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Fixed Point Theorems for Weakly Contractive Mappings on a Metric Space Endowed with a Graph

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Abstract. In this paper we define the notion of weakly G-contractive mappings and obtain some fixed point theorems for such mappings. Our theorems generalize, extend and unify some recent results by Harjani and Sadarangani and those contained therein. Moreover, we also furnish with an application and an example to substantiate the validity of our results.

1. Introduction and Preliminaries

Banach [3] showed that every contraction mapping on a complete metric space always possess a unique fixed point. Edelstein [8] defined the notion of contractive mappings. The mapping $f : X \to X$ is said to be contractive if

$$d(fx, fy) < d(x, y) \ \forall x, y \in X \text{ with } x \neq y. \tag{1}$$

In order to obtain a fixed point of a contractive map we have to add further assumptions such as there exists a point $x \in X$ for which $\{f^n x\}$ contains a convergent subsequence or the space is compact. The mappings $f : X \to X$ is said to be nonexpansive if

$$d(fx, fy) \leq d(x, y) \ \forall x, y \in X. \tag{2}$$

To obtain a fixed point of nonexpansive map we also need to impose some certain assumptions such as uniform normal structure or compactness of the space. The mapping $f : X \to X$ is said to be weakly contractive if

$$d(fx, fy) \leq d(x, y) - \psi(d(x, y)) \ \forall x, y \in X, \tag{3}$$

where $\psi : [0, \infty) \to [0, \infty)$ is continuous, nondecreasing such that ψ is positive on $(0, \infty)$, $\psi(0) = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$. Alber and Guerre-Delabriere [1] introduced the notion of weakly contractive mapping when the underlying space was taken to be a Hilbert spaces. They proved that every weakly contractive mapping defined on a Hilbert space posses a unique fixed point point, with out any additional assumption. Later on, Rhoades [19] showed that this result also holds for general metric spaces. From the definitions it is clear that weakly contractive maps lie between contraction maps and contractive maps.

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Theorem 1.1. *[19, Theorm 1] Let (X, d) be a complete metric space. Assume the mapping $f : X \to X$ is weakly contractive. Then f has a unique fixed point ξ in X.*

Ran and Reurings [18] initiated the study of fixed points of self mappings on a metric space (X, d) endowed with the partial ordering $\preceq$. In this context, many interesting results have been established by some authors for partially ordered set endowed with a complete metric (see e.g., [5, 6, 9, 14–18]). Recently, Harjani and Sadarangani [10] obtained some fixed point results for weakly contractive mappings defined on partially ordered set endowed with a complete metric space.

By using the platform of graph theory Jachymski [11] obtained some useful fixed point results for mappings defined on a complete metric space endowed with a graph instead of partial ordering. Later on many authors undertook further investigations in this direction by weakening contractive condition and analyzing connectivity condition of graph (see, [4, 20]). Motivated by Jachymski [11] in this paper, using the language of graph theory, we obtain some fixed point results that unify and extend above results by Harjani and Sadarangani [10]. In particular, we show that Theorems 1.2, 1.3, 1.4, 1.5 and 1.6 are special cases of our results. Consequently, as an application of our results we obtain a fixed point result for weakly contractive cyclic mappings. An example has been established to demonstrate the degree of generality of our result over some pre-existing results.
Let $(X, \preceq)$ is a partially ordered set. Let f be a mapping from X into X, then f is said to be nonincreasing if $x, y \in X, x \preceq y \Rightarrow f(x) \succeq f(y)$. The mapping f is nondecreasing if $x, y \in X, x \preceq y \Rightarrow f(x) \preceq f(y)$. Let $G = (V(G), E(G))$ be a directed graph. Let us denote with G^{-1} the graph which is formed by reversing the direction of all edges in G and let $\widetilde{G}$ denote the undirected graph which is formed by ignoring direction of all edges. It is rather easier to treat $\widetilde{G}$ as a directed graph having symmetric set of edges or equivalently, $E(\widetilde{G}) = E(G) \cup E(G^{-1})$. Let $x, y \in V(G)$. A path in G from x to y of length N is a sequence of $N + 1$ vertices $\{x_i\}_{i=0}^{N}$ such that $x_0 = x, x_N = y$ and $(x_{i-1}, x_i) \in E(G)$ for all $i = 1, \cdots, N$. A graph G is connected if there exists a path between any two vertices. It is weakly connected if at least $\widetilde{G}$ is connected. For a graph G such that $E(G)$ is symmetric and $x \in V(G)$, the subgraph G_x represents a component of G containing x which consists of all vertices and edges that are traversed in some path beginning at x. Then it elicits $V(G_x) = [x]_{\tilde{G}}$, where $[x]_{\widetilde{G}}$ is the equivalence class of the relation R defined on $V(G)$ by the rule: uRv if there exists a path in G from u to v. It is easily seen that G_x is connected. Let (X, d) be a metric space and f be a mapping from X into X. We say that f is a Picard operator [16] if f has a unique fixed point ξ and $\lim_{n\to\infty} f^n x = \xi$ for all $x \in X$.

