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Ideal Convergence in Locally Solid Riesz Spaces

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Dedicated to Professor Hari M. Srivastava

Abstract. An ideal I is a family of subsets of positive integers $\mathbb{N}$ which is closed under taking finite unions and subsets of its elements. In this paper, we introduce the concepts of ideal τ-convergence, ideal τ-Cauchy and ideal τ-bounded sequence in locally solid Riesz space endowed with the topology τ. Some basic properties of these concepts has been investigated. We also examine the ideal τ-continuity of a mapping defined on locally solid Riesz space.

1. Introduction

The notion of statistical convergence was introduced by Fast [10] and Steinhaus [32] independently in the same year 1951. Actually the idea of statistical convergence was formerly given under the name "almost convergence" by Zygmund in the first edition of his celebrated monograph published in Warsaw in 1935 [37]. The concept was formally introduced by Fast [10] and later was reintroduced by Schoenberg [31], and also independently by Buck [4]. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on it was further investigated from various points of view. For example, statistical convergence has been investigated in summability theory by (Boos [3], Fridy [11], Šalát [27], topological groups (Çakalli [5, 6]), topological spaces (Di Maio and Kočinac [22]), function spaces (Caserta and Kočinac [7], Caserta, Di Maio and Kočinac [8]), locally convex spaces (Maddox[21]), measure theory (Cheng et al., [9], Millar[23]), fuzzy mathematics (Nuray and Savaş [24], Savaş [30]).

The notion of I-convergence(I denotes the ideal of subsets of $\mathbb{N}$) was initially introduced by Kostyrko et al. [18] as a generalization of statistical convergence. More applications of ideals can be seen in ([12–14], [15, 16, 28, 29, 33, 35]).

A family of sets $I \subset P(\mathbb{N})$ (power sets of $\mathbb{N}$) is called an *ideal* if and only if for each $A, B \in I$, we have $A \cup B \in I$ and for each $A \in I$ and each $B \subset A$, we have $B \in I$. A non-empty family of sets $\mathcal{F} \subset P(\mathbb{N})$ is a *filter* on $\mathbb{N}$ if and only if $\phi \notin \mathcal{F}$, for each $A, B \in \mathcal{F}$, we have $A \cap B \in \mathcal{F}$ and each $A \in \mathcal{F}$ and each $B \supset A$, we have $B \in \mathcal{F}$. An ideal I is called non-trivial ideal if $I \neq \phi$ and $\mathbb{N} \notin I$. Clearly $I \subset P(\mathbb{N})$ is a non-trivial ideal

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if and only if $\mathcal{F} = \mathcal{F}(I) = \{\mathbb{N} - A : A \in I\}$ is a filter on $\mathbb{N}$. A non-trivial ideal $I \subset P(\mathbb{N})$ is called *admissible* if and only if $\{\{x\} : x \in \mathbb{N}\} \subset I$. A non-trivial ideal I is *maximal* if there cannot exists any non-trivial ideal $J \neq I$ containing I as a subset. Further details on ideals of $P(\mathbb{N})$ can be found in Kostyrko, et.al [18].

If we take $I = I_f = \{A \subseteq \mathbb{N} : A$ is a finite subset $\}$. Then I_f is a non-trivial admissible ideal of $\mathbb{N}$ and the corresponding convergence coincide with the usual convergence. If we take $I = I_\delta = \{A \subseteq \mathbb{N} : \delta(A) = 0\}$ where $\delta(A)$ denote the asyptotic density of the set A. Then I_δ is a non-trivial admissible ideal of $\mathbb{N}$ and the corresponding convergence coincide with the statistical convergence.

A Riesz space is an ordered vector space which is a lattice at the same time. It was first introduced by F. Riesz [25] in 1928. Riesz spaces have many applications in measure theory, operator theory and optimization. They have also some applications in economics (see [2]), and we refer to ([1, 17, 20, 36]) for more details.

Now we recall some of the basic concepts related to statistical convergence and ideal convergence.

Let $E \subseteq \mathbb{N}$. Then the natural density of E is denoted by $\delta(E)$ and is defined by

$$\delta(E) = \lim_{n\to\infty} |\{k \in E : k \leq n\}|,$$

where the vertical bar denotes the cardinality of the respective set.

Definition 1.1.([22]) A sequence $x = (x_k)$ in a topological space X is said to be *statistically convergent* to x_0 if for every neighbourhood V of x_0

$$\delta\left(\{k \in \mathbb{N} : x_k \notin V\}\right) = 0.$$

In this case, we write $S-\lim x = x_0$ or $(x_k) \xrightarrow{S} x_0$ and S denotes the set of all statistically convergent sequences.

Definition 1.2.([19]) A sequence $x = (x_k)$ in a topological space X is said to be *I- convergent* to x_0 if for every neighbourhood V of x_0

$$\{k \in \mathbb{N} : x_k \notin V\} \in I.$$

In this case, we write $I - \lim x = x_0$ or $(x_k) \xrightarrow{I} x_0$ and $\mathcal{I}$ denotes the set of all ideally convergent sequences.

