fixed point. Moreover, if $\alpha(x)\geq 1$ and $\beta(y)\geq 1$ for all $x,y\in Fix(f)$, then f has a unique fixed point.

Proof. Define a sequence $\{x_n\}$ by $x_n=f^nx_0=fx_{n-1}$ for all $n\in\mathbb{N}$. Since f is a cyclic (α,β)-admissible mapping and $\alpha(x_0)\geq 1$ then $\beta(x_1)=\beta(fx_0)\geq 1$ which implies $\alpha(x_2)=\alpha(fx_1)\geq 1$. By continuing this process, we get $\alpha(x_{2n})\geq 1$ and $\beta(x_{2n-1})\geq 1$ for all $n\in\mathbb{N}$. Again, since f is a cyclic (α,β)-admissible mapping and $\beta(x_0)\geq 1$, by the similar method, we have $\beta(x_{2n})\geq 1$ and $\alpha(x_{2n-1})\geq 1$ for all $n\in\mathbb{N}$. That is, $\alpha(x_n)\geq 1$ and $\beta(x_n)\geq 1$ for all $n\in\mathbb{N}\cup\{0\}$. Equivalently, $\alpha(x_{n-1})\beta(x_n)\geq 1$ for all $n\in\mathbb{N}$. Therefore by (1), we have

$$\psi(d(x_n,x_{n+1}))\leq\psi(d(x_{n-1},x_n))-\phi(d(x_{n-1},x_n))\leq\psi(d(x_{n-1},x_n)),\tag{2}$$

and since ψ is increasing, we get

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n),$$

for all $n \in \mathbb{N}$. So, $\{d_n := d(x_n, x_{n+1})\}$ is a non-increasing sequence of positive real numbers. Then there exist $r \geq 0$ such that $\lim_{n\to\infty} d_n = r$. We shall show that $r = 0$.

By taking the limsup on both sides of (2), we have

$$\psi(r) \leq \psi(r) - \phi(r).$$

Hence $\phi(r) = 0$. That is $r = 0$. Then

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3}$$

Now, we want to show that $\{x_n\}$ is a Cauchy sequence. Suppose, to the contrary, that $\{x_n\}$ is not a Cauchy sequence. Then there are $\varepsilon > 0$ and sequences $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k,

$$n(k) > m(k) > k, \quad d(x_{n(k)}, x_{m(k)}) \geq \varepsilon \qquad \text{and} \quad d(x_{n(k)}, x_{m(k)-1}) < \varepsilon.$$

Now for all $k \in \mathbb{N}$, we have

$$\begin{aligned} \varepsilon \leq d(x_{n(k)}, x_{m(k)}) &\leq d(x_{n(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)}) \\ &< \varepsilon + d(x_{m(k)-1}, x_{m(k)}). \end{aligned}$$

Taking the limit as $k \to +\infty$ in the above inequality and using (3), we get

$$\lim_{k\to+\infty} d(x_{n(k)}, x_{m(k)}) = \varepsilon. \tag{4}$$

Since,

$$d(x_{n(k)}, x_{m(k)}) \leq d(x_{m(k)}, x_{m(k)+1}) + d(x_{m(k)+1}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{n(k)}),$$

and

$$d(x_{n(k)+1}, x_{m(k)+1}) \leq d(x_{m(k)}, x_{m(k)+1}) + d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)+1}, x_{n(k)}),$$

then by taking the limit as $k \to +\infty$ in above inequalities and using (3) and (4), we deduce that

$$\lim_{k\to+\infty} d(x_{n(k)+1}, x_{m(k)+1}) = \varepsilon. \tag{5}$$

Now, by (1), we get

$$\psi(d(x_{n(k)+1}, x_{m(k)+1})) \leq \psi(d(x_{n(k)}, x_{m(k)})) - \phi(d(x_{n(k)}, x_{m(k)})), \tag{6}$$

since, $\alpha(x_{n(k)})\beta(x_{m(k)}) \geq 1$ for all $k \in \mathbb{N}$. By taking the limsup on both sides of (6) and applying (4) and (5), we obtain

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon).$$

That is, $\varepsilon = 0$ which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete metric space, then there is $z \in X$ such that $x_n \to z$ as $n \to \infty$.

