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Bounding the Paired-Domination Number of a Tree in Terms of its Annihilation Number

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Abstract. A paired-dominating set of a graph $G = (V, E)$ with no isolated vertex is a dominating set of vertices whose induced subgraph has a perfect matching. The paired-domination number of G, denoted by $\gamma_{pr}(G)$, is the minimum cardinality of a paired-dominating set of G. The annihilation number $a(G)$ is the largest integer k such that the sum of the first k terms of the non-decreasing degree sequence of G is at most the number of edges in G. In this paper, we prove that for any tree T of order $n \geq 2$, $\gamma_{pr}(T) \leq \frac{4a(T)+2}{3}$ and we characterize the trees achieving this bound.

1. Introduction

In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V(G)$, the *open neighborhood* $N_G(v) = N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = \deg(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. For a subset $S \subseteq V(G)$, we let

$$\sum(S, G) = \sum_{v \in S} \deg_G(v).$$

A *leaf* of a tree T is a vertex of degree 1, a *support vertex* is a vertex adjacent to a leaf and a *strong support vertex* is a vertex adjacent to at least two leaves. For a vertex v in a rooted tree T, let $C(v)$ denote the set of children of v. Let $D(v)$ denote the set of descendants of v and $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v. We write P_n for a path of order n.

A paired-dominating set, abbreviated PDS, of a graph G is a set S of vertices of G such that every vertex is adjacent to some vertex in S and the subgraph $G[S]$ induced by S contains a perfect matching (not necessary induced). Every graph without isolated vertices has a PDS since the end-vertices of any maximal matching form such a set. The paired-domination number of G, denoted by $\gamma_{pr}(G)$, is the minimum cardinality of a PDS. A PDS of cardinality $\gamma_{pr}(G)$ is called a $\gamma_{pr}(G)$-set. Paired-domination was introduced by Haynes and

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Slater (1995, 1998) as a model for assigning backups to guards for security purposes, since this time many results have been obtained on this parameter (see for instance [2–5, 7–11].

Let $d_1 \leq d_2 \leq \ldots \leq d_n$ be the degree sequence of a graph G. Pepper [13] defined the annihilation number of G, denoted $a(G)$, to be the largest integer k such that the sum of the first k terms of the degree sequence is at most half the sum of the degrees in the sequence. Equivalently, the annihilation number is the largest integer k such that

$$\sum_{i=1}^{k} d_i \leq \sum_{i=k+1}^{n} d_i.$$

We observe that if G has m edges and annihilation number k, then $\sum_{i=1}^{k} d_i \leq m$. As an immediate consequence of the definition of the annihilation number, we observe that

$$a(G) \geq \lfloor \frac{n}{2} \rfloor. \quad (1)$$

The relation between annihilation number and some graph parameters have been studied by several authors (see for example [1, 6, 12]).

If G is a connected graph of order $n \geq 6$ with $\delta(G) \geq 2$, then it is known ([8]) that $\gamma_{pr}(G) \leq \frac{2n}{3}$. Hence if G is a connected graph of order $n \geq 6$ with minimum degree at least 2, then

$$\gamma_{pr}(G) \leq \frac{4a(G) + 2}{3}.$$

Our purpose in this paper is to establish the above upper bound on the paired-domination number for trees.

We make use of the following results in this paper.

Proposition A. ([8]) For $n \geq 3$,

$$\gamma_{pr}(P_n) = 2\left\lceil \frac{n}{4} \right\rceil.$$

Proposition B. For $n \geq 2$,

$$a(P_n) = \left\lceil \frac{n}{2} \right\rceil.$$

Corollary 1.1. For $n \geq 3$,

$$\gamma_{pr}(P_n) \leq \frac{4a(P_n)}{3}$$

with equality if and only if $T = P_5$ or P_6.

