Rapid Variability and Karamata's Integral Theorem

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Abstract. In this paper some important variations and generalizations of the well-known Karamata's integral theorem are proved. The property of rapid variability is the central argument for the results presented in this paper.

1. Introduction

Let $f : [a, +\infty) \mapsto (0, +\infty)$, $(a > 0)$, be a measurable function.

$1°$ f is *rapidly varying* in the sense of de Haan with the index of variability $+\infty$ (see e.g. [1]) if

$$\lim_{x \to +\infty} \frac{f(\lambda x)}{f(x)} = +\infty, \tag{1}$$

for all $\lambda > 1$. The class of all these functions is denoted by R_∞ and $f \in R_\infty$ holds if and only if $\lim_{x \to +\infty} \frac{\inf\{f(t) | t \in [\lambda x, +\infty)\}}{f(x)} = +\infty$ for all $\lambda > 1$ (see e.g. [2]). In the paper [2] it is shown that $\int_a^x f(t)\mathrm{d}t$ is an element of the class R_∞ (as well as $1/\int_x^{+\infty} \frac{\mathrm{d}t}{f(t)}$) whenever $f \in R_\infty$. Also, in the same paper it is shown that for all $\lambda > 1$ it holds $\int_a^x f(t)\mathrm{d}t \sim \int_{x/\lambda}^x f(t)\mathrm{d}t$, $x \to +\infty$, and that $\int_x^{+\infty} \frac{\mathrm{d}t}{f(t)} \sim \int_x^{\lambda x} \frac{\mathrm{d}t}{f(t)}$, for $x \to +\infty$, whenever $f \in R_\infty$ ($\sim$ is the relation of strong asymptotic equivalence [1]).

$2°$ f is *bounded increasing* (see e.g. [1]) if

$$\varlimsup_{x \to +\infty} \frac{\sup\{f(t) \mid t \in [x, \lambda x]\}}{f(x)} < +\infty \tag{2}$$

for all $\lambda > 1$. The class of these functions is denoted by *BI*, and it holds that a function f is bounded decreasing (denoted by $f \in BD$) if and only if the function $1/f$ is bounded increasing (see [1]). Also,

we have $ORV = BI \cap BD$ and $MR_\infty = R_\infty \cap BD$, where ORV is the class of O-regularly varying functions in the Karamata's sense (see [1]), and MR_∞ is a class of functions (a proper subclass of the class R_∞ (see [1])) for which the lower Karamata's index is $+\infty$ (see [1]).

3° f is *O-regularly varying* with continuous index function (see e.g. [3]) if

$$\lim_{\substack{x \to +\infty \\ \lambda \to 1}} \frac{f(\lambda x)}{f(x)} = 1. \tag{3}$$

The class of these functions is denoted by CRV (or IRV [3]).

2. Results

Proposition 2.1. *Let $f \in R_\infty$ and $g \in CRV$. Then:*

(a) $\int\limits_a^x f(t)g(t)\mathrm{d}t \sim g(x) \int\limits_a^x f(t)\mathrm{d}t$, *for $x \to +\infty$, whenever f and g are locally bounded on $[a, +\infty)$;*

(b) $\int\limits_x^{+\infty} \frac{g(t)}{f(t)}\mathrm{d}t \sim g(x) \int\limits_x^{+\infty} \frac{\mathrm{d}t}{f(t)}$, *for $x \to +\infty$.*

Remark 2.2. Proposition 2.1(a) is a variation (and a sort of generalization) of the result from the famous Karamata's integral theorem (see e.g. [4]). In relation to this one can find results by S. Simić [4].

Proposition 2.3. *Let $f \in R_\infty$ and $g \in ORV$. Then:*

(a) $\int\limits_a^x f(t)g(t)\mathrm{d}t \asymp g(x) \cdot \int\limits_a^x f(t)\mathrm{d}t$, *for $x \to +\infty$, whenever the functions f and g are locally bounded on $[a, +\infty)$;*

(b) $\int\limits_x^{+\infty} \frac{g(t)}{f(t)}\mathrm{d}t \asymp g(x) \cdot \int\limits_x^{+\infty} \frac{\mathrm{d}t}{f(t)}$, *for $x \to +\infty$.*

Remark 2.4. Symbol $\asymp$ represents the relation of weak asymptotic equivalence [1].