First, we assume that f is continuous. Hence, we deduce

$$fz = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} x_{n+1} = z.$$

So z is a fixed point of f.

Now, assume that (c) is held, That is, $\alpha(x_n)\beta(z) \geq 1$. From (1) we have

$$\psi(d(x_{n+1}, fz)) \leq \psi(d(x_n, z)) - \phi(d(x_n, z)). \tag{7}$$

By taking the limsup on both sides of (7), we get $\psi(d(z, fz)) = 0$. Then $d(z, fz) = 0$. i.e., $z = fz$.

To prove the uniqueness of fixed point, suppose that z and z^* are two fixed points of f. Since $\alpha(z)\beta(z^*) \geq 1$, it follows from (1) that

$$\psi(d(z, z^*)) \leq \psi(d(z, z^*)) - \phi(d(z, z^*)).$$

So $\phi(d(z, z^*)) = 0$ and hence $d(z, z^*) = 0$ i.e., $z = z^*$. □

Example 2.5. *Let $X = \mathbb{R}$ endowed with the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$ and $f : X \to X$ be defined by*

$$fx = \begin{cases} -\frac{1}{2}x, & \text{if} \quad x \in [-1, 1] \\ 2x, & \text{if} \quad x \in \mathbb{R}\setminus[-1, 1] \end{cases},$$

and $\alpha, \beta : X \to [0, +\infty)$ be given by

$$\alpha(x) = \begin{cases} 1, & \text{if} \quad x \in [-1, 0] \\ 0, & \text{otherwise} \end{cases} \quad \& \quad \beta(x) = \begin{cases} 1, & \text{if} \quad x \in [0, 1] \\ 0, & \text{otherwise} \end{cases}.$$

Also define $\psi, \phi : [0, \infty) \to [0, \infty)$ by

$$\psi(t) = t \quad \text{and} \quad \phi(t) = \frac{1}{4}t.$$

Now, we prove that f is a cyclic (α, β)-admissible mapping and the hypotheses (a) and (c) of Theorem 2.4 are satisfied by f and hence f has a fixed point. Moreover, the result of Dutta and Choudhury [12] can not be applied to f.

Let $\alpha(x) \geq 1$ for some $x \in X$. Then $x \in [-1, 0]$ and so $fx \in [0, 1]$. Therefore, $\beta(fx) \geq 1$. Similarly, if $\beta(x) \geq 1$ then $\alpha(fx) \geq 1$. Then f is a cyclic (α, β)-admissible mapping.

Now, if $\{x_n\}$ is a sequence in X such that $\beta(x_n) \geq 1$ and $x_n \to x$ as $n \to \infty$. Therefore, $x_n \in [0, 1]$. Hence $x \in [0, 1]$, i.e., $\beta(x) \geq 1$.

Let $\alpha(x)\beta(y) \geq 1$. Then $x \in [-1, 0]$ and $y \in [0, 1]$ and so we have

$$\begin{aligned} \psi(d(fx, fy)) &= |fx - fy| = \frac{1}{2}|x - y| \leq \frac{3}{4}|x - y| \\ &= |x - y| - \frac{1}{4}|x - y| = \psi(d(x, y)) - \phi(d(x, y)). \end{aligned}$$

So the hypotheses of Theorem 2.4 hold and therefore, f has a fixed point. But, if $x = 2$ and $y = 3$ then

$$\psi(d(f2, f3)) = 2 > \frac{3}{4} = \psi(d(2, 3)) - \phi(d(2, 3))$$

That is, Theorem 1.1 can not be applied to f.

Corollary 2.6. *Let (X, d) be a complete metric space and $f : X \to X$ be a cyclic (α, β)-admissible mapping such that*

$$\alpha(x)\beta(y)\psi(d(fx, fy)) \leq \psi(d(x, y)) - \phi(d(x, y)), \tag{8}$$

for all $x, y \in X$ where $\psi \in \Psi$ and $\phi \in \Phi$. Suppose that the following assertions hold:

(a) there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$;

(b) f is continuous, or

(c) if $\{x_n\}$ is a sequence in X such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all n, then $\beta(x) \geq 1$;

then f has a fixed point. Moreover, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ for all $x, y \in Fix(f)$, then f has a unique fixed point.