2. Main result

A *subdivision* of an edge uv is obtained by replacing the edge uv with a path uwv, where w is a new vertex. The *subdivision graph* $S(G)$ is the graph obtained from G by subdividing each edge of G. The subdivision star $S(K_{1,t})$ for $t \geq 2$, is called a *healthy spider* S_t. A *wounded spider* S_t is the graph formed by subdividing at most $t-1$ of the edges of a star $K_{1,t}$ for $t \geq 2$. Note that stars are wounded spiders. A *spider* is a healthy or wounded spider.

Lemma 2.1. If T is a spider, then $\gamma_{pr}(T) \leq \frac{4a(T)+2}{3}$ with equality if and only if T is a healthy spider S_t, where t is odd.

Proof. First let $T = S_t$ be a healthy spider for some $t \geq 2$. Then obviously $\gamma_{pr}(T) = 2t$. If t is even, then $a(T) = t + \frac{t}{2}$ and hence $\gamma_{pr}(T) = 2t = \frac{4a(T)}{3} < \frac{4a(T)+2}{3}$. If t is odd, then $a(T) = t + \frac{t-1}{2}$ and hence $\gamma_{pr}(T) = 2t = \frac{4a(T)+2}{3}$.

Now let T be a wounded spider obtained from $K_{1,t}$ $(t \geq 2)$ by subdividing $0 \leq s \leq t-1$ edges. If $s = 0$, then T is a star and we have $\gamma_{pr}(T) = 2$ and $a(T) = t$. Hence $\gamma_{pr}(T) = 2 < \frac{4a(T)+2}{3}$. Suppose $s > 0$. If $(t, s) = (2, 1)$, then $T = P_4$ and the result follows from Corollary 1.1. If $(t, s) \neq (2, 1)$, then $\gamma_{pr}(T) = 2s$ and $a(T) = t + \lfloor \frac{s}{2} \rfloor$. It follows that $\gamma_{pr}(T) = 2s < \frac{4a(T)+2}{3}$. This completes the proof. □

Theorem 2.2. If T is a tree of order $n \geq 2$, then

$$\gamma_{pr}(T) \leq \frac{4a(T) + 2}{3}.$$

This bound is sharp for healthy spider S_t, where t is odd.

Proof. The proof is by induction on n. The statement holds for all trees of order $n = 2, 3, 4$. For the inductive hypothesis, let $n \geq 5$ and suppose that for every nontrivial tree T of order less than n the result is true. Let T be a tree of order n. We may assume that T is not a path for otherwise the result follows by Corollary 1.1. If $\mathrm{diam}(T) = 2$, then T is a star and so $\gamma_{pr}(T) < \frac{4a(T)+2}{3}$ by Lemma 2.1. If $\mathrm{diam}(T) = 3$, then T is a double star $S(r, s)$. In this case, $a(T) = r + s \geq 3$ and $\gamma_{pr}(T) = 2$, implying that $\gamma_{pr}(T) < \frac{4a(T)+2}{3}$. Hence we may assume that $\mathrm{diam}(T) \geq 4$.

In what follows, we will consider trees T' formed from T by removing a set of vertices. For such a tree T' of order n', let $d'_1, d'_2, \ldots, d'_{n'}$ be the non-decreasing degree sequence of T', and let S' be a set of vertices corresponding to the first $a(T')$ terms in the degree sequence of T'. We denote the size of T' by m'. We proceed further with a series of claims that we may assume satisfied by the tree.

Claim 1. *T has no strong support vertex such as u that the graph obtained from T by removing u and the leaves adjacent to u is connected.*

Let T have a strong support vertex u such that the graph obtained from T by removing u and the leaves adjacent to u is connected. Suppose w is a vertex in T with maximum distance from u. Root T at w and let v be the parent of u. Assume $T' = T - T_u$. It is easy to see that $\gamma_{pr}(T) \leq \gamma_{pr}(T') + 2$. If $v \notin S'$, then $\sum(S', T) = \sum(S', T')$ and if $v \in S'$, then $\sum(S', T) = \sum(S', T') + 1$. Thus, $\sum(S', T) \leq \sum(S', T') + 1 \leq m' + 1 \leq m - 2$. Let z_1, z_2 be two leaves adjacent to u and assume $S = S' \cup \{z_1, z_2\}$. Then $\sum(S, T) = \sum(S', T) + 2 \leq m$ implying that $a(T) \geq a(T') + 2$. By inductive hypothesis, we obtain

$$\gamma_{pr}(T) \leq \gamma_{pr}(T') + 2 \leq \frac{4a(T') + 2}{3} + 2 \leq \frac{4(a(T) - 2) + 2}{3} + 2 < \frac{4a(T) + 2}{3},$$

as desired. (■)