2. Preleminaries

Let X be a real vector space and $\leq$ be a partial order on this space. Then X is said to be an *ordered vector space* if it satifies the following properties:

(i) if $x, y \in X$ and $y \leq x$, then $y + z \leq x + z$ for each $z \in X$.
(ii) if $x, y \in X$ and $y \leq x$, then $ay \leq ax$ for each $a \geq 0$.

If, in addition, X is a lattice with respect to the partially ordered, then X is said to be a *Riesz space* (or a *vector lattice*)(see[36]), if for each pair of elements $x, y \in X$ the supremum and infimum of the set $\{x, y\}$ both exist in X.

We shall write

$$x \vee y = \sup\{x, y\} \text{ and } x \wedge y = \inf\{x, y\}.$$

For an element x of a Risez space X, the *positive part* of x is defined by $x^+ = x \vee \overline{\theta}$, the *negative part* of x by $x^- = -x \vee \overline{\theta}$ and the *absolute value* of x by $|x| = x \vee (-x)$, where $\overline{\theta}$ is the zero element of X.

A subset S of a Riesz space X is said to be solid if $y \in S$ and $|y| \leq |x|$ implies $x \in S$.

A *topological vector space* (X, τ) is a vector space X which has a topology (linear) τ, such that the algebraic operations of addition and scalar multiplication in X are continuous. Continuity of addition means that the function $f : X \times X \to X$ defined by $f(x, y) = x + y$ is continuous on $X \times X$, and cnotinuity of scalar multiplication means that the function $f : \mathbb{C} \times X \to X$ defined by $f(a, x) = ax$ is continuous on $\mathbb{C} \times X$.

Every linear topology τ on a vector space X has a base N for the neighborhoods of $\overline{\theta}$ satisfying the following properties:

(1) Each $Y \in N$ is a *balanced set*, that is, $ax \in Y$ holds for all $x \in Y$ and for every $a \in \mathbb{R}$ with $|a| \leq 1$.

(2) Each $Y \in N$ is an *absorbing set* , that is , for every $x \in X$, there exists $a > 0$ such that $ax \in Y$.

(3) For each $Y \in N$ there exists some $E \in N$ with $E + E \subseteq Y$.

A linear topology τ on a Riesz space X is said to be *locally solid* (see[26]) if τ has a base at zero consisting of solid sets. A *locally solid Riesz space* (X, τ) is a Riesz space equipped with a locally solid topology τ.

Recall that a first countable space is a topological space satisfying the "first axiom of countability". Specifically , a space X is said to be first countable if each point has a countable neighbourhood basis(local base). That is, for each point x in X there exists a sequence $V_1, V_2, ...$ of open neighbourhoods of x such that for any open neighbourhood V of x there exists an integer j with V_j cointained in V.

The purpose of this article is to give certain characterizations of ideal convergent sequences in locally solid Riesz spaces and investigate some basic properties of the notions ideal τ-convergence, ideal τ-Cauchy, ideal τ-bounded sequence and ideal τ-continuity of a mapping in locally solid Riesz spaces. Finally we prove a Tauberian theorem to the locally solid Riesz spaces.

Throughout the article, the symbol N_{sol} we will denote any base at zero consistiong of solid sets and satisfying the conditions (1), (2) and (3) in a locally solid topology. Also we assume I is a non-trivial admissible ideal of $\mathbb{N}$.

3. Ideal topological convergence in locally solid Riesz spaces

Throughout the article X will denote the Hausdorff locally solid Riesz space, which satisfies the first axiom of countability.

Recently, in [1], Albayrak and Pehlivan introduced the notion of statiatical convergence in locally solid Riesz spaces as follows:

Definition 3.1.([1]) Let (X, τ) be a locally solid Riesz space. A sequence (x_k) of points in X is said to be $S(\tau)$-*convergent* to an element x_0 of X if for each τ-neighbourhood V of zero,

$$\delta(\{k \in \mathbb{N} : x_k - x_0 \notin V\}) = 0$$

i.e.,

$$\lim_{m} \frac{1}{m} |\{k \leq m : x_k - x_0 \notin V\}| = 0.$$

In this case, we write $S(\tau) - \lim_{k\to\infty} x_k = x_0$ or $(x_k) \xrightarrow{S(\tau)} x_0$.

Now we give the definitions of $I(\tau)$-convergence and $I(\tau)$-bounded in locally solid Riesz spaces.

Definition 3.2. Let (X, τ) be a locally solid Riesz space. A sequence (x_k) of points in X is said to be $I(\tau)$-*convergent* to an element x_0 of X if for each τ-neighbourhood V of zero,

$$\{k \in \mathbb{N} : x_k - x_0 \notin V\} \in I.$$

i.e.,

$$\{k \in \mathbb{N} : x_k - x_0 \in V\} \in \mathcal{F}.$$

In this case, we write $I(\tau) - \lim_{k\to\infty} x_k = x_0$ or $(x_k) \xrightarrow{I(\tau)} x_0$.