Proof. Let $\alpha(x)\beta(y) \geq 1$ for $x, y \in X$. Then by (8), we get

$$\psi(d(fx, fy)) \leq \psi(d(x, y)) - \phi(d(x, y)).$$

This implies that the inequality (1) holds. Therefore, the proof follows from Theorem 2.4. □

Corollary 2.7. *Let (X, d) be a complete metric space and $f : X \to X$ be a cyclic (α, β)-admissible mapping such that*

$$\Big(\alpha(x)\beta(y) + 1\Big)^{\psi(d(fx, fy))} \leq 2^{\psi(d(x, y)) - \phi(d(x, y))},$$

for all $x, y \in X$ where $\psi \in \Psi$ and $\phi \in \Phi$. Suppose that the following assertions hold:

(a) there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$;

(b) f is continuous, or

(c) if $\{x_n\}$ is a sequence in X such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all n, then $\beta(x) \geq 1$;

then f has a fixed point. Moreover, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ for all $x, y \in Fix(f)$, then f has a unique fixed point.

Example 2.8. *Let $X = \mathbb{R}$ endowed with the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$ and $f : X \to X$ be defined by*

$$fx = \begin{cases} -\dfrac{1}{6}(x + x^3), & \text{if} \quad x \in [-1, 1] \\ \\ \ln|x|, & \text{if} \quad x \in \mathbb{R}\backslash[-1, 1] \end{cases},$$

and $\alpha, \beta : X \to [0, +\infty)$ be given by

$$\alpha(x) = \begin{cases} 1, & \text{if} \quad x \in [-1, 0] \\ \\ 0, & \text{otherwise} \end{cases} \quad \& \quad \beta(x) = \begin{cases} 1, & \text{if} \quad x \in [0, 1] \\ \\ 0, & \text{otherwise} \end{cases}.$$

Also, define $\psi, \phi : [0, \infty) \to [0, \infty)$ by

$$\psi(t) = t \quad \text{and} \quad \phi(t) = \frac{1}{2}t.$$

Now, we prove that the hypotheses (a) and (c) of Corollary 2.7 are satisfied by f and hence f has a fixed point.

Proceeding as in the Example 2.5, we deduce that f is a cyclic (α, β)-admissible mapping and that the hypotheses (a) and (c) of Corollary 2.7 hold.

Now, for all $x \in [-1, 0]$ and all $y \in [0, 1]$, we get

$$\begin{aligned} \Big(\alpha(x)\beta(y) + 1\Big)^{\psi(d(fx,fy))} &= 2^{|fx - fy|} \\ &= 2^{\frac{1}{6}|x-y||1+x^2+xy+y^2|} \\ &\leq 2^{\frac{1}{6}|x-y||1+x^2+y^2|} \\ &\leq 2^{\frac{1}{2}|x-y|} = 2^{|x-y| - \frac{1}{2}|x-y|} \\ &= 2^{\psi(d(x,y)) - \phi(d(x,y))}. \end{aligned}$$

Otherwise, $\alpha(x)\beta(y) = 0$ *and so we have*

$$\left(\alpha(x)\beta(y) + 1\right)^{\psi(d(fx,fy))} = 1 \leq 2^{\psi(d(x,y))-\phi(d(x,y))}.$$

Therefore, Corollary 2.7 implies that f *has a fixed point.*

Corollary 2.9. *Let* (X, d) *be a complete metric space and* $f : X \to X$ *be a cyclic* (α, β)*-admissible mapping. Assume that there exists* $\ell > 1$ *such that*

$$\left(\psi(d(fx, fy)) + \ell\right)^{\alpha(x)\beta(y)} \leq \psi(d(x, y)) - \phi(d(x, y)) + \ell,$$

for all $x, y \in X$ *where* $\psi \in \Psi$ *and* $\phi \in \Phi$*. Suppose that the following assertions hold:*