Let $v_1 v_2 \ldots v_D$ be a diametral path in T and root T at v_D. If $\mathrm{diam}(T) = 4$, then T is a spider by Claim 1, and the result follows by Lemma 2.1. Assume $\mathrm{diam}(T) \geq 5$. It follows from Claim 1 that T_{v_3} is a spider.

Claim 2. $\deg(v_3) \leq 3$.

Let $\deg(v_3) \geq 4$. We consider three cases.

Case 2.1 T_{v_3} is a healthy spider S_t, where t is even.

Assume $T' = T - T_{v_3}$. Then obviously $\gamma_{pr}(T) \leq \gamma_{pr}(T') + 2t$. As above, we have $\sum(S', T) \leq \sum(S', T') + 1$. Let S be the set obtained from S' by adding all the leaves and half of the support vertices of T_{v_3}. Then $\sum(S, T) \leq m$. Therefore, $a(T) \geq |S| = |S'| + \frac{3t}{2} = a(T') + \frac{3t}{2}$. By inductive hypothesis, we obtain

$$\gamma_{pr}(T) \leq \gamma_{pr}(T') + 2t \leq \frac{4a(T') + 2}{3} + 2t \leq \frac{4(a(T) - \frac{3t}{2}) + 2}{3} + 2t = \frac{4a(T) + 2}{3}.$$

Case 2.2 T_{v_3} is a healthy spider S_t, where t is odd.

First let $\deg(v_4) = 2$. In this case, assume $T' = T - T_{v_4}$. Then obviously $\gamma_{pr}(T) \leq \gamma_{pr}(T') + 2t$. As above, we have $\sum(S', T) \leq \sum(S', T') + 1$. Let S be the set obtained from S' by adding all the leaves and $\frac{t+1}{2}$ of the support vertices of T_{v_3}. Then $\sum(S,T) \le m$ and hence $a(T) \ge |S| = |S'| + \frac{3t+1}{2} = a(T') + \frac{3t+1}{2}$. It follows from inductive hypothesis that

$$\gamma_{pr}(T) \le \gamma_{pr}(T') + 2t \le \frac{4a(T')+2}{3} + 2t \le \frac{4(a(T) - \frac{3t+1}{2}) + 2}{3} + 2t < \frac{4a(T)+2}{3}.$$

Now let $\deg(v_4) \ge 3$. Assume $T' = T - T_{v_3}$. Then $\gamma_{pr}(T) \le \gamma_{pr}(T') + 2t$. If $v_4 \notin S'$, then let S be the set obtained from S' by adding all the leaves and $\frac{t+1}{2}$ of the support vertices of T_{v_3}. If $v_4 \in S'$, then let S be the set obtained from $S' - \{v_4\}$ by adding all the leaves and $\frac{t+3}{2}$ of the support vertices of T_{v_3}. Then $\sum(S,T) \le m$ and hence $a(T) \ge |S| = |S'| + \frac{3t+1}{2} = a(T') + \frac{3t+1}{2}$. By inductive hypothesis, we obtain $\gamma_{pr}(T) < \frac{4a(T)+2}{3}$.