We state following results from Harjani and Sadarangani [10] for convenience.

Theorem 1.2. *[10, Theorem 2] Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $f : X \to X$ be continuous and nondecreasing mapping such that*

$$d(fx, fy) \le d(x, y) - \psi(d(x, y)) \text{ for } x \succeq y, \tag{4}$$

where $\psi : [0, \infty) \to [0, \infty)$ is continuous nondecreasing such that ψ is positive in $(0, \infty)$, $\psi(0) = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$. If there exists $x_0 \in X$ with $x_0 \preceq fx_0$ then f has a fixed point.

Following hypothesis was appeared in Nieto and Rodríguez-López [14, Theorem 1].
If $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$ then $x_n \preceq x$ for all $n \in \mathbb{N}$.

(5)

Theorem 1.3. *[10, Theorem 3] Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Assume that X satisfies (5). Let $f : X \to X$ be nondecreasing mapping such that*

$$d(fx, fy) \le d(x, y) - \psi(d(x, y)) \text{ for } x \succeq y, \tag{6}$$

where $\psi : [0, \infty) \to [0, \infty)$ is continuous nondecreasing such that ψ is positive in $(0, \infty)$, $\psi(0) = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$. If there exists $x_0 \in X$ with $x_0 \preceq fx_0$ then f has a fixed point.

It was shown in Nieto and Rodríguez-López [14] that the following two conditions are equivalent.

$$\text{For } x, y \in (X, \preceq) \text{ there exists a lower or an upper bound.} \tag{7}$$

$$\text{For } x, y \in (X, \preceq) \text{ there exists } z \in X \text{ which is comparable to } x \text{ and } y. \tag{8}$$

Theorem 1.4. *[10, Theorem 4] Adding condition (8) to the hypothesis of Theorem 1.2 (resp. Theorem 1.3) we obtain the uniqueness of the fixed point.*

Theorem 1.5. *[10, Theorem 5] Let $(X, \preceq)$ be a partially ordered set satisfying (8) and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $f : X \to X$ be nonincreasing mapping such that*

$$d(fx, fy) \le d(x, y) - \psi(d(x, y)) \text{ for } x \succeq y, \tag{9}$$

where $\psi : [0, \infty) \to [0, \infty)$ is continuous nondecreasing such that ψ is positive in $(0, \infty)$, $\psi(0) = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$. Suppose also that either
f is continuous, or
X is such that if $x_n \to x$ is a sequence in X whose consecutive terms are comparable, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that every term is comparable to the limit x.

(10)

If there exists $x_0 \in X$ with $x_0 \preceq fx_0$ or $x_0 \succeq fx_0$ then f has a unique fixed point.

Theorem 1.6. *[10, Theorem 6] Let $(X, \preceq)$ be a partially ordered set and suppose that (8) holds and that there exists a metric d in X such that (X, d) is a complete metric space. Let f maps comparable elements to comparable elements, that is,*

$$\text{for } x, y \in X, x \preceq y \Rightarrow fx \preceq fy \text{ or } fx \succeq fy$$

and such that for $x, y \in X$ with $x \succeq y$

$$d(fx, fy) \le d(x, y) - \psi(d(x, y))$$

where $\psi : [0, \infty) \to [0, \infty)$ is continuous nondecreasing such that ψ is positive in $(0, \infty)$, $\psi(0) = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$. Suppose that either f is continuous or X is such that condition (10) holds. If there exists $x_0 \in X$ with x_0 comparable to fx_0 then f has a unique fixed point ξ. Moreover, for $x \in X$, $\lim f^n x = \xi$.

2. Main Results

Motivated by Jachymski [11] we introduce the following definition. Let (X, d) be a metric space. Let Δ denote the diagonal of the cartesian product $X \times X$. G is a directed graph such that the set of its vertices $V(G)$ coincides with X, and the set of its edges $E(G)$ contains all loops, that is, $E(G) \supseteq \Delta$. We assume that G has no parallel edges. We may treat G as a weighted graph by attributing to each edge the distance between its vertices. The sequences $\{x_n\}$ and $\{y_n\}$ in X are equivalent if $d(x_n, y_n) \to 0$ [11]. Further, we note that if either of the sequences is Cauchy then the other sequence is Cauchy as well. In that case we call them Cauchy equivalent.

Definition 2.1. *Let (X, d) be a metric space endowed with the a graph G (as discuss above). A mapping $f : X \to X$ is called weakly G-contractive if it satisfies the following two conditions. For $x, y \in X$,*

$$(fx, fy) \in E(G) \text{ whenever } (x, y) \in E(G), \tag{11}$$

$$d(fx, fy) \le d(x, y) - \psi(d(x, y)) \text{ whenever } (x, y) \in E(G), \tag{12}$$

where $\psi : [0, \infty) \to [0, \infty)$ is continuous nondecreasing such that ψ is positive on $(0, \infty)$ and $\psi(0) = 0$.