Example 3.1. Let us consider the locally solid Riesz space $(\mathbb{R}^2, \|.\|)$ with the Euclidean norm $\|.\|$ and coordinatewise ordering. In this space, let us define a sequence (x_k) by

$$x_k = \begin{cases} (1 + \frac{1}{k+1}, 2 + \frac{5}{k+1}), & \text{if } k \neq n^2; \\ (4, 4), & \text{if } k = n^2. \end{cases}$$

for each $n \in \mathbb{N}$.The family N_{sol} of all U_ε defined by

$$U_\varepsilon = \{x \in \mathbb{R}^2 : \|x\| < \varepsilon\},$$

where $0 < \varepsilon \in \mathbb{R}$ constitutes a base at zero $(\overline{\theta} = (0,0))$. For $x_0 = (1, 2)$, we have

$$x_k - x_0 = \begin{cases} (\frac{1}{k+1}, \frac{5}{k+1}), & \text{if } k \neq n^2; \\ (3, 2), & \text{if } k = n^2. \end{cases}$$

For each τ-neighbourhood V of zero, there exists some $U_\varepsilon \in N_{sol}$, $\varepsilon > 0$ such that $U_\varepsilon \subseteq V$ and

$$\{k \in \mathbb{N} : x_k - x_0 \notin U_\varepsilon\} \subseteq K \cup \{1, 4, 9, 16, ..., n^2, ...\},$$

where K is a finite set. Then, we have

$$\{k \in \mathbb{N} : x_k - x_0 \notin V\} \subseteq \{k \in \mathbb{N} : x_k - x_0 \notin U_\varepsilon\}$$

$$i.e.\{k \in \mathbb{N} : x_k - x_0 \notin V\} \subseteq K \cup \{1, 4, 9, 16, ..., n^2, ...\}.$$

Since I is admissible, so we have

$$\{k \in \mathbb{N} : x_k - x_0 \notin V\} \in I.$$

Hence $I(\tau) - \lim_k x_k = (1, 2)$.

Definition 3.3. Let (X, τ) be a locally solid Riesz space. A sequence (x_k) of points in X is said to be $I(\tau)$-*bounded* in X if for each τ-neighbourhood V of zero, there is some $a > 0$,

$$\{k \in \mathbb{N} : ax_k \notin V\} \in I.$$

Theorem 3.1. *Let (X, τ) be a locally solid Riesz space. Every $I(\tau)$-convergent sequences in X has only one limit.*

Proof. Suppose that $x = (x_k)$ is a sequence in X such that $I(\tau)$-$\lim_k x_k = x_0$ and $I(\tau)$-$\lim_k x_k = y_0$.

Let V be any τ-neighbourhood of zero. Also for each τ-neighbourhood V of zero there exists $Y \in N_{sol}$ such that $Y \subseteq V$. Choose any $W \in N_{sol}$ such that $W + W \subseteq Y$. We define the following sets:

$$A_1 = \{k \in \mathbb{N} : x_k - x_0 \in W\}$$

$$A_2 = \{k \in \mathbb{N} : x_k - y_0 \in W\}.$$

Since $I(\tau) - \lim x_k = x_0$ and $I(\tau) - \lim x_k = y_0$, we get $A_1, A_2 \in \mathcal{F}$.
Now, let $A = A_1 \cap A_2$. Then we have

$$x_0 - y_0 = x_0 - x_k + x_k - y_0 \in W + W \subseteq Y \subseteq V.$$

Hence for each τ-neighbourhood V of zero we have $x_0 - y_0 \in V$. Since (X, τ) is Hausdorff, the intersection of all τ-neighbourhoods V of zero is the singleton set $\{\overline{\theta}\}$. Thus we get $x_0 - y_0 = 0$ i.e., $x_0 = y_0$.

Theorem 3.2. *Let (X, τ) be a locally solid Riesz space and let (x_k) and (y_k) be two sequences of points in X. Then the following hold:*

(i) *If $I(\tau)\text{-}\lim_k x_k = x_0$ and $I(\tau)\text{-}\lim_k y_k = y_0$ then $I(\tau)\text{-}\lim_k(x_k + y_k) = x_0 + y_0$.*

(ii) *If $I(\tau)\text{-}\lim_k x_k = x_0$ then $I(\tau)\text{-}\lim_k ax_k = ax_0$ for $a \in \mathbb{R}$.*

Proof. Let V be an arbitrary τ-neighbourhood of zero. Then there exists $Y \in N_{sol}$ such that $Y \subseteq V$. Choose $W \in N_{sol}$ such that $W + W \subseteq Y$. Since $I(\tau)\text{-}\lim_k x_k = x_0$ and $I(\tau)\text{-}\lim_k y_k = y_0$. We write

$$B_1 = \{k \in \mathbb{N} : x_k - x_0 \in W\}$$

$$B_2 = \{k \in \mathbb{N} : y_k - y_0 \in W\}.$$

Then we have $B_1, B_2 \in \mathcal{F}$.