(a) there exists $x_0 \in X$ *such that* $\alpha(x_0) \geq 1$ *and* $\beta(x_0) \geq 1$*;*

(b) f *is continuous, or*

(c) if $\{x_n\}$ *is a sequence in* X *such that* $x_n \to x$ *and* $\beta(x_n) \geq 1$ *for all* n*, then* $\beta(x) \geq 1$*;*

then f *has a fixed point. Moreover, if* $\alpha(x) \geq 1$ *and* $\beta(y) \geq 1$ *for all* $x, y \in Fix(f)$*, then* f *has a unique fixed point.*

Example 2.10. *Let* $X = [0, +\infty)$ *endowed with the usual metric* $d(x, y) = |x - y|$ *for all* $x, y \in X$ *and* $f : X \to X$ *be defined by*

$$fx = \begin{cases} \frac{1}{8}(x + x^2), & \text{if} \quad x \in [0, 1] \\ \\ 3x, & \text{if} \quad x \in (1, +\infty) \end{cases},$$

and $\alpha, \beta : X \to [0, +\infty)$ *be given by*

$$\alpha(x) = \beta(x) = \begin{cases} 1, & \text{if} \quad x \in [0, 1] \\ 0, & \text{otherwise} \end{cases}.$$

Also, define $\psi, \phi : [0, \infty) \to [0, \infty)$ *by*

$$\psi(t) = \frac{1}{2}t \quad \text{and} \quad \phi(t) = \frac{1}{8}t.$$

Now, we prove that the Corollary 2.9 can be applied to f*.*

At first, proceeding as in the proof of Example 2.5, we deduce that f *is a cyclic* (α, β)*-admissible mapping and that the conditions (a) and (c) of Corollary 2.9 hold. Now, suppose that* $\ell > 1$*. Let* $x, y \in [0, 1]$*. Then,*

$$\begin{aligned} \left(\psi(d(fx, fy)) + \ell\right)^{\alpha(x)\beta(y)} &= \frac{1}{2}|fx - fy| + \ell \\ &= \frac{1}{16}|x - y||1 + x + y| + \ell \\ &\leq \frac{3}{8}|x - y| + \ell \\ &= \frac{1}{2}|x - y| - \frac{1}{8}|x - y| + \ell \\ &= \psi(d(x, y)) - \phi(d(x, y)) + \ell. \end{aligned}$$

Otherwise, $\alpha(x)\beta(y) = 0$ and hence we get

$$\Big(\psi(d(fx, fy)) + \ell\Big)^{\alpha(x)\beta(y)} = 1 \leq \psi(d(x, y)) - \phi(d(x, y)) + \ell$$

So, it follows from Corollary 2.9 that f has a fixed point.

Definition 2.11. *Let (X, d) be a metric space and $f : X \to X$ be a cyclic (α, β)-admissible mapping. We say that f is a weak $\alpha - \beta - \psi-$ rational contraction if $\alpha(x)\beta(y) \geq 1$ for some $x, y \in X$ implies*

$$d(fx, fy) \leq M(x, y) - \psi\Big(M(x, y)\Big),$$

where $\psi \in \Psi$ and

$$M(x, y) = \max\left\{d(x, y), \frac{[1 + d(x, Tx)]d(y, Ty)}{d(x, y) + 1}\right\}.$$

Theorem 2.12. *Let (X, d) be a complete metric space and $f : X \to X$ be a weak $\alpha - \beta - \psi$- rational contraction. Assume that the following assertions hold:*

(a) there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$;

(b) f is continuous, or

(c) if $\{x_n\}$ is a sequence in X such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all n, then $\beta(x) \geq 1$;

then f has a fixed point. Moreover, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ for all $x, y \in Fix(f)$, then f has a unique fixed point.

Proof. Similar to the proof of Theorem 2.4, we define a sequence $\{x_n\}$ by $x_n = f^n x_0$ and see that, $\alpha(x_{n-1})\beta(x_n) \geq 1$ for all $n \in \mathbb{N}$. Also we assume that $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$.