Example 2.2. *Let (X, d) be a metric space. Consider the graph G_0 defined by $G_0 = X \times X$. Then any weakly contractive map is weakly G-contractive.*

Let us denote by $r(x, y) = \sum_{i=1}^{m} d(z_{i-1}, z_i)$, where $\{z_i\}_{i=0}^{m}$ is a path from x to y in G.

Lemma 2.3. *Let (X, d) be a metric space and $f : X \to X$ be a weakly G-contractive map. Then for any $x \in X$ and $y \in [x]_{\tilde{G}}$ we have*

$$\lim_{n\to\infty} d(f^n x, f^n y) = \lim_{n\to\infty} r(f^n x, f^n y) = 0. \tag{13}$$

Proof. Let $x \in X$ and $y \in [x]_{\tilde{G}}$. Then there exists a path $x = z_0, z_1, \cdots, z_l = y$ in the graph G. As f is weakly G-contractive from (11) and (12) we get

$$(f^n z_{i-1}, f^n z_i) \in E(G)\ \forall i = 1, 2, \cdots, l, \forall n \in \mathbb{N} \tag{14}$$

and

$$\begin{aligned} d(f^n z_{i-1}, f^n z_i) &\leq d(f^{n-1} z_{i-1}, f^{n-1} z_i) - \psi(d(f^{n-1} z_{i-1}, f^{n-1} z_i)) && (15)\\ &\leq d(f^{n-1} z_{i-1}, f^{n-1} z_i)\ \forall i = 1, 2, \cdots, l, \forall n \in \mathbb{N}. && (16) \end{aligned}$$

This shows that $\{d(f^n z_{i-1}, f^n z_i)\}$ is a nonincreasing sequence of nonnegative real numbers, bounded below by 0, thus convergent. Let $d(f^n z_{i-1}, f^n z_i) \to \gamma$. Taking limit as $n \to \infty$ from (15) we get $\gamma \leq \gamma - \psi(\gamma) \leq \gamma$. Therefore, $\psi(\gamma) = 0$ and using properties of ψ we get $\gamma = 0$. Thus

$$\lim_{n\to\infty} d(f^n z_{i-1}, f^n z_i) = 0\ \forall i = 1, 2, \cdots, l, \forall n \in \mathbb{N}. \tag{17}$$

By triangular inequality we have

$$\begin{aligned} d(f^n x, f^n y) &= d(f^n z_0, f^n z_k)\\ &\leq d(f^n z_0, f^n z_1) + d(f^n z_1, f^n z_2) + \cdots + d(f^n z_{k-1}, f^n z_l). \end{aligned}$$

Taking limit as $n \to \infty$ and using (17) we get

$$\lim_{n\to\infty} d(f^n x, f^n y) \leq \lim_{n\to\infty} r(f^n x, f^n y) = 0.$$

□

In [2, 11] authors put forth the following:

($\mathcal{P}$) for any $\{x_n\}$ in X such that $x_n \to x$ with $(x_{n+1}, x_n) \in E(G)$ for all $n \geq 1$ there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x, x_{n_k}) \in E(G)$ [11];

($\mathcal{Q}$) for any $\{x_n\}$ in X such that $x_n \to x \in X$ with $x_n \in [x]_{\widetilde{G}}$ for all $n \geq 1$ then $r(x_n, x) \to 0$ [2].

It has been shown that properties ($\mathcal{P}$) and ($\mathcal{Q}$) are independent (see, [2, Exmaples 2.1, 2.2]). In this context we define the following properties for a graph G. Let $f : X \to X$:

($\mathcal{P}'$) for any $\{f^n x\}$ in X such that $f^n x \to y \in X$ with $(f^{n+1} x, f^n x) \in E(G)$ there exists a subsequence $\{f^{n_k} x\}$ of $\{f^n x\}$ and $n_0 \in \mathbb{N}$ such that $(y, f^{n_k} x) \in E(G)$ for all $k \geq n_0$;

($\mathcal{Q}'$) for any $\{f^n x\}$ in X such that $f^n x \to y \in X$ with $f^n x \in [y]_{\widetilde{G}}$ for all $n \geq 1$ then $r(f^n x, y) \to 0$.

It is easily seen that in a graph G property ($\mathcal{P}$) subsumes property ($\mathcal{P}'$) for any self-mapping f on X but converse may not hold as shown below.