Let $B = B_1 \cap B_2$. Hence we have $B \in \mathcal{F}$ and

$$(x_k + y_k) - (x_0 + y_0) = (x_k - x_0) + (y_k - y_0) \in W + W \subseteq Y \subseteq V.$$

Therefore

$$\{k \in \mathbb{N} : (x_k + y_k) - (x_0 + y_0) \in V\} \in \mathcal{F}.$$

Since V is arbitary, we have $I(\tau) - \lim(x_k + y_k) = x_0 + y_0$.

(ii) Let V be an arbitrary τ-neighbourhood of zero and $I(\tau)\text{-}\lim_k x_k = x_0$. Then there exists $Y \in N_{sol}$ such that $Y \subseteq V$ and we have

$$\{k \in \mathbb{N} : x_k - x_0 \in Y\} \in \mathcal{F}.$$

Since Y is balanced, $x_k - x_0 \in Y$ implies that $a(x_k - x_0) \in Y$ for every $a \in \mathbb{R}$ with $|a| \leq 1$. Hence

$$\{k \in \mathbb{N} : x_k - x_0 \in Y\} \subseteq \{k \in \mathbb{N} : ax_k - ax_0 \in Y\} \subseteq \{k \in \mathbb{N} : x_k - x_0 \in V\}.$$

Thus, we have

$$\{k \in \mathbb{N} : x_k - x_0 \in V\} \in \mathcal{F}$$

for each τ-neighbourhood V of zero. Noew let $|a| > 1$ and $[|a|]$ be the smallest integer greater than or equal to $|a|$. Then there exists $W \in N_{sol}$ such that $[|a|]W \subseteq Y$. Since $I(\tau)\text{-}\lim_k x_k = x_0$ we have the set

$$K = \{k \in \mathbb{N} : x_k - x_0 \in W\} \in \mathcal{F}.$$

Therefore

$$|ax_k - ax_0| = |a||x_k - x_0| \leq [|a|]|x_k - x_0| \in [|a|]W \subseteq Y \subseteq V.$$

Since Y is solid, we have $ax_k - ax_0 \in Y$. This implies that $ax_k - ax_0 \in V$. Thus,

$$\{k \in \mathbb{N} : ax_k - ax_0 \in V\} \in \mathcal{F},$$

for each τ-neighbourhood V of zero. Hence $I(\tau)\text{-}\lim_k ax_k = ax_0$. This completes the proof of the theorem.

Theorem 3.3. *Let (X,τ) be a locally solid Riesz space. If a sequence (x_k) in X is $I(\tau)$-convergent, then it is $I(\tau)$-bounded.*

Proof. Suppose that (x_k) is $I(\tau)$-convergent to a point x_0 in X. Let V be an arbitrary τ-neighbourhood of zero, there exists $Y \in N_{sol}$ such that $Y \subseteq V$. We choose $W \in N_{sol}$ such that $W+W \subseteq Y$. Since $I(\tau)\text{-}\lim_{k\to\infty} x_k = x_0$, the set

$$A = \{k \in \mathbb{N} : x_k - x_0 \notin W\} \in I.$$

Since W is absorbing, there exists $a > 0$ such that$ax_0 \in W$. Let b be such that $|b| \leq 1$ and $b \leq a$. Since W is solid and $|bx_0| \leq |ax_0|$, we have $bx_0 \in W$. Also, since W is balanced, $x_k - x_0 \in W$ implies $b(x_k - x_0) \in W$. Then we have

$$bx_k = b(x_k - x_0) + bx_0 \in W + W \subseteq V, \text{ for } \text{ each } k \in \mathbb{N} - A.$$

Thus

$$\{k \in \mathbb{N} : bx_k \notin W\} \in I.$$

Hence (x_k) is $I(\tau)$-bounded.

Theorem 3.4. *Let (X,τ) be a locally solid Riesz space and let (x_k), (y_k) and (z_k) be three sequences of points in X such that*

(i) $x_k \leq y_k \leq z_k$, *for all* $k \in \mathbb{N}$,

(ii) $I(\tau)\text{-}\lim_k x_k = x_0 = I(\tau)\text{-}\lim_k z_k$,
then $I(\tau)\text{-}\lim_k y_k = x_0$.

Proof. Let V be an arbitrary τ-neighbourhood of zero, there exists $Y \in N_{sol}$ such that $Y \subseteq V$. We choose $W \in N_{sol}$ such that $W + W \subseteq Y$. From given condition (ii), we have $P, Q \in F$, where

$$P = \{k \in \mathbb{N} : x_k - x_0 \in W\}$$

and

$$Q = \{k \in \mathbb{N} : z_k - x_0 \in W\}.$$

Also from the given condition (i), we have

$$x_k - x_0 \leq y_k - x_0 \leq z_k - x_0$$

$$\Rightarrow |y_k - x_0| \leq |x_k - x_0| + |z_k - x_0| \in W + W \subseteq Y.$$

Since Y is solid, we have $y_k - x_0 \in Y \subseteq V$. Thus,

$$\{k \in \mathbb{N} : y_k - x_0 \in V\} \in \mathcal{F},$$

for each τ-neighbourhood V of zero. Hence $I(\tau)\text{-}\lim_k y_k = x_0$. This completes the proof of the theorem.