Since f is a weak $\alpha - \beta - \psi-$ rational contraction, we obtain

$$d(x_n, x_{n+1}) \leq M(x_{n-1}, x_n) - \psi\Big(M(x_{n-1}, x_n)\Big), \tag{9}$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\left\{d(x_{n-1}, x_n), \frac{[d(x_{n-1}, x_n) + 1]d(x_n, x_{n+1})}{d(x_{n-1}, x_n) + 1}\right\} \\ &= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \end{aligned}$$

Now, suppose that there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, x_{n_0+1}) > d(x_{n_0-1}, x_{n_0})$. Therefore $M(x_{n_0-1}, x_{n_0}, x_{n_0}, x_{n_0+1}) = d(x_{n_0}, x_{n_0+1})$ and so from (9), we get

$$d(x_{n_0}, x_{n_0+1}) \leq d(x_{n_0}, x_{n_0+1}) - \psi\Big(d(x_{n_0}, x_{n_0+1})\Big)$$

This implies that $\psi\Big(d(x_{n_0}, x_{n_0+1})\Big) = 0$, i.e., $d(x_{n_0}, x_{n_0+1}) = 0$, which is a contradiction. Hence, $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$. Therefore the sequence $\{d_n := d(x_n, x_{n+1})\}$ is decreasing and so there exists $d \in \mathbb{R}_+$ such that $d_n \to d$ as $n \to \infty$. Taking the limit as $n \to \infty$ in (9), we have

$$d \leq d - \psi(d).$$

This yields that $\psi(d) = 0$. Therefore, the property of ψ implies that $d = 0$. That is,

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{10}$$

We next prove that $\{x_n\}$ is a Cauchy sequence. Assume toward a contradiction that $\{x_n\}$ is not a Cauchy sequence. Then there are $\varepsilon > 0$ and sequences $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k,

$$n(k) > m(k) > k, \quad d(x_{n(k)}, x_{m(k)}) \geq \varepsilon \qquad \text{and} \quad d(x_{n(k)}, x_{m(k)-1}) < \varepsilon.$$

Now for all $k \in \mathbb{N}$, we have

$$\begin{aligned}\varepsilon \le d(x_{n(k)}, x_{m(k)}) &\le d(x_{n(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)})\\ &< \varepsilon + d(x_{m(k)-1}, x_{m(k)}).\end{aligned}$$

Taking the limit as $k \to +\infty$ in the above inequality and using (10) we get

$$\lim_{k\to+\infty} d(x_{n(k)}, x_{m(k)}) = \varepsilon. \tag{11}$$

On the other hand,

$$d(x_{n(k)}, x_{m(k)}) \le d(x_{m(k)}, x_{m(k)+1}) + d(x_{m(k)+1}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{n(k)}),$$

and

$$d(x_{n(k)+1}, x_{m(k)+1}) \le d(x_{m(k)}, x_{m(k)+1}) + d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)+1}, x_{n(k)}).$$

Taking the limit as $k \to +\infty$ in above two inequalities, by using (10) and (11), we deduce that

$$\lim_{k\to+\infty} d(x_{n(k)+1}, x_{m(k)+1}) = \varepsilon. \tag{12}$$

Again since f is a weak $\alpha - \beta - \psi-$ rational contraction and $\alpha(x_{n(k)})\alpha(x_{m(k)}) \ge 1$, then

$$d(x_{n(k)+1}, x_{m(k)+1}) \le M(x_{n(k)}, x_{m(k)}) - \psi\Big(M(x_{n(k)}, x_{m(k)})\Big), \tag{13}$$

where

$$M(x_{n(k)}, x_{m(k)}) = \max\left\{d(x_{n(k)}, x_{m(k)}), \frac{[1 + d(x_{n(k)}, x_{n(k)+1})]d(x_{m(k)}, x_{m(k)+1})}{d(x_{n(k)}, x_{m(k)}) + 1}\right\}.$$