Example 2.4. *Let $X = [0,1]$ endowed with usual metric $d(x,y) = |x-y|$. Consider a graph G consisting of $V(G) := X$ and $E(G) := \left\{\left(\frac{n}{n+1}, \frac{n+1}{n+2}\right) : n \in \mathbb{N}\right\} \cup \left\{\left(\frac{x}{2^n}, \frac{x}{2^{n+1}}\right) : n \in \mathbb{N}, x \in [0,1]\right\} \cup \left\{\left(\frac{x}{2^{2n}}, 0\right) : n \in \mathbb{N}, x \in [0,1]\right\}$. Note that G does not satisfy property $(\mathcal{P})$ as $\frac{n}{n+1} \to 1$. Whereas by defining $f : X \to X$ as $fx = \frac{x}{2}$, G satisfies property $(\mathcal{P}')$. Since, $f^n x = \frac{x}{2^n} \to 0$ as $n \to \infty$.*

Similarly, any graph G satisfying property $(\mathcal{Q})$ also satisfies $(\mathcal{Q}')$ whereas next example shows converse may not hold.

Example 2.5. *Let $X = [0,1]$ endowed with usual metric $d(x,y) = |x-y|$. Consider a graph G consisting of $V(G) := X$ and $E(G) := \left\{\left(\frac{n}{n+1}, 0\right) : n \in \mathbb{N}\right\} \cup \left\{\left(\frac{x}{2^n}, 0\right) : n \in \mathbb{N}, x \in [0,1]\right\} \cup \{(0,1)\}$. Since, $\frac{n}{n+1} \to 1$ but $r(\frac{n}{n+1}, 1) = |\frac{n}{n+1} - 0| + |0-1| \nrightarrow 0$. Thus property $(\mathcal{Q})$ does not hold in G. On the other hand by defining $f : X \to X$ as $fx = \frac{x}{2}$ then G satisfies $(\mathcal{Q}')$.*

Assume that $x \in X$ and a is any positive real number. For convenience we define the following:

$$\mathfrak{B}_a(x) = \{y \in [x]_{\widetilde{G}} : r(x,y) < a, \text{for at least one path between } x \text{ and } y \text{ in } \widetilde{G}\}$$

and

$$\overline{\mathfrak{B}_a(x)} = \{y \in [x]_{\widetilde{G}} : r(x,y) \leq a, \text{for at least one path between } x \text{ and } y \text{ in } \widetilde{G}\}.$$

Proposition 2.6. *Let (X,d) be a metric space and f be a weakly G-contractive mapping from X into X. Let there exists $x_0 \in X$ such that $fx_0 \in [x_0]_{\widetilde{G}}$ then the sequence $\{f^n x_0\}$ is Cauchy.*

Proof. Using Lemma 2.3 as $fx_0 \in [x_0]_{\tilde{G}}$, we obtain

$$\lim_{n\to\infty} r(f^{n+1}x_0, f^n x_0) = 0. \tag{18}$$

Now, we will show that $\{f^n x_0\}$ is a Cauchy sequence. Since $\lim_{n\to\infty} r(f^{n+1}x_0, f^n x_0) = 0$, for $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$r(f^{n_0+1}x_0, f^{n_0}x_0) \leq \inf_j \left\{\frac{\epsilon}{2}, \psi\Big(d(x_j, f^{n_0}x_0)\Big)\right\}, \tag{19}$$

where the vertex $x_j \in X$ is adjacent to $f^{n_0}x_0$ with a direct edge. Since, $fx_0 \in [x_0]_{\tilde{G}}$ then by induction there exists a path between $f^{n_0}x_0$ and $f^{n_0+1}x_0$ in $\widetilde{G}$. Which evokes the existence of at least one vertex $x_j \in X$ adjacent to $f^{n_0}x_0$. We claim that $f\Big(\overline{\mathfrak{B}_\epsilon(f^{n_0}x_0)}\Big) \subset \overline{\mathfrak{B}_\epsilon(f^{n_0}x_0)}$. Let $z \in \overline{\mathfrak{B}_\epsilon(f^{n_0}x_0)}$. Let $\{y_i\}_{i=0}^l$ be a path between z and $f^{n_0}x_0$ such that $y_0 = z$ and $y_l = f^{n_0}x_0$ then $\{fy_i\}_{i=0}^l$ is a path between fz and $ff^{n_0}x_0$. So that $fz \in [f^{n_0+1}x_0]_{\widetilde{G}} = [f^{n_0}x_0]_{\widetilde{G}}$.

Then two cases arise:

Case 1. If $0 < r(z, f^{n_0}x_0) \leq \frac{\epsilon}{2}$.