Case 2.3 T_{v_3} is a wounded spider obtained from $K_{1,t}$ by subdividing $1 \le s \le t-1$ edges.
As in Lemma 2.1, we can see that $\gamma_{pr}(T) \le \gamma_{pr}(T') + 2s$ and $a(T) \ge a(T') + t + \lfloor \frac{s}{2} \rfloor$. It follows from inductive hypothesis that

$$\gamma_{pr}(T) \le \gamma_{pr}(T') + 2s \le \frac{4a(T')+2}{3} + 2s \le \frac{4(a(T) - t - \lfloor \frac{s}{2} \rfloor) + 2}{3} + 2s < \frac{4a(T)+2}{3}. \qquad (\blacksquare)$$

Claim 3. $\deg(v_3) = 2$.
Assume that $\deg(v_3) = 3$. First let v_3 is adjacent to a support vertex $z_2 \ne v_2$. Suppose z_1 is the leaf adjacent to z_2 and let $T' = T - T_{v_3}$. Then every $\gamma_{pr}(T')$-set can be extended to a PDS of T by adding v_1, v_2, z_1, z_2, implying that $\gamma_{pr}(T) \le \gamma_{pr}(T') + 4$. If $v_4 \notin S'$, then $\sum(S',T) = \sum(S',T')$ and if $v_4 \in S'$, then $\sum(S',T) = \sum(S',T') + 1$. Thus, $\sum(S',T) \le \sum(S',T') + 1 \le m' + 1 = m - 4$. Let $S = S' \cup \{v_1, v_2, z_1\}$. Then $\sum(S,T) = \sum(S',T) + \deg_T(v_1) + \deg_T(v_2) + \deg_T(z_1) \le m$. Therefore, $a(T) \ge |S| = |S'| + 3 = a(T') + 3$. It follows from inductive hypothesis that

$$\gamma_{pr}(T) \le \gamma_{pr}(T') + 4 \le \frac{4a(T')+2}{3} + 4 \le \frac{4(a(T)-3)+2}{3} + 4 = \frac{4a(T)+2}{3}.$$

Now let v_3 is adjacent to a leaf w. Suppose $T' = T - T_{v_3}$. Then every $\gamma_{pr}(T')$-set can be extended to a PDS of T by adding v_3 and v_2, implying that $\gamma_{pr}(T) \le \gamma_{pr}(T') + 2$. Now let $S = S' \cup \{v_1, v_2\}$. Then we have $\sum(S,T) = \sum(S',T) + \deg_T(v_1) + \deg_T(v_2) \le m' + 4 = m$, which implies that $a(T) \ge a(T') + 2$. By inductive hypothesis,

$$\gamma_{pr}(T) \le \gamma_{pr}(T') + 2 \le \frac{4a(T')+2}{3} + 2 \le \frac{4(a(T)-2)+2}{3} + 2 < \frac{4a(T)+2}{3}. \qquad (\blacksquare)$$

Claim 4. $\deg(v_4) = 2$.
Assume that $\deg(v_4) \ge 3$. Let $T' = T - T_{v_3}$. Then every $\gamma_{pr}(T')$-set can be extended to a PDS of T by adding v_2 and v_3. Thus $\gamma_{pr}(T) \le \gamma_{pr}(T') + 2$. Suppose that $v_4 \notin S'$. Then $\sum(S',T) = \sum(S',T')$. In this case, let $S = S' \cup \{v_1, v_2\}$. Then $\sum(S,T) = \sum(S',T) + \deg_T(v_1) + \deg_T(v_2) \le m' + 3 = m$, implying that $a(T) \ge |S| = |S'| + 2 = a(T') + 2$. Applying inductive hypothesis we obtain

$$\gamma_{pr}(T) \le \gamma_{pr}(T') + 2 \le \frac{4a(T')+2}{3} + 2 \le \frac{4(a(T)-2)+2}{3} + 2 < \frac{4a(T)+2}{3},$$

as desired. Now we may assume $v_4 \in S'$. In this case, let $S = (S' - \{v_4\}) \cup \{v_1, v_2, v_3\}$. Since $\deg_T(v_3) = 2 \le \deg_{T'}(v_4)$, we have $\sum(S,T) = \sum(S',T) - \deg_{T'}(v_4) + \deg_T(v_1) + \deg_T(v_2) + \deg_T(v_3) \le m$. Therefore, $a(T) \ge |S| = |S'| + 2 = a(T') + 2$. It follows from inductive hypothesis that

$$\gamma_{pr}(T) \le \gamma_{pr}(T') + 2 \le \frac{4a(T')+2}{3} + 2 \le \frac{4(a(T)-2)+2}{3} + 2 < \frac{4a(T)+2}{3},$$

as desired. $(\blacksquare)$

We now return to the proof of theorem. Let $T' = T - T_{v_4}$, and hence $m' = m - 4$. Every $\gamma_{pr}(T')$-set can be extended to a PDS of T by adding v_3, v_2, which implies that $\gamma_{pr}(T) \le \gamma_{pr}(T') + 2$. Let $S = S' \cup \{v_1, v_2\}$.