Proposition 2.5. *Let $f \in BI$. Then:*

(a) $\int\limits_a^x f(t)\mathrm{d}t$ *is an element of the class CRV whenever the function f is locally bounded on $[a, +\infty)$;*

(b) $\int\limits_x^{+\infty} \frac{\mathrm{d}t}{f(t)}$ *is an element of the class CRV whenever this integral is convergent.*

Proposition 2.6. *Let $f \in MR_\infty$ and $g \in BI$. Then:*

(a) $\int\limits_a^x f(t)g(t)\mathrm{d}t = o\left(f(x) \cdot \int\limits_a^x g(t)\mathrm{d}t\right)$, *for $x \to +\infty$, whenever the functions f and g are locally bounded on $[a, +\infty)$;*

(b) $\int\limits_x^{+\infty} \frac{1}{f(t) \cdot g(t)}\mathrm{d}t = o\left(\frac{1}{f(x)} \cdot \int\limits_x^{+\infty} \frac{\mathrm{d}t}{g(t)}\right)$, *for $x \to +\infty$, whenever the integral $\int\limits_x^{+\infty} \frac{\mathrm{d}t}{g(t)}$ is convergent.*

Remark 2.7. Symbol o is the Landau symbol [1].

The next proposition is a direct consequence of Proposition 2.6, while (a) in Proposition 2.6 is proved in [2] as a separate proposition.

Corollary 2.8. *Let $f \in MR_\infty$. Then:*

(a) $\dfrac{1}{x}\displaystyle\int_a^x f(t)\mathrm{d}t << f(x) << \dfrac{1}{x\displaystyle\int_x^{+\infty}\frac{\mathrm{d}t}{t^2 f(t)}}$, *for $x \to +\infty$, whenever the function f is locally bounded on $[a, +\infty)$;*

(b) $\dfrac{\displaystyle\int_a^x f(t)g(t)\mathrm{d}t}{\displaystyle\int_a^x g(t)\mathrm{d}t} << f(x) << \dfrac{\displaystyle\int_x^{+\infty}\frac{\mathrm{d}t}{t^2 g(t)}}{\displaystyle\int_x^{+\infty}\frac{\mathrm{d}t}{t^2 f(t)g(t)}}$, *for $x \to +\infty$, whenever the function g is non-increasing on $[a, +\infty)$.*

Remark 2.9. For positive functions s and p on $[a, +\infty)$ the relation $s(x) << p(x)$, for $x \to +\infty$, represents the fact that $s(x) = o(p(x)$ for $x \to +\infty$.

3. Proofs

Proof. [Proof of Proposition 2.1] (a) Let $\mu > 1$. Let us choose $\lambda_0 > 1$ and $x_0 \geq a\lambda_0$ such that for $\lambda \in [1, \lambda_0]$ and $x > x_0$, $\frac{1}{\sqrt{\mu}} \leq \frac{g(\frac{x}{\lambda})}{g(x)} \leq \sqrt{\mu}$ holds. Let $x_1 > x_0$ be such that

$$\int_a^x f(t)g(t)\mathrm{d}t \leq \sqrt{\mu}\int_{x/\lambda_0}^x f(t)g(t)\mathrm{d}t \quad \text{and} \quad \int_a^x f(t)\mathrm{d}t \leq \sqrt{\mu}\int_{x/\lambda_0}^x f(t)\mathrm{d}t, \qquad \text{for all } x > x_1.$$

From the fact that the functions f and $f \cdot g$ are elements of the class R_∞ and from the results given in [2] it follows that x_1 exists. So, for $x > x_1$ it follows

$$\frac{1}{\mu}\int_a^x f(t)g(t)\mathrm{d}t \leq \frac{1}{\sqrt{\mu}}\int_{x/\lambda_0}^x f(t)g(t)\mathrm{d}t \leq g(x)\int_{x/\lambda_0}^x f(t)\mathrm{d}t \leq g(x)\int_a^x f(t)\mathrm{d}t,$$

$$\sqrt{\mu}g(x)\int_{x/\lambda_0}^x f(t)\mathrm{d}t \leq \mu\int_{x/\lambda_0}^x f(t)g(t)\mathrm{d}t \leq \mu\int_a^x f(t)g(t)\mathrm{d}t.$$