Letting $k \to \infty$ in (13) and using (10), (11) and (12), we get

$$\varepsilon \le \varepsilon - \psi(\varepsilon).$$

So $\varepsilon = 0$, which is a contradiction. Hence, $\{x_n\}$ is a Cauchy sequence. Since X is complete, then there exists $z \in X$ such that $x_n \to z$. Suppose that (c) is held. That is, $\alpha(x_n)\beta(z) \ge 1$. Since f is a weak $\alpha - \beta - \psi-$ rational contraction, then

$$d(x_{n+1}, fz) \le M(x_n, z) - \psi\Big(M(x_n, z)\Big), \tag{14}$$

where

$$M(x_n, z) = \max\left\{d(x_n, z), \frac{[1 + d(x_n, x_{n+1})]d(z, Tz)}{d(x_n, z) + 1}\right\}.$$

Taking the limit as $n \to \infty$ in (14), we get $z = fz$.

To prove the uniqueness of fixed point, suppose that z and z^* are two fixed points of f. Since $\alpha(z)\beta(z^*) \ge 1$, it follows that

$$\psi(d(z, z^*)) \le \psi(M(z, z^*)) - \phi(M(z, z^*)),$$

where

$$M(z, z^*) = \max\left\{d(z, y), \frac{[1 + d(z, Tz)]d(z^*, Tz^*)}{d(z, z^*) + 1}\right\}.$$

So $\phi(d(z, z^*)) = 0$ and hence, $d(z, z^*) = 0$ i.e., $z = z^*$. $\square$

Also we can obtain the following corollaries from Theorem 2.12.

Corollary 2.13. *Let (X,d) be a complete metric space and $f : X \to X$ be a cyclic (α,β)-admissible mapping such that*

$$\alpha(x)\beta(y)d(fx,fy) \le M(x,y) - \psi(M(x,y)), \tag{15}$$

for all $x,y \in X$ where $\psi \in \Psi$. Suppose that the following assertions hold:

(a) there exists $x_0 \in X$ such that $\alpha(x_0) \ge 1$ and $\beta(x_0) \ge 1$;

(b) f is continuous, or

(c) if $\{x_n\}$ is a sequence in X such that $x_n \to x$ and $\beta(x_n) \ge 1$ for all n, then $\beta(x) \ge 1$;

then f has a fixed point. Moreover, if $\alpha(x) \ge 1$ and $\beta(y) \ge 1$ for all $x,y \in Fix(f)$, then f has a unique fixed point.

Corollary 2.14. *Let (X,d) be a complete metric space and $f : X \to X$ be a cyclic (α,β)-admissible mapping such that*

$$\Big(\alpha(x)\beta(y)+1\Big)^{d(fx,fy)} \le 2^{M(x,y)-\psi(M(x,y))},$$

for all $x,y \in X$ where $\psi \in \Psi$. Suppose that the following assertions hold:

(a) there exists $x_0 \in X$ such that $\alpha(x_0) \ge 1$ and $\beta(x_0) \ge 1$;

(b) f is continuous, or

(c) if $\{x_n\}$ is a sequence in X such that $x_n \to x$ and $\beta(x_n) \ge 1$ for all n, then $\beta(x) \ge 1$;

then f has a fixed point. Moreover, if $\alpha(x) \ge 1$ and $\beta(y) \ge 1$ for all $x,y \in Fix(f)$, then f has a unique fixed point.

Corollary 2.15. *Let (X,d) be a complete metric space and $f : X \to X$ be a cyclic (α,β)-admissible mapping. Assume that there exists $\ell > 1$ such that*

$$\Big(d(fx,fy)+\ell\Big)^{\alpha(x)\beta(y)} \le M(x,y) - \psi(M(x,y)) + \ell,$$

for all $x,y \in X$ where $\psi \in \Psi$ and $\ell > 0$. Suppose that the following assertions hold:

(a) there exists $x_0 \in X$ such that $\alpha(x_0) \ge 1$ and $\beta(x_0) \ge 1$;

(b) f is continuous, or

(c) if $\{x_n\}$ is a sequence in X such that $x_n \to x$ and $\beta(x_n) \ge 1$ for all n, then $\beta(x) \ge 1$;

then f has a fixed point. Moreover, if $\alpha(x) \ge 1$ and $\beta(y) \ge 1$ for all $x,y \in Fix(f)$, then f has a unique fixed point.