Then $\sum(S,T) = \sum(S',T) + \deg_T(v_1) + \deg_T(v_2) \le m' + 4 = m$, implying that $a(T) \ge |S| = |S'| + 2 = a(T') + 2$. Applying inductive hypothesis,

$$\gamma_{pr}(T) \le \gamma_{pr}(T') + 2 \le \frac{4a(T') + 2}{3} + 2 \le \frac{4(a(T) - 2) + 2}{3} + 2 < \frac{4a(T) + 2}{3},$$

as desired. This completes the proof. □

To characterize all trees achieving the bound in Theorem 2.2 we start with the following propositions.

Proposition 2.3. Let T be a tree of order $n \ge 2$ with diam$(T) \le 4$. Then $\gamma_{pr}(T) = \frac{4a(T)+2}{3}$ if and only if $T = P_2$ or T is a healthy spider S_t, where t is odd.

Proof. If $T = P_2$, then obviously $\gamma_{pr}(T) = 2$ and $a(T) = 1$. Hence, $\gamma_{pr}(T) = \frac{4a(T)+2}{3}$. If T is a healthy spider S_t where t is odd, then the result follows by Lemma 2.1.

Conversely, let $\gamma_{pr}(T) = \frac{4a(T)+2}{3}$. If diam$(T) = 1$, then $T = P_2$ and we are done. If diam$(T) = 2$, then T is a star and it follows from Lemma 2.1 that $\gamma_{pr}(T) < \frac{4a(T)+2}{3}$, a contradiction. Suppose diam$(T) = 3$. Then T is a double star $S(r,s)$. In this case, $a(T) = r + s$ and $\gamma_{pr}(T) = 2$. If $r + s = 2$, then $T = P_4$, which leads to a contradiction by Corollary 1.1. If $r + s \ge 3$, then we have $\gamma_{pr}(T) < \frac{4a(T)+2}{3}$, which is a contradiction again. Finally, let diam$(T) = 4$. By the proof of Claim 1 in Theorem 2.2, we may assume that the degree of each support vertex on a diametral path of T is two and hence T is a spider. Since $\gamma_{pr}(T) = \frac{4a(T)+2}{3}$, it follows from Lemma 2.1 that T is a healthy spider S_t, where t is odd. This completes the proof. □

Proposition 2.4. If T is a tree of order n with diam$(T) = 5$, then $\gamma_{pr}(T) < \frac{4a(T)+2}{3}$

Proof. Let $v_1v_2\dots v_6$ be a diametral path in T and root T at v_6 (at v_1, respectively). By a closer look at the proof of Theorem 2.2 we may assume T_{v_3} and T_{v_4} are spiders S_t (if the root is v_6) and S_r (if the root is v_1) for some even integers t and r, respectively. It is easy to see that $\gamma_{pr}(T) = 2t + 2r$ and $a(T) = \frac{3t}{2} + \frac{3r}{2}$ and hence $\gamma_{pr}(T) \le \frac{4a(T)}{3} < \frac{4a(T)+2}{3}$, as desired. □

Proposition 2.5. If T is a tree of order n with diam$(T) = 6$, then $\gamma_{pr}(T) < \frac{4a(T)+2}{3}$