4. $I(\tau)$-Cauchy and $I^*(\tau)$-convergence in locally solid Riesz spaces

Definition 4.1. Let (X,τ) be a locally solid Riesz space. A sequence (x_k) of points in X is said to be $I(\tau)$-*Cauchy* in X if for each τ-neighbourhood V of zero, there is an integer $n \in \mathbb{N}$,

$$\{k \in \mathbb{N} : x_k - x_n \notin V\} \in I.$$

Theorem 4.1. *Let (X, τ) be a locally solid Riesz space. A sequence (x_k) is $I(\tau)$-convergent to x_0 in X if and only if for each τ-neighbourhood V of zero there exists a subsequence $(x_{k'(r)})$ of (x_k) such that $\lim_{r\to\infty} x_{k'(r)} = x_0$ and*

$$\{k \in \mathbb{N} : x_k - x_{k'(r)} \notin V\} \in I.$$

Proof. Let $x = (x_k)$ be a sequence in X such that $I(\tau) - \lim_{k\to\infty} x_k = x_0$. Let V be an arbitrary τ-neighbourhood of zero. Let $\{V_n\}$ be a sequence of nested base of τ-neighbourhoods of zero. We write

$$E^{(i)} = \{k \in \mathbb{N} : x_k - x_0 \notin V_i\},$$

for any positive integer i. Then for each i, we have $E^{(i+1)} \subset E^{(i)}$ and $E^{(i)} \in F$.Choose $n(1)$ such that $r > n(1)$, then $E^{(1)} \neq \phi$. Then for each positive integer r such that $n(1) \le r < n(2)$, choose $k'(r) \in E^{(i)}$ i.e. $x_{k'(r)} - x_0 \in V_1$. In general, choose $n(p+1) > n(p)$ such that $r > n(p+1)$, then $E^{(p+1)} \neq \phi$. Then for all r satisfying $n(p) \le r < n(p+1)$, choose $k'(r) \in E^{(p)}$ i.e. $x_{k'(r)} - x_0 \in V_p$. Hence it follows that $\lim_r x_{k'(r)} = x_0$.

Since V is an arbitrary τ-neighbourhood of zero, there exists $Y \in N_{sol}$ such that $Y \subseteq V$. Choose $W \in N_{sol}$ such that $W + W \subseteq Y$. Now we have

$$x_k - x_{k'(r)} = x_k - x_0 + x_{k'(r)} - x_0 \in W + W \subseteq Y \subseteq V.$$

Since $I(\tau) - \lim_{k\to\infty} x_k = x_0$ and $\lim_{r\to\infty} x_{k'(r)} = x_0$ implies that

$$\{k \in \mathbb{N} : x_k - x_{k'(r)} \notin V\} \in I.$$

Next suppose for an arbitrary τ-neighbourhood V of zero there exists a subsequence $(x_{k'(r)})$ of (x_k) such that $\lim_{r\to\infty} x_{k'(r)} = x_0$ and

$$\{k \in \mathbb{N} : x_k - x_{k'(r)} \notin V\} \in I.$$

Since V is any τ-neighbourhood of zero, we choose $W \in N_{sol}$ such that $W + W \subseteq V$. Then we have

$$x_k - x_0 = x_k - x_{k'(r)} + x_{k'(r)} - x_0 \in W + W \subseteq V.$$

i.e.

$$\{k \in \mathbb{N} : x_k - x_0 \notin V\} \subseteq \{k \in \mathbb{N} : x_k - x_{k'(r)} \notin W\} \cup \{r \in \mathbb{N} : x_{k'(r)} - x_0 \notin W\}.$$

Therefore

$$\{k \in \mathbb{N} : x_k - x_0 \notin V\} \in I.$$

This completes the proof of the theorem.

Theorem 4.2. *If $\lim_{k\to\infty} x_k = x_0$ and $I(\tau) - \lim_{k\to\infty} y_k = 0$, then $I(\tau) - \lim_{k\to\infty}(x_k + y_k) = \lim_{k\to\infty} x_k$.*

Proof. Let V be any τ-neighbourhood of 0. Then there exists $Y \in N_{sol}$ such that $Y \subseteq V$. Choose $W \in N_{sol}$ such that $W + W \subseteq Y$. Since $\lim_{k\to\infty} x_k = x_0$, then there exists an integer n_0 such that $k \ge n_0$ implies that $x_k - x_0 \in W$. Hence

$$\{k \in \mathbb{N} : x_k - x_0 \notin W\} \subseteq \mathbb{N} - \{n_0\}.$$

By assumption $I(\tau) - \lim_{k\to\infty} y_k = 0$, then we have $\{k \in \mathbb{N} : y_k \notin W\} \in I$. Thus

$$\{k \in \mathbb{N} : (x_k - x_0) + y_k \notin V\} \subseteq \{k \in \mathbb{N} : x_k - x_0 \notin W\} \cup \{k \in \mathbb{N} : y_k \notin W\}.$$

i.e.

$$\{k \in \mathbb{N} : (x_k - x_0) + y_k \notin V\} \in I.$$

This implies that $I(\tau) - \lim_{k\to\infty}(x_k + y_k) = \lim_{k\to\infty} x_k$.