Since $z \in [f^{n_0}x_0]$, using (12) & (19) along the path $\{fz, \cdots, f^{n_0+1}x_0, \cdots, f^{n_0}x_0\}$, we have

$$\begin{aligned}
r(fz, f^{n_0}x_0) &= r(fz, f^{n_0+1}x_0) + r(f^{n_0+1}x_0, f^{n_0}x_0) \\
&= \sum_{i=1}^{l} d(fy_{i-1}, fy_i) + r(f^{n_0+1}x_0, f^{n_0}x_0) \\
&\leq \sum_{i=1}^{l} d(y_{i-1}, y_i) - \sum_{i=1}^{l} \psi(d(y_{i-1}, y_i)) + r(f^{n_0+1}x_0, f^{n_0}x_0) \\
&\leq r(z, f^{n_0}x_0) + r(f^{n_0+1}x_0, f^{n_0}x_0) \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Case 2. If $\frac{\epsilon}{2} < r(z, x_{n_0}) \leq \epsilon$.

In this case, again using (12) & (19) along the path $\{fz, \cdots, f^{n_0+1}x_0, \cdots, f^{n_0}x_0\}$, we have

$$\begin{aligned} r(fz, f^{n_0}x_0) &= r(fz, f^{n_0+1}x_0) + r(f^{n_0+1}x_0, f^{n_0}x_0) \\ &= \sum_{i=1}^{l} d(fy_{i-1}, fy_i) + r(f^{n_0+1}x_0, f^{n_0}x_0) \\ &\leq \sum_{i=1}^{l} d(y_{i-1}, y_i) - \sum_{i=1}^{l} \psi(d(y_{i-1}, y_i)) + r(f^{n_0+1}x_0, f^{n_0}x_0) \\ &\leq r(z, f^{n_0}x_0) - \psi(d(y_{l-1}, y_l)) + r(f^{n_0+1}x_0, f^{n_0}x_0) \\ &\leq r(z, f^{n_0}x_0) - \psi(d(y_{l-1}, f^{n_0}x_0)) + \psi(d(y_{l-1}, f^{n_0}x_0)) \leq \epsilon. \end{aligned}$$

This proves our claim. As $f^{n_0+1}x_0 \in \overline{\mathfrak{B}_\epsilon(f^{n_0}x_0)}$. Therefore, $f(f^{n_0+1}x_0) \in \overline{\mathfrak{B}_\epsilon(f^{n_0}x_0)}$. Since, $f^{n_0+k}x_0 \in [f^{n_0}x_0]_{\widetilde{G}}$ for $k = 1, 2, \cdots$. Repeating the same procedure it follows that $f^n x_0 \in \overline{\mathfrak{B}_\epsilon(f^{n_0}x_0)}$, for $n \geq n_0$. Which asserts that for each $n \geq n_0$

$$d(f^n x_0, f^{n_0}x_0) \leq r(f^n x_0, f^{n_0}x_0) \leq \epsilon.$$

It simply yields $f^n x_0 \in \overline{B}(f^{n_0}x_0; \epsilon)$ for all $n \geq n_0$. Finally, it vindicates that $\{f^n x_0\}$ is a Cauchy sequence in X. □

Theorem 2.7. *Let (X, d) be a complete metric space endowed with a graph G and f be a weakly G-contractive mapping from X into X. Suppose that the following conditions holds*

(*i*) *G satisfies property $(\mathcal{P}')$,*

(*ii*) *there exists some $x_0 \in X_f := \{x \in X : (x, fx) \in E(G)\}$.*

Then $f|_{[x_0]_{\widetilde{G}}}$ has a unique fixed point $\xi \in [x_0]_{\widetilde{G}}$ and $f^n y \to \xi$ for any $y \in [x_0]_{\widetilde{G}}$.

Proof. Let $x_0 \in X_f$ i.e, $(fx_0, x_0) \in E(G)$ then $fx_0 \in [x_0]_{\widetilde{G}}$. Thus Proposition 2.6 yields $\{f^n x_0\}$ is Cauchy. Since X is complete there exists $\xi \in X$ such that $f^n x_0 \to \xi$.
Suppose condition (*i*) holds. Then there exists a subsequence $\{f^{n_k}x_0\}$ of $\{f^n x_0\}$ and $n_0 \in \mathbb{N}$ such that $(\xi, f^{n_k}x_0) \in E(G)$ for all $k \in \mathbb{N}$ and $k \geq n_0$. Now, using (12) we have for all $k \geq n_0$

$$\begin{aligned} d(f\xi, \xi) &\leq d(f\xi, f^{n_k+1}x_0) + d(f^{n_k+1}x_0, \xi) \\ &\leq d(\xi, f^{n_k}x_0) - \psi(d(\xi, f^{n_k}x_0)) + d(f^{n_k+1}x_0, \xi) \\ &\leq d(\xi, f^{n_k}x_0) + d(f^{n_k+1}x_0, \xi). \end{aligned}$$

Letting $k \to \infty$ we get $d(f\xi, \xi) = 0$. Thus ξ is fixed point of f. We observe that $\{x_0, fx_0, \cdots, f^{n_1}x_0, \cdots, f^{n_0}x_0, \xi\}$ is path from x_0 to ξ in $\widetilde{G}$. It vindicates $\xi \in [x_0]_{\widetilde{G}}$. Now let $y \in [x_0]_{\widetilde{G}}$ be arbitrary. Then by Lemma 2.3 we have

$$\lim_{n \to \infty} d(f^n y, f^n x_0) = 0.$$

Thus $\lim_{n \to \infty} f^n(y) = \xi$.