Proof. Let $v_1v_2\dots v_7$ be a diametral path in T and root T at v_7 (at v_1, respectively). As in Proposition 2.4, we may assume T_{v_3} and T_{v_5} are healthy spider S_t (if the root is v_7) and S_r (if the root is v_1) for some even integers t and r, respectively. Let u_i (w_j, respectively) be the leaves of T_{v_3} (T_{v_5}, respectively) and u'_i (w'_j, respectively) be the support vertices of T_{v_3} (T_{v_5}, respectively). If $\deg(v_4) = 2$, then obviously $\gamma_{pr}(T) = 2t + 2r$ and $a(T) = \frac{3t}{2} + \frac{3r}{2} + 1$ and hence $\gamma_{pr}(T) < \frac{4a(T)}{3}$. If $\deg(v_4) \ge 4$, then let $T' = T - (T_{v_3} \cup T_{v_5})$. Clearly $\gamma_{pr}(T) \le \gamma_{pr}(T') + 2t + 2r$. Let $S = S' \cup (\{u_i \mid 1 \le i \le t\} \cup \{u'_i \mid 1 \le i \le \frac{t}{2} + 1\} \cup \{w_j \mid 1 \le j \le r\} \cup \{w'_j \mid 1 \le i \le \frac{r}{2}\})$ if $v_4 \notin S'$ and $S = (S' - \{v_4\}) \cup (\{u_i \mid 1 \le i \le t\} \cup \{u'_i \mid 1 \le i \le \frac{t}{2} + 1\} \cup \{w_j \mid 1 \le j \le r\} \cup \{w'_j \mid 1 \le j \le \frac{r}{2} + 1\})$ when $v_4 \in S'$. It is easy to see that $\sum(S,T) \le m$, implying that $a(T) \ge a(T') + \frac{3t}{2} + \frac{3s}{2} + 1$. It follows from Theorem 2.2 that

$$\begin{array}{rcccl} \gamma_{pr}(T) & \le & \gamma_{pr}(T') + 2r + 2s & \le & \frac{4a(T')+2}{3} + 2r + 2s \\ & \le & \frac{4(a(T) - \frac{3t}{2} - \frac{3s}{2} - 1)+2}{3} + 2r + 2s & < & \frac{4(a(T)+2}{3}. \end{array}$$

Now let $\deg(v_4) = 3$. If v_4 is adjacent to a leaf, a support vertex whose all neighbors are leaves except v_4, or there is a path $v_4z_1z_2z_3$ in T such that all neighbors of z_1 except v_4 and z_2 are leaves, then obviously $\gamma_{pr}(T) = 2r + 2s + 2$ and $a(T) \ge \frac{3t}{2} + \frac{3s}{2} + 2$. Hence $\gamma_{pr}(T) < \frac{4a(T)+2}{3}$. Thus as above, we may assume T_{z_1} is a healthy spider S_k for some even integer k. In this case, we can see that $\gamma_{pr}(T) = 2t + 2r + 2k$ and $a(T) = \frac{3(t+r+k)}{2} + 1$ implying that $\gamma_{pr}(T) < \frac{4a(T)+2}{3}$. This completes the proof. □

Proposition 2.6. If T is a tree of order n with diam$(T) \ge 7$, then $\gamma_{pr}(T) < \frac{4a(T)+2}{3}$

Proof. Let $v_1v_2\dots v_D$ be a diametral path in T and root T at v_D (at v_1, respectively). As in Proposition 2.4 we may assume T_{v_3} and $T_{v_{D-2}}$ are spiders S_t (if the root is v_D) and S_r (if the root is v_1) for some even integers t and r, respectively. Suppose u_i $(1 \le i \le t)$ are the leaves of T_{v_3} and u'_i is the support vertex of u_i in T_{v_3} for each i. Similarly, assume w_j $(1 \le j \le r)$ are the leaves of $T_{v_{D-2}}$ and w'_j is the support vertex of w_j in $T_{v_{D-2}}$ for each j.