3. Some cyclic contraction via cyclic (α,β)-admissible mapping

In this section, in a natural way, we apply the Theorem 2.4 for proving a fixed point theorem involving a cyclic mapping. For more results see [19, 25, 26, 28, 29, 32].

Theorem 3.1. *Let A and B be two closed subsets of complete metric space (X,d) such that $A \cap B \ne \emptyset$ and $f : A \cup B \to A \cup B$ be a mapping such that $fA \subseteq B$ and $fB \subseteq A$. Assume that*

$$\psi(d(fx,fy)) \le \psi(d(x,y)) - \phi(d(x,y)), \tag{16}$$

for all $x \in A$ and $y \in B$ where $\psi \in \Psi$ and $\phi \in \Phi$. Then f has a unique fixed point in $A \cap B$.

Proof. Define $\alpha, \beta : X \to [0, +\infty)$ by

$$\alpha(x) = \begin{cases} 1, & \text{if} \quad x \in A \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \beta(x) = \begin{cases} 1, & \text{if} \quad x \in B \\ 0, & \text{otherwise} \end{cases}.$$

Let $\alpha(x)\beta(y) \geq 1$. Then $x \in A$ and $y \in B$. Hence, by (16) we have

$$\psi(d(fx, fy)) \leq \psi(d(x, y)) - \phi(d(x, y)),$$

for all $x, y \in A \cup B$. Let $\alpha(x) \geq 1$ for some $x \in X$, then $x \in A$. Hence, $fx \in B$ and so $\beta(fx) \geq 1$. Now, let $\beta(x) \geq 1$ for some $x \in X$, so $x \in B$. Hence, $fx \in A$ and then $\alpha(fx) \geq 1$. Therefore f is a cyclic (α, β)-admissible mapping. Since $A \cap B$ is nonempty, then there exists $x_0 \in A \cap B$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$.

Now, Let $\{x_n\}$ be a sequence in X such that $\beta(x_n) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then $x_n \in B$ for all $n \in \mathbb{N}$. Therefore $x \in B$. This implies that $\beta(x) \geq 1$. So the conditions (a) and (c) of Theorem 2.4 hold. So f has a fixed point in $A \cup B$, say z. Since $z \in A$, then $z = fz \in B$ and since $z \in B$, then $z = fz \in A$. Therefore $z \in A \cap B$.

The uniqueness of the fixed point follows easily from (16). □

Example 3.2. *Let $X = \mathbb{R}$ endowed with the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$ and $f : A \cup B \to A \cup B$ be defined by $fx = -x/5$ where $A = [-1, 0]$ and $B = [0, 1]$. Also define $\psi, \phi : [0, \infty) \to [0, \infty)$ by $\psi(t) = t$ and $\phi(t) = \frac{4}{5}t$. Then f has a unique fixed point (here, $x = 0$ is a unique fixed point of f). Indeed, for all $x \in A$ and all $y \in B$, we have*

$$\psi(d(fx, fy)) = |fx - fy| = \frac{1}{5}|x - y| \leq \frac{1}{5}|x - y| = \psi(d(x, y)) - \phi(d(x, y)).$$

Therefore, the conditions of Theorem 3.1 hold and f has a unique fixed point.

Similarly, we can prove the following Theorem.

Theorem 3.3. *Let A and B be two closed subsets of complete metric space (X, d) such that $A \cap B \neq \emptyset$ and $f : A \cup B \to A \cup B$ be a mapping such that $fA \subseteq B$ and $fB \subseteq A$. Assume that*

$$d(fx, fy) \leq M(x, y) - \psi(M(x, y)) \tag{17}$$

for all $x \in A$ and all $y \in B$ where $\psi \in \Psi$. Then f has a unique fixed point in $A \cap B$.