Uniqueness: Suppose f has two fixed points ξ and η. Then it follows from Lemma 2.3 that

$$d(\xi, \eta) = d(f^n \xi, f^n \eta).$$

Taking limit as $n \to \infty$ we get $\xi = \eta$. □

Remark 2.8. *Indeed f is a Picard operator on X if G is weakly connected because $X := [x_0]_{\widetilde{G}}$.*

Theorem 2.9. *Let (X, d) be a complete metric space endowed with a graph G and f be a weakly G-contractive mapping from X into X. Assume that G is weakly connected satisfying property (Q'). Then f is Picard operator.*

Proof. Let G is weakly connected and $x_0 \in X$ then there exists a path between x_0 and fx_0 in $\widetilde{G}$ or equivalently $fx_0 \in [x_0]_{\widetilde{G}}$. By Proposition 2.6, $\{f^n x_0\}$ is Cauchy. Since, X is complete then $f^n x_0 \to \xi \in X$. Since, G is weakly connected so that for each n there exists a path of finite length from $f^n x_0$ to ξ. Let $\{z_i^n\}_{i=0}^m$ be a path between $f^n x_0$ to ξ with $z_0^n = f^n x_0$ and $z_m^n = \xi$ then

$$\begin{aligned} d(\xi, f\xi) &\leq d(\xi, f^{n+1}x_0) + d(f^{n+1}x_0, f\xi) \\ &\leq d(\xi, f^{n+1}x_0) + \sum_{i=1}^{m} d(fz_{i-1}^n, fz_i^n) \\ &\leq d(\xi, f^{n+1}x_0) + \sum_{i=1}^{m} d(z_{i-1}^n, z_i^n) - \sum_{i=1}^{m} \psi(d(z_{i-1}^n, z_i^n)) \\ &\leq d(\xi, f^{n+1}x_0) + r(f^n x_0, \xi). \end{aligned}$$

Since, $f^n x_0 \in [\xi]_{\widetilde{G}}$ as G is weakly connected, then R.H.S $\to 0$ as $n \to \infty$. We conclude $f\xi = \xi$. Let $y \in X := [x_0]$ be arbitrary then by Lemma 2.3, $f^n y \to \xi$. Uniqueness of fixed point can be proved similarly as in Theorem 2.7. □

A self-mapping f on X is called orbitally continuous if for all $x, y \in X$ and any sequence $\{k_n\}_{n\in\mathbb{N}}$ of positive integers, $f^{k_n} x \to y$ implies $f(f^{k_n} x) \to fy$ as $n \to \infty$. A self-mapping f on X is called orbitally G-continuous if for all $x, y \in X$ and any sequence $\{k_n\}_{n\in\mathbb{N}}$ of positive integers, $f^{k_n} x \to y$ and $(f^{k_n} x, f^{k_n+1} x) \in E(G)$ $\forall n \in \mathbb{N}$ imply $f(f^{k_n} x) \to fy$ [11].

Theorem 2.10. *Let (X, d) be a complete metric space endowed with a graph G and f be a weakly G-contractive mapping from X into X. Suppose that the following conditions holds*

(*i*) *f is orbital G-continuous,*

(*ii*) *there exists some $x_0 \in X_f := \{x \in X : (x, fx) \in E(G)\}$.*

Then f has a fixed point $\xi \in X$ and $f^n y \to \xi$ for any $y \in [x_0]_{\widetilde{G}}$. Moreover, if G is weakly connected then f is Picard operator.

Proof. Since, $(x_0, fx_0) \in E(G)$ imply $fx_0 \in [x_0]_{\widetilde{G}}$ then from Proposition 2.6 $\{f^n x_0\}$ is Cauchy. Since, X is complete there exists $\xi \in X$ such that $f^n x_0 \to \xi$. By orbital G-continuity as $(f^n x_0, f^{n+1} x_0) \in E(G)$ for all $n \geq 1$ we obtain $ff^n x_0 \to f\xi$. Hence, $f\xi = \xi$. Let $y \in [x_0]_{\widetilde{G}}$ be arbitrary then from Lemma 2.3, $f^n y \to \xi$. Uniqueness can be easily followed. □

Theorem 2.11. *Let (X, d) be a complete metric space endowed with a graph G and f be a weakly G-contractive mapping from X into X. Suppose that the following conditions holds*

(*i*) *f is orbitally continuous,*

(*ii*) *there exists some $x_0 \in X$ such that $fx_0 \in [x_0]_{\widetilde{G}}$.*

Then f has a fixed point $\xi \in X$ and $f^n y \to \xi$ for any $y \in [x_0]_{\widetilde{G}}$. Moreover, if G is weakly connected then f is Picard operator.