Since the previous inequalities hold for all $\mu > 1$, it follows

$$\int_a^x f(t)g(t)\mathrm{d}t \sim g(x)\int_a^x f(t)\mathrm{d}t, \text{for all } x \to +\infty.$$

(b) Let $\mu > 1$. Choose $\lambda_0 > 1$ and $x_0 \geq a$ such that $\frac{1}{\sqrt{\mu}} \leq \frac{g(\lambda x)}{g(x)} \geq \sqrt{\mu}$ for $\lambda \in [1, \lambda_0]$ and $x > x_0$. Let $x_1 > x_0$ be such that

$$\int_x^{+\infty}\frac{g(t)}{f(t)}\mathrm{d}t \leq \sqrt{\mu}\int_x^{\lambda_0 x}\frac{g(t)}{f(t)}\mathrm{d}t$$

and

$$\int_x^{+\infty}\frac{\mathrm{d}t}{f(t)} \leq \sqrt{\mu}\int_x^{\lambda_0 x}\frac{\mathrm{d}t}{f(t)},$$

for all $x > x_1$. From the fact that the functions f and f/g are elements of the class R_∞ and from results in [2] it follows that x_1 exists. So, for $x > x_1$ it follows

$$\frac{1}{\mu} \int_x^{+\infty} \frac{g(t)}{f(t)} dt \le \frac{1}{\sqrt{\mu}} \int_x^{\lambda_0 x} \frac{g(t)}{f(t)} dt \le$$

$$\le g(x) \int_x^{\lambda_0 x} \frac{dt}{f(t)} \le g(x) \int_x^{+\infty} \frac{dt}{f(t)} \le$$

$$\le \sqrt{\mu} g(x) \int_x^{\lambda_0 x} \frac{dt}{f(t)} \le \mu \int_x^{\lambda_0 x} \frac{g(t)}{f(t)} \le \mu \int_x^{+\infty} \frac{g(t)}{f(t)} dt.$$

Since the previous inequality holds for all $\mu > 1$, it follows

$$\int_x^{+\infty} \frac{g(t)}{f(t)} dt \sim g(x) \int_x^{+\infty} \frac{dt}{f(t)}, \quad \text{for } x \to +\infty.$$

$\square$

Proof. [Proof od Proposition 2.3] (a) There exist $C > 0$ and $x_0 \ge 2a$ such that for all $x > x_0$ and for all $\lambda \in [1, 2]$, $\frac{2}{c} \le \frac{g(x)}{g(x/\lambda)} \le \frac{c}{2}$ holds. Also there exists $x_1 > x_0$ such that for all $x > x_1$ it holds $\int_a^x f(t) dt \le 2 \int_{x/2}^x f(t) dt$ and $\int_a^x f(t)g(t) dt \le 2 \int_{x/2}^x f(t)g(t) dt$ (because the functions f and fg are elements of the class R_∞). For $x > x_1$ we have

$$\frac{1}{c} \int_a^x f(t)g(t) dt \le \frac{2}{c} \int_{x/2}^x f(t)g(t) dt \le g(x) \int_{x/2}^x f(t) dt \le$$

$$\le g(x) \int_a^x f(t) dt \le 2g(x) \int_{x/2}^x f(t) dt \le$$

$$\le c \int_{x/2}^x f(t)g(t) dt \le c \int_a^x f(t)g(t) dt.$$

The previous inequalities prove Proposition 2.3(a).

(b) Combining ideas from the proof of Proposition 2.1(b) and the proof of Proposition 2.3(a) we get the proof of Proposition 2.3(b). Due to these obvious analogies the proof is not given step by step. $\square$

Proof. [Proof of Proposition 2.5] (a) Let us prove that $\displaystyle\lim_{\substack{x \to +\infty \\ \lambda \to 1}} \frac{\int_{x/\lambda}^x f(t) dt}{\int_a^x f(t) dt} = 0$, and this fact is equal to the part (a) of the proposition. Respectively, let us show that for all $M > 0$ there exist $x_0 \ge a$ and $\lambda_0 > 1$ such that for all $x > x_0$ and for all $\lambda \in [1, \lambda_0]$ it is true $\int_a^x f(t) dt > M \int_{x/\lambda}^x f(t) dt$. This procedure (observe only the values of λ to the right side of 1) is sufficient since $f(t) > 0$ for $t \ge a$, so the function $\int_a^x f(t) dt$ is increasing.