4. Application to the existence of solutions of integral equations

Let $X = C([0, T], \mathbb{R})$ be the set of real continuous functions defined on $[0, T]$ and $d : X \times X \to [0, +\infty)$ be defined by

$$d(x, y) = \|x - y\|_\infty,$$

for all $x, y \in X$. Then (X, d) is a complete metric space.

Consider the integral equation

$$x(t) = p(t) + \int_0^T S(t, s) f(s, x(s)) ds, \tag{18}$$

and the mapping $F : X \to X$ defined by

$$Fx(t) = p(t) + \int_0^T S(t, s) f(s, x(s)) ds, \tag{19}$$

where

(A) $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous,

(B) $p : [0, T] \to \mathbb{R}$ is continuous,

(C) $S : [0, T] \times [0, T] \to [0, +\infty)$ is continuous,

(D) there exist $\theta, \vartheta, \pi : X \to \mathbb{R}$ such that if $\theta(x) \geq 0$ and $\vartheta(y) \geq 0$ for some $x, y \in X$, then for every $s \in [0, T]$ we have

$$0 \leq |f(s, x(s)) - f(s, y(s))| \leq |\pi(y)||x(s) - y(s)|,$$

(F)

$$\left\| \int_0^T S(t, s)|\pi(y)|ds \right\|_\infty < 1,$$

(G) there exists $x_0 \in X$ such that $\theta(x_0) \geq 0$ and $\vartheta(x_0) \geq 0$,

(H)

$$\theta(x) \geq 0 \quad \text{for some} \quad x \in X \quad \text{implies} \quad \vartheta(Fx) \geq 0,$$

and

$$\vartheta(x) \geq 0 \quad \text{for some} \quad x \in X \quad \text{implies} \quad \theta(Fx) \geq 0,$$

(I) if $\{x_n\}$ is a sequence in X such that $\theta(x_n) \geq 0$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, then $\theta(x) \geq 0$.

Theorem 4.1. *Under the assumptions* (*A*)-(*I*), *the integral equation* (18) *has a solution in* $X = C([0, T], \mathbb{R})$.

Proof. Let $F : X \to X$ be defined by (19) and let $x, y \in X$ be such that $\theta(x) \geq 0$ and $\vartheta(y) \geq 0$. By the condition (*D*), we deduce that

$$\begin{aligned}
|Fx(t) - Fy(t)| &= \left| \int_0^T S(t, s)[f(s, x(s)) - f(s, y(s))]ds \right| \\
&\leq \int_0^T S(t, s)|f(s, x(s)) - f(s, y(s))|ds \\
&\leq \int_0^T S(t, s)|\pi(y(s))||x(s) - y(s)|ds \\
&\leq \int_0^T S(t, s)|\pi(y(s))|||x - y||_\infty ds \\
&= ||x - y||_\infty \left(\int_0^T S(t, s)|\pi(y(s))|ds \right).
\end{aligned}$$

Then

$$||Fx - Fy||_\infty \leq \left\| \int_0^T S(t, s)|\pi(y)|ds \right\|_\infty ||x - y||_\infty.$$

Now, define $\alpha, \beta : X \to [0, +\infty)$ by

$$\alpha(x) = \begin{cases} 1, & \text{if} \quad \theta(x) \geq 0 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \beta(y) = \begin{cases} 1, & \text{if} \quad \vartheta(y) \geq 0 \\ 0, & \text{otherwise} \end{cases}.$$

Also, define $\psi, \phi : [0, +\infty) \to [0, +\infty)$ by

$$\psi(t) = t \quad \text{and} \quad \phi(t) = \left(1 - \left\| \int_0^T S(t,s)|\pi(y)|ds \right\|_\infty \right) t.$$

Consequently, for all $x, y \in X$ we have

$$\alpha(x)\beta(y)d(F(x), F(y)) \le \psi(d(x,y)) - \phi(d(x,y)).$$

It easily shows that all the hypotheses of Corollary 2.6 are satisfied and hence the mapping F has a fixed point which is a solution of the integral equation (18) in $X = C([0,T], \mathbb{R})$. $\square$

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