Proof. Since, $fx_0 \in [x_0]_{\widetilde{G}}$ then from Proposition 2.6 $\{f^n x_0\}$ is Cauchy. Completeness of X $f^n x_0 \to \xi \in X$. Since, f is orbitally continuous then $ff^n x_0 \to f\xi$. Hence, $f\xi = \xi$. Let $y \in [x_0]_{\widetilde{G}}$ be arbitrary then from Lemma 2.3, $f^n y \to \xi$. □

Remark 2.12. *Let (X, d) be a metric space and $\geq$ be a partial order in X. Define the graph G_1 by*

$$E(G_1) = \{(x, y) \in X \times X : x \geq y\}.$$

Note that for this graph, condition (11) means f is nondecreasing with respect to this order. Furthermore for graph G_1 property $(\mathcal{P})$ is equivalent to the statement; that any nondecreasing sequences $\{x_n\}$, with $x_n \to x$ has a subsequence $\{x_{n_k}\}$ such that $x \geq x_{n_k}$ for all $k \in \mathbb{N}$, which is certainly weaker than condition (5). Further more, the weak connectivity of the graph gives condition (8). Therefore, Theorems 1.2, 1.3 and 1.4 are special cases of Theorem 2.7 when $G = G_1$ which satisfies property $(\mathcal{P})$.

Proposition 2.13. *Let (X, d) be a metric space endowed with a graph G and $f : X \to X$ be a weakly G-contractive mapping. Then f is weakly G^{-1}-contractive as well as weakly $\tilde{G}$-contractive.*

Proof. Let $(x, y) \in E(G^{-1})$, then $(y, x) \in E(G)$. Since f is weakly G-contractive, $(fy, fx) \in E(G)$. Thus $(fx, fy) \in E(G^{-1})$. Therefore, condition (11) is satisfied for the graph G^{-1}. As $(y, x) \in E(G)$ and f is weakly G-contractive, from (12) we get $d(fy, fx) \leq d(y, x) - \psi(d(y, x))$ and symmetry of d implies that $d(fx, fy) \leq d(x, y) - \psi(d(x, y))$. Thus condition (12) also holds for the graph G^{-1}. Hence f is weakly G^{-1}-contractive mapping. Similar argument shows that f is weakly $\tilde{G}$-contractive mapping.

Remark 2.14. *Consider the following graph in the metric space (X, d)*

$$E(G_2) = \{(x, y) \in X \times X : x \geq y \vee y \geq x\}.$$

For this graph (11) holds if f is monotone with respect to the order. Moreover, $G_2 = \tilde{G_1}$ and it follows from above proposition that if f is weakly G_1-contractive it is weakly G_2-contractive. Therefore Theorems 1.5 and 1.6 are special cases of Theorem 2.7 when $G = G_2$ which satisfies property $(\mathcal{P})$.

Lemma 2.15. *Let (X, d) be a metric space endowed with a graph and $f : X \to X$ be a G-contractive map. Suppose for some $x_0 \in X$, $fx_0 \in [x_0]_{\tilde{G}}$. Then*

(i) $f([x_0]_{\tilde{G}}) \subseteq [x_0]_{\tilde{G}}$,

(ii) $f|_{[x_0]_{\tilde{G}}}$ is a $\tilde{G}_{x_0}$-contractive.

Proof. Let $x \in [x_0]_{\tilde{G}}$. Then there is a path $x = z_0, z_1, \cdots, z_l = x_0$ between x and x_0. Since f is G-contractive, $(fz_{i-1}, fz_i) \in E(G) \forall i = 1, 2, \cdots, l$. Thus $fx \in [fx_0]_{\tilde{G}} = [x_0]_{\tilde{G}}$.
Suppose $(x, y) \in E(\tilde{G}_{x_0})$. Then $(fx, fy) \in G$, since f is G-contractive. But $[x_0]_{\tilde{G}}$ is f invariant, so we conclude that $(fx, fy) \in E(\tilde{G}_{x_0})$. Condition (12) is satisfied automatically, since $\tilde{G}_{x_0}$ is a subgraph of G. □

In [13] authors introduced the notions of cyclic representations and cyclic contractions. Let X be a nonempty set, m a positive integer and $\{A_i\}_{i=1}^m$ be nonempty closed subsets of X and $f : \cup_{i=1}^m A_i \to \cup_{i=1}^m A_i$ be an operator. Then $X := \cup_{i=1}^m A_i$ is known as cyclic representation of X with respect to f if;

$$f(A_1) \subset A_2, \cdots, f(A_{m-1}) \subset A_m, f(A_m) \subset A_1 \tag{20}$$

and operator f is known as cyclic operator.
Now from Theorem 2.7 one can easily invoke [12, Theorem 6] as follows.