Thus: let $M > 0$ and let $\lambda_1 > 1$, $x_0 > a\lambda_1$ and $K > 0$ be such that for $t > \frac{x_0}{\lambda_1}$ and $\mu \in [1, \lambda_1]$ one has $f(\mu t) < K f(t)$. Let $n \in \mathbb{N}$ be such that $n > M K \lambda_1$ and let $\lambda_0 = \sqrt[n]{\lambda_1}$. For $x > x_0$ and $\lambda \in [1, \lambda_0]$ we have

$$\int_a^x f(t)\,dt \geq \int_{x/\lambda^n}^x f(t)\,dt = \sum_{k=0}^{n-1} \int_{x/\lambda^{k+1}}^{x/\lambda^k} f(t)\,dt =$$

$$= \sum_{k=0}^{n-1} \frac{1}{\lambda^k} \int_{x/\lambda}^x f\left(\frac{t}{\lambda^k}\right)dt > \sum_{k=0}^{n-1} \frac{1}{\lambda^k} \int_{x/\lambda}^x \frac{1}{K} f(t)\,dt >$$

$$> n \cdot \frac{1}{\lambda_1} \cdot \frac{1}{K} \int_{x/\lambda}^x f(t)\,dt > M \int_{x/\lambda}^x f(t)\,dt.$$

This completes the proof of (a).

(b) Let us show that $\lim\limits_{\substack{x \to +\infty \\ \lambda \to 1}} \dfrac{\displaystyle\int_x^{\lambda x} \frac{dt}{f(t)}}{\displaystyle\int_x^{+\infty} \frac{dt}{f(t)}} = 0$, and this is equal to the part (b) of the proposition. Respectively,

let us show that for all $M > 0$ there exist $x_0 > a$ and $\lambda_0 > 1$ such that for all $x > x_0$ and for all $\lambda \in [1, \lambda_0]$ the inequality $\int_x^{+\infty} \frac{dt}{f(t)} > M \int_x^{\lambda x} \frac{dt}{f(t)}$ holds. This procedure (we observe only the values of λ to the right side of 1) is

sufficient since $f(t) > 0$ for $t \geq a$, so the function $\int_x^{+\infty} \frac{dt}{f(t)}$ is decreasing.

Thus: let $M > 0$ and let $\lambda_1 > 1$, $x_0 > a$ and $K > 0$ be such that for $t > x_0$ and $\mu \in [1, \lambda_1]$ it holds $f(\mu t) < K f(t)$. Let $n \in \mathbb{N}$ be such that $n > MK$ and let $\lambda_0 = \sqrt[n]{\lambda_1}$. For $x > x_0$ and $\lambda \in [1, \lambda_0]$ we have

$$\int_x^{+\infty} \frac{d(t)}{f(t)} > \int_x^{\lambda^n x} \frac{d(t)}{f(t)} = \sum_{k=0}^{n-1} \int_{\lambda^k x}^{\lambda^{k+1} x} \frac{dt}{f(t)} = \sum_{k=0}^{n-1} \lambda^k \int_x^{\lambda x} \frac{dt}{f(\lambda^k t)} >$$

$$> \sum_{k=0}^{n-1} \lambda^k \int_x^{\lambda x} \frac{1}{K} \frac{dt}{f(t)} > n \cdot \frac{1}{K} \cdot \int_x^{\lambda x} \frac{dt}{f(t)} > M \int_x^{\lambda x} \frac{dt}{f(t)}.$$

With this (b) is proved. $\square$

Proof. [Proof of Proposition 2.6] (a) Let us show that for all $M > 0$ we have $M \int_a^x f(t)g(t)\,dt < f(x) \int_a^x g(t)\,dt$, for sufficiently large x. Since $f \in BD$, it follows that for some $C < +\infty$ it holds $\varlimsup\limits_{x \to +\infty} \frac{\sup\{f(t)|t \in [x/2, x]\}}{f(x)} < C$. Also, it holds $C > 1$. According to Proposition 2.5(a) there exist $x_0 \geq a$ and $\lambda_0 > 1$ such that for all $x > x_0$ and for all $\lambda \in [1, \lambda_0]$ it holds $\int_{x/\lambda}^x g(t)\,dt < \frac{1}{2MC} \int_a^x g(t)\,dt$. Also, for the same λ we have (take $\lambda_0 < 2$):