First let $\deg(v_4) = \deg(v_{D-3}) = 2$. If $\mathrm{diam}(T) = 7$, then $v_{D-3} = v_5$ and it is easy to see that $\gamma_{pr}(T) = 2t + 2r$ and $a(T) = \frac{3(t+r)}{2} + 1$. Hence, $\gamma_{pr}(T) < \frac{4a(T)+2}{3}$. If $\mathrm{diam}(T) = 8$ and $\deg(v_5) \ge 3$ or $\mathrm{diam}(T) \ge 9$, then let $T' = T - (T_{v_4} \cup T_{v_{D-3}})$. It is easy to check that $\gamma_{pr}(T) \le \gamma_{pr}(T') + 2t + 2r$ and $a(T) \ge a(T') + \frac{3(t+r)}{2} + 1$. It follows from Theorem 2.2 that $\gamma_{pr}(T) < \frac{4a(T)+2}{3}$. Assume now that $\mathrm{diam}(T) = 8$ and $\deg(v_5) = 2$. Then one can see that $\gamma_{pr}(T) = 2t + 2r + 2$ and $a(T) = \frac{3(t+r)}{2} + 2$ and so $\gamma_{pr}(T) < \frac{4a(T)+2}{3}$.

Now let $\deg(v_4) \ge 3$ and $\deg(v_{D-3}) = 2$ (the case $\deg(v_4) = 2$ and $\deg(v_{D-3}) \ge 3$ is similar). Let $T' = T - (T_{v_3} \cup T_{v_{D-3}})$. It is easy to see that $\gamma_{pr}(T) \le \gamma_{pr}(T') + 2t + 2r$. If $v_4 \notin S'$, then let $S = S' \cup (\{u_i \mid 1 \le i \le t\} \cup \{u'_i \mid 1 \le i \le \frac{t}{2} + 1\} \cup \{w_j \mid 1 \le j \le r\} \cup \{w'_j \mid 1 \le j \le \frac{r}{2}\})$. If $v_4 \in S'$, then let $S = (S' - \{v_4\}) \cup (\{u_i \mid 1 \le i \le t\} \cup \{u'_i \mid 1 \le i \le \frac{t}{2} + 1\} \cup \{w_j \mid 1 \le j \le r\} \cup \{w'_j \mid 1 \le j \le \frac{r}{2} + 1\})$. In each case, we have $\sum(S,T) \le m$, implying that $a(T) \ge a(T') + \frac{3(t+r)}{2} + 1$, hence by Theorem 2.2, $\gamma_{pr}(T) < \frac{4a(T)+2}{3}$.

Finally, let $\min\{\deg(v_4), \deg(v_{D-3})\} \ge 3$. Consider two cases.

Case 1. Assume that $t \ge 4$ (the case $r \ge 4$ is similar).
Let $T' = T - (T_{v_3} \cup T_{v_{D-2}})$. Then every $\gamma_{pr}(T')$-set can be extended to a PDS of T by adding all leaves and their support vertices of $T_{v_3} \cup T_{v_{D-2}}$, hence $\gamma_{pr}(T) \le \gamma_{pr}(T') + 2t + 2r$. If $v_4, v_{D-3} \notin S'$, then let $S = S' \cup (\{u_i \mid 1 \le i \le t\} \cup \{u'_i \mid 1 \le i \le \frac{t}{2} + 1\} \cup \{w_j \mid 1 \le j \le r\} \cup \{w'_j \mid 1 \le j \le \frac{r}{2}\})$. If $v_4, v_{D-3} \in S'$, then let $S = (S' - \{v_4, v_{D-3}\}) \cup (\{u_i \mid 1 \le i \le t\} \cup \{u'_i \mid 1 \le i \le \frac{t}{2} + 2\} \cup \{w_j \mid 1 \le j \le r\} \cup \{w'_j \mid 1 \le j \le \frac{r}{2} + 1\})$. Finally, if $v_4 \in S'$ and $v_{D-3} \notin S'$ (the case $v_4 \notin S'$ and $v_{D-3} \in S'$ is similar), then let $S = (S' - \{v_4\}) \cup (\{u_i \mid 1 \le i \le t\} \cup \{u'_i \mid 1 \le i \le \frac{t}{2} + 2\} \cup \{w_j \mid 1 \le j \le r\} \cup \{w'_j \mid 1 \le j \le \frac{r}{2}\})$. In all cases, we have $\sum(S,T) \le m$, implying that $a(T) \ge a(T') + \frac{3(t+r)}{2} + 1$. By Theorem 2.2, we have

$$\gamma_{pr}(T) \le \gamma_{pr}(T') + 2t + 2r \le \frac{4a(T')+2}{3} + 2t + 2r \le \frac{4(a(T) - \frac{3(t+r)}{2} - 1) + 2}{3} + 2t + 2r < \frac{4a(T)+2}{3}.$$

Case 2. Assume that $t = r = 2$.
Consider two subcases.