Theorem 4.3. *Let (X, τ) be a locally solid Riesz space and let $x = (x_k)$ be a sequence in X. If there is a $I(\tau)$-convergent sequence $y = (y_k)$ in X such that $\{k \in \mathbb{N} : y_k \neq x_k \notin V\} \in I$ then x is also $I(\tau)$-convergent.*

Proof. Suppose that $\{k \in \mathbb{N} : y_k \neq x_k \notin V\} \in I$ and $I(\tau)-\lim_k y_k = x_0$. Then for an arbitrary τ-neighborhood V of zero, we have

$$\{k \in \mathbb{N} : y_k - x_0 \notin V\} \in I.$$

Now,

$$\{k \in \mathbb{N} : x_k - x_0 \notin V\} \subseteq \{k \in \mathbb{N} : y_k \neq x_k \notin V\} \cup \{k \in \mathbb{N} : y_k - x_0 \notin V\}.$$

Therefore we have

$$\{k \in \mathbb{N} : x_k - x_0 \notin V\} \in I.$$

This completes the proof of the theorem.

Definition 4.2. Let (X, τ) be a locally solid Riesz space. A sequence $x = (x_k)$ in X is said to be $I^*(\tau)$-convergent to x_0 if there exists a set $K = \{k_1 < k_2 < \cdots < k_r < \dots\} \subseteq \mathbb{N}$ with $K \in F$ suh that $\lim_k x_k = x_0$. In this case we write $I^*(\tau)$-$\lim_k x_k = x_0$.

Theorem 4.4. *Let (X, τ) be a locally solid Riesz space. A sequence $x = (x_k)$ in X is $I(\tau)$-convergent to x_0 if and only if it is $I^*(\tau)$-convergent to x_0.*

Proof. Suppose that $I^*(\tau)$-$\lim_k x_k = x_0$. Let V be an arbitrary τ-neighbourhood V of zero. Since $I^*(\tau)$-$\lim_k x_k = x_0$, there is a set $K = \{k_1 < k_2 < ...\} \subset \mathbb{N}$ with $K \in F$ and $n \in \mathbb{N}$ such that $k \geq n$ and $k \in K$ imply $x_k - x_0 \in V$. Then

$$K_1 = \{k \in \mathbb{N} : x_k - x_0 \notin V\} \subseteq \mathbb{N} - \{k_{n+1}, k_{n+2}, ...\}.$$

Therefore

$$K_1 \in I.$$

Hence x is $I(\tau)$-convergent to x_0.

Next suppose that x is $I(\tau)$-convergent to x_0. For a fix countable local base $V_1 \supset V_2 \supset .....$ at x_0. For each $j \in \mathbb{N}$, we put

$$K_j = \{k \in \mathbb{N} : x_k - x_0 \notin V_j\}$$

and

$$M_j = \{k \in \mathbb{N} : x_k - x_0 \in V_j\}.$$

Then $K_j \in I$ and

$$M_1 \supset M_2 \supset ... \supset M_j \supset M_{j+1} \supset ... \tag{1}$$

and

$$M_j \in F, \quad j = 1, 2, 3.... \tag{2}$$

Now we show that for $k \in M_j$, (x_k) is convergent to x_0. Suppose that (x_k) is not convergent to x_0. Therefore $x_k - x_0 \notin V_j$ for infinitely many terms. Let

$$M_i = \{k \in \mathbb{N} : x_k - x_0 \in V_i\} \ \ (i > j).$$

Then $M_i \in I$, by using (1) we get $M_j \subset M_i$. Hence $M_j \in I$ which contardicts (2). Therefore (x_k) is convergent to x_0. Hence x is $I^*(\tau)$-convergent to x_0. This completes the proof of the theorem.

Theorem 4.5. *The sequential method $I(\tau)$ is regular, i.e. if $\tau - \lim x_k = x_0$ then $I(\tau) - \lim x_k = x_0$.*

Proof. Proof of the theorem is straightforward.

Theorem 4.6. *The sequential method $I(\tau)$ is subsequential.*

Proof. Proof of the theorem follows from the Theorem 4.4.

5. *I*-sequentially continuous in locally solid Riesz spaces

Definition 5.1. Let (X_1, τ_1) and (X_2, τ_2) be locally solid Riesz spaces and $S \subset X_1$. A function $f : S \to X_2$ is said to be *I-sequentially continuous at a point $x_0 \in S$*, if $x_k \overset{I(\tau_1)}{\to} x_0$ in S implies that $f(x_k) \overset{I(\tau_2)}{\to} f(x_0)$ in X_2.