(1_a°) $\frac{f(x)}{\sup\{f(t)|t \in [a, \frac{x}{\lambda}]\}} > 2M$ for sufficiently large x (because $f \in R_\infty$), so for the same λ and x we have

$$M \int_a^{x/\lambda} f(t)g(t)\,dt < \frac{1}{2} f(x) \int_a^{x/\lambda} g(t)\,dt \leq \frac{1}{2} f(x) \int_a^x g(t)\,dt;$$

(2°_a) $\sup\{f(t) \mid t \in [x/\lambda, x]\} < C \cdot f(x)$ for sufficiently large x, so for the same λ and x (also it should be $x \geq x_0$) it holds

$$M \int\limits_{x/\lambda}^{x} f(t)g(t)\mathrm{d}t < MCf(x) \int\limits_{x/\lambda}^{x} g(t)\mathrm{d}t < \frac{1}{2}f(x) \int\limits_{a}^{x} g(t)\mathrm{d}t.$$

For the same λ and sufficiently large x, by using (1°_a) and (2°_a), (a) can be obtained.

(b) We prove that for all $M > 0$ it holds $M \int\limits_{x}^{+\infty} \frac{\mathrm{d}t}{f(t)g(t)} < \frac{1}{f(x)} \int\limits_{x}^{+\infty} \frac{\mathrm{d}t}{g(t)}$, for sufficiently large x. Since $f \in BD$, then for some $c > 0$, $\varliminf\limits_{x\to+\infty} \frac{\inf\{f(t)|t\in[x,2x]\}}{f(x)} > c$ holds. Also, $c < 1$.

According to Proposition 2.5(b), there exist $x_0 \geq a$ and $\lambda_0 > 1$ such that for all $x > x_0$ and for all $\lambda \in [1, \lambda_0]$ it holds $\int\limits_{x}^{\lambda x} \frac{\mathrm{d}t}{g(t)} < \frac{c}{2M} \int\limits_{x}^{+\infty} \frac{\mathrm{d}t}{g(t)}$.

Also, for the same λ (take $\lambda_0 < 2$) we have

(1°_b) $\frac{\inf\{f(t)|t\in[\lambda x,+\infty)\}}{f(x)} > 2M$ for sufficiently large x (because $f \in R_\infty$), so for the same λ and x it holds

$$M \int\limits_{\lambda x}^{+\infty} \frac{\mathrm{d}t}{f(t)g(t)} < \frac{1}{2}\frac{1}{f(x)} \int\limits_{\lambda x}^{+\infty} \frac{\mathrm{d}t}{g(t)} < \frac{1}{2}\frac{1}{f(x)} \int\limits_{x}^{+\infty} \frac{\mathrm{d}t}{g(t)};$$

(2°_b) $\sup\{\frac{1}{f(t)} \mid t \in [x, \lambda x]\} < \frac{1}{c} \cdot \frac{1}{f(x)}$ for sufficiently large x, so for the same λ and x (also it should be $x \geq x_0$) it holds

$$M \int\limits_{x}^{\lambda x} \frac{\mathrm{d}t}{f(t)g(t)} < \frac{M}{c} \cdot \frac{1}{f(x)} \int\limits_{x}^{\lambda x} \frac{\mathrm{d}t}{g(t)} < \frac{1}{2}\frac{1}{f(x)} \int\limits_{x}^{+\infty} \frac{\mathrm{d}t}{g(t)}.$$

For the same λ and sufficiently large x, by using (1°_b) and (2°_b), (b) is obtained.

$\square$

References

[1] N.H. Bingham, C.M. Goldie, J.L. Teugels, Regular Variation, Cambridge University Press, Cambridge, 1987.
[2] N. Elez, D. Djurčić, Some properties of rapidly varying functions, J. Math. Anal. Appl. 401 (2013) 888–895.
[3] D. Djurčić, A. Torgašev, Strong asymptotic equivalence and inversion of functions in the class K_c, J. Math. Anal. Appl. 255 (2001) 383–390.
[4] S. Simić, On a new class of rapidly varying functions, Mathematica 47(70) (2005) 217–222.