Subcase 2.1 $\max\{\deg(v_4), \deg(v_{D-3})\} \ge 4$.
Let $T' = T - (T_{v_3} \cup T_{v_{D-2}})$. Then clearly $\gamma_{pr}(T) \le \gamma_{pr}(T') + 8$. If $v_4, v_{D-3} \notin S'$, then let $S = S' \cup \{u_1, u_2, w_1, w_2, u'_1, u'_2, w'_1\}$. If $v_4, v_{D-3} \in S'$, then let $S = (S' - \{v_4, v_{D-3}\}) \cup (\{v_3\} \cup \{u_1, u_2, w_1, w_2, u'_1, u'_2, w'_1, w'_2\})$. Finally, if $v_4 \in S'$ and $v_{D-3} \notin S'$ (the case $v_4 \notin S'$ and $v_{D-3} \in S'$ is similar), then let $S = (S' - \{v_4\}) \cup \{u_i, w_i, u'_i, w'_i \mid i = 1, 2\}$. In all cases, we have $\sum(S,T) \le m$, implying that $a(T) \ge a(T') + \frac{3(t+r)}{2} + 7$. It follows from Theorem 2.2 that $\gamma_{pr}(T) < \frac{4a(T)+2}{3}$.

Subcase 2.2 $\deg(v_4) = \deg(v_{D-3}) = 3$.
If v_4 is adjacent to a support vertex or there is a path $v_4z_1z_2z_3$ in T such that all neighbors of z_1 except z_2 and v_4 are leaves and $\deg(z_3) = 1$, then let $T' = T - T_{v_4}$. It is easy to see that $\gamma_{pr}(T) \le \gamma_{pr}(T') + 6$ and $a(T) \ge a(T') + 5$. It follows from Theorem 2.2 that $\gamma_{pr}(T) < \frac{4a(T)+2}{3}$. Let $z \in N(v_4) - \{v_5, v_3\}$. If T_z is a spider, then we may assume $T_z = P_5$, for otherwise the result follows as above. Let $T' = T - T_{v_4}$. Then clearly $\gamma_{pr}(T) \le \gamma_{pr}(T') + 8$ and $a(T) \ge a(T') + 7$, and by Theorem 2.2 we have $\gamma_{pr}(T) < \frac{4a(T)+2}{3}$. Now let z be a leaf. Assume $T' = T - (T_{v_4} \cup T_{v_{D-2}})$. Then $\gamma_{pr}(T) \le \gamma_{pr}(T') + 10$. If $v_{D-3} \notin S'$, then let $S = S' \cup \{z, u_1, u_2, w_1, w_2, u'_1, u'_2, w'_1\}$. If $v_{D-3} \in S'$, then let $S = (S' - \{v_{D-3}\}) \cup \{z, u_1, u_2, w_1, w_2, u'_1, u'_2, w'_1, w'_2\}$. In each case, we have $\sum(S,T) \le m$, implying that $a(T) \ge a(T') + 8$ and it follows from Theorem 2.2 that $\gamma_{pr}(T) < \frac{4a(T)+2}{3}$. This completes the proof. □

Next result is an immediate consequence of Lemma 2.1 and Propositions 2.3, 2.4, 2.5 and 2.6.

Theorem 2.7. Let T be a tree of order $n \geq 2$. Then $\gamma_{pr}(T) = \frac{4a(T)+2}{3}$ if and only if T is P_2 or T is a healthy spider S_t, where t is odd.

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