Theorem 5.1. *Let (X, τ) be a locally solid Riesz space. Any I-sequentially continuous function at a point x_0 is τ-continuous at x_0.*

Proof. Let f be any I-sequentially continuous function at a point x_0. Since any proper admissible ideal is a regular subsequential method by Theorem 4.5 and 4.6, it follows that f is τ-continuous.

Corollary 5.2. *Let (X, τ) be a locally solid Riesz space. Any I-sequentially continuous function at a point x_0 is I-continuous at x_0.*

As statistical limit is an I-sequential method we get:

Corollary 5.3. *Let (X, τ) be a locally solid Riesz space. A function is statistically continuous at a point x_0 if and only if it is τ-continuous at x_0.*

Theorem 5.4. *Let (X_1, τ_1) and (X_2, τ_2) be locally solid Riesz spaces. If a function $f : X_1 \to X_2$ is uniformly continuous, then f is I-continuous.*

Proof. Let the function $f : X_1 \to X_2$ be uniformly continuous and $x_k \overset{I(\tau_1)}{\to} x_0$ holds in X_1. Let us denote the zeros of X_1 and X_2 by $\overline{\theta}_1$ and $\overline{\theta}_2$, respectively. Let V be an arbitrary τ_2-neighbourhood of $\overline{\theta}_2$. Since f is uniformly continuous, there exists some τ_1-neighborhood W of $\overline{\theta}_1$ such that

$$x - y \in W \Rightarrow f(x) - f(y) \in V \text{ for all } x, y \in X_1. \tag{3}$$

Since $x_k \overset{I(\tau_1)}{\to} x_0$, we have $A \in \mathcal{F}$, where $A = \{k \in \mathbb{N} : x_k - x_0 \in W\}$. Using (3) we have

$$f(x_k) - f(x_0) \in V \text{ for each } k \in A.$$

Then we get

$$A \subseteq \{k \in \mathbb{N} : f(x_k) - f(x_0) \in V\},$$

and hence

$$\{k \in \mathbb{N} : f(x_k) - f(x_0) \in V\} \in \mathcal{F}.$$

Thus $f(x_k) \overset{I(\tau_2)}{\to} f(x_0)$. This shows that f is I-continuous.

Theorem 5.5. *Let (X,τ) be a locally solid Riesz space. Then the following mappings*
(i) $(X,\tau)\times(X,\tau) \to (X,\tau) : (x,y) \longmapsto x \vee y$,
(ii) $(X,\tau)\times(X,\tau) \to (X,\tau) : (x,y) \longmapsto x \wedge y$,
(iii) $(X,\tau) \to (X,\tau) : (x,y) \longmapsto |x|$,
(iv) $(X,\tau) \to (X,\tau) : (x,y) \longmapsto x^-$,
(v) $(X,\tau) \to (X,\tau) : (x,y) \longmapsto x^+$
are all I-continuous.

Proof. (i) Let $I(\tau\times\tau) - \lim(x_k, y_k) = (x, y)$ and V be an arbitrary τ-neighbourhood of zero in X. Then there exists a $Y \in N_{sol}$ such that $Y \subseteq V$. Let us choose $W \in N_{sol}$ such that $W+W \subseteq Y$. Since $I(\tau\times\tau)-\lim(x_k, y_k) = (x, y)$, we have

$$A = \{k \in \mathbb{N} : (x_k - x, y_k - y) \in W \times W\} \in \mathcal{F}.$$

Also we have

$$|x_k \vee y_k - x \vee y| \leq |x_k - x| + |y_k - y| \in W + W \subseteq Y \text{ for each } k \in A.$$

Since Y is solid, we have

$$x_k \vee y_k - x \vee y \in Y, \text{ for each } k \in A.$$

Then we get

$$A \subseteq \{k \in \mathbb{N} : x_k \vee y_n - x \vee y \in V\}$$

and hence

$$\{k \in \mathbb{N} : x_n \vee y_k - x \vee y \in V\} \in \mathcal{F}.$$

Therefore we have $I(\tau) - \lim(x_k \vee y_k) = (x \vee y)$.

(ii) Let V be an arbitrary τ-neighbourhood of zero in X. Then there exists a $Y \in N_{sol}$ such that $Y \subseteq V$. Let us choose $W \in N_{sol}$ such that $W + W \subseteq Y$. Let $I(\tau \times \tau) - \lim(x_k, y_k) = (x, y)$. Then we have

$$A = \{k \in \mathbb{N} : (x_k - x, y_n - y) \in W \times W\} \in \mathcal{F}.$$

Also we have

$$|x_k \wedge y_k - x \wedge y| = | - [(-x_k) \vee (-y_k)] + [(-x) \vee (-y)]|$$
$$\leq |(-x) - (-x_k)| + |(-y) - (-y_k)|$$
$$= |x_k - x| + |y_k - y| \in W + W \subseteq Y \text{ for each } k \in A.$$

Since Y is solid, we have

$$x_k \wedge y_k - x \wedge y \in Y, \text{ for each } k \in A.$$

Then we get

$$A \subseteq \{k \in \mathbb{N} : x_k \wedge y_k - x \wedge y \in V\}$$

and hence

$$\{k \in \mathbb{N} : x_k \wedge y_k - x \wedge y \in V\} \in \mathcal{F}.$$

Therefore we have $I(\tau) - \lim(x_k \wedge y_k) = (x \wedge y)$.