Corollary 2.16. *([12, Theorem 6]) Let (X, d) be complete metric space. Let m be positive integer, $\{A_i\}_{i=1}^m$ be nonempty closed subsets of X, $Y := \cup_{i=1}^m A_i$ and $f : Y \to Y$. Assume that $\cup_{i=1}^m A_i$ is cyclic representation of Y w.r.t. f and*

(i) there exists $\psi : [0, \infty) \to [0, \infty)$ where ψ is continuous, nondecreasing, positive on $(0, \infty)$, $\psi(0) = 0$ and the following holds:

$$d(fx, fy) \leq d(x, y) - \psi(d(x, y)) \text{ for } x \in A_i, y \in A_{i+1};\ A_{m+1} = A_1.$$

Then f has a unique fixed point $t \in \cap_{i=1}^m A_i$ and $f^n y \to t$ for any $y \in \cup_{i=1}^m A_i$.

Proof. Since, A_i, $i \in \{1, \cdots, m\}$ are closed then (Y, d) is complete metric space. Consider a graph G consisting of $V(G) := Y$ and $E(G) := \Delta \cup \{(x, y) \in Y \times Y : x \in A_i, y \in A_{i+1}; i = 1, \cdots, m\}$. Relation (20) invokes f preserves edges and thus By (i) it follows that f is G-contractive. Now let $f^n x \to x^* \in Y$ such that $(f^n x, f^{n+1} x) \in E(G)$ for all $n \geq 1$ then in view of (20), sequence $\{f^n x\}$ has infinitely many terms in each A_i so that one can easily extract a subsequence of $\{f^n x\}$ converging to x^* in each A_i, since A_i's are closed then $x^* \in \cap_{i=1}^m A_i$. Now it is easy to form a subsequence $\{f^{n_k} x\}$ in some A_j, $j \in \{1, \cdots, m\}$ such that $(f^{n_k} x, x^*) \in E(G)$ for $k \geq 1$. Thus G is weakly connected and satisfies property $(\mathcal{P})$. Hence, conclusion follows from Theorem 2.7. □

Definition 2.17. *([7, 8]) A metric space (X, d) is said to be ϵ-chainable, for some $\epsilon > 0$, if for $x, y \in X$ there exist $x_i \in X$; $i = 0, 1, 2, \cdots, l$ with $x_0 = x, x_l = y$ such that $d(x_{i-1}, x_i) < \epsilon$ for $i = 1, 2, \cdots, l$.*

Now we state and prove our last fixed point result.

Theorem 2.18. *Let (X, d) be a complete ϵ-chainable metric space. Let $\psi : [0, \infty) \to [0, \infty)$ is continuous, nondecreasing, positive on $(0, \infty)$ and $\psi(0) = 0$. Assume that $f : X \to X$ satisfies,*

$$d(x, y) < \epsilon \quad \Longrightarrow \quad d(fx, fy) \leq d(x, y) - \psi(d(x, y)), \tag{21}$$

for all $x, y \in X$. Then f is a Picard operator.

Proof. Let $G := (V(G), E(G))$ such that $V(G) := X$ and $E(G) := \{(x, y) \in X \times X : d(x, y) < \epsilon\}$. Since X is ϵ-chainable which shows G is weakly connected. Let $(x, y) \in E(G)$, from (21) we have,

$$d(fx, fy) \leq d(x, y) - \psi(d(x, y)) \leq d(x, y) < \epsilon.$$

Thus $(fx, fy) \in E(G)$ so that f is weakly G-contractive. Further, (21) implies that f is continuous. The conclusion follows from Theorem 2.11. □

Example 2.19. *Let $X := [0, \infty)$ equipped with usual metric d. Let $\psi : [0, \infty) \to [0, \infty)$ be defined as $\psi(t) = \frac{t^2}{2}$. Define a mapping $f : X \to X$ as*

$$fx = \begin{cases} x - \frac{x^2}{2} & ; x \in [0, 1] \setminus \{1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots\} \\ 2x & ; \textit{otherwise}. \end{cases}$$

Consider the graph G such that $V(G) := X$ and $E(G) := \Delta \cup \left\{(0, x) : x \in [0, 1] \setminus \{1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots\}\right\}$.

It can be easily seen that G satisfies (11) and (12). Theorem 2.7 yields 0 is a fixed point of f. On the other hand one cannot invoke Theorem 1.1 (specifically by taking $x = 0, y = 1$, (3) does not hold). Also we note that f is not a Banach G-contraction. Since, $\{(0, \frac{1}{n})\}_{n \geq 3} \subset E(G)$, then

$$d(f0, f\frac{1}{n}) = |\frac{1}{n} - \frac{1}{2n^2}| \leq c\,|\frac{1}{n}| = c\, d(0, \frac{1}{n}). \tag{22}$$

Letting $n \to \infty$, (22) yields $c \geq 1$.

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