(iii) Let V be an arbitrary τ-neighbourhood of zero in X. Then there exists a $Y \in N_{sol}$ such that $Y \subseteq V$. Let us choose $W \in N_{sol}$ such that $W + W \subseteq Y$. Let $I(\tau) - \lim x_k = x$. Then we have

$$A = \{k \in \mathbb{N} : x_k - x \in W\} \in \mathcal{F}.$$

Also we have

$$|x_k| - |x| = |[x_k \vee (-x_k)] - [x \vee (-x)]|$$

$$\leq |x_k - x| + |(-x_k) - (-x)| \in W + W \subseteq Y \text{ for each } k \in A.$$

Since Y is solid, we have

$$|x_k| - |x| \in Y, \text{ for each } k \in A.$$

Then we get

$$A \subseteq \{k \in \mathbb{N} : |x_k| - |x| \in V\}$$

and hence

$$\{k \in \mathbb{N} : |x_k| - |x| \in V\} \in \mathcal{F}.$$

Therefore we have $I(\tau) - \lim |x_k| = |x|$.

(iv) Let V be an arbitrary τ-neighbourhood of zero in X. Then there exists a $Y \in N_{sol}$ such that $Y \subseteq V$. Let us choose $W \in N_{sol}$ such that $W + W \subseteq Y$. Let $I(\tau) - \lim x_k = x$. Then we have

$$A = \{k \in \mathbb{N} : x_k - x \in W\} \in \mathcal{F}.$$

Also we have

$$|x_k^- - x^-| = |[(-x_k) \vee 0] - [(-x) \vee 0]|$$

$$\leq |(-x_k) - (-x)| + |0 + 0| = |x_k - x| \in W \subseteq Y \text{ for each } k \in A.$$

Since Y is solid, we have

$$x_k^- - x^- \in Y, \text{ for each } k \in A.$$

Then we get

$$A \subseteq \{k \in \mathbb{N} : x_k^- - x^- \in V\}$$

and hence

$$\{k \in \mathbb{N} : x_k^- - x^- \in V\} \in \mathcal{F}.$$

Therefore we have $I(\tau) - \lim x_k^- = x^-$.

(v) Let V be an arbitrary τ-neighbourhood of zero in X. Then there exists a $Y \in N_{sol}$ such that $Y \subseteq V$. Let us choose $W \in N_{sol}$ such that $W + W \subseteq Y$. Let $I(\tau) - \lim x_k = x$. Then we have

$$A = \{k \in \mathbb{N} : x_k - x \in W\} \in \mathcal{F}.$$

Also we have

$$|x_k^+ - x^+| = |(x_k \vee 0) - (x \vee 0)|$$

$$\leq |x_k - x| + |0 + 0| = |x_k - x| \in W \subseteq Y \text{ for each } k \in A.$$

Since Y is solid, we have

$$x_k^+ - x^+ \in Y, \text{ for each } k \in A.$$

Then we get

$$A \subseteq \{k \in \mathbb{N} : x_k^+ - x^+ \in V\}$$

and hence

$$\{k \in \mathbb{N} : x_k^+ - x^+ \in V\} \in \mathcal{F}.$$

Therefore we have $I(\tau) - \lim x_k^+ = x^+$.

Definition 5.2. Let (X, τ) be a locally solid Riesz space. A sequence (x_k) in X is called *slowly oscillating* if , for each τ-neighbourhood V of zero, there exists a positive integer n_0 and $\delta > 0$ such that if $n_0 \leq k \leq n \leq (1 + \delta)k$, then $x_k - x_n \in V$.

Now we give a Tauberian theorem.

Theorem 5.6. *Let (X, τ) be a locally solid Riesz space. If (x_k) is statistically convergent and slowly oscillating, then it is convergent.*

Proof. Let $S(\tau) - \lim x_k = x_0$. Then we have a subsequence (i_m) with $1 \leq i_1 \leq i_2 \leq ... \leq i_m \leq ...$ of those indices n for which $y_n = x_n$. Since

$$\lim_{k\to\infty} \frac{1}{k} |\{n \leq k : x_n \neq y_n\}| = 0.$$

Then we have

$$\lim_{m\to\infty} \frac{1}{i_m} |\{n \leq i_m : x_n = y_n\}| = \lim_{m\to\infty} \frac{m}{i_m} = 1.$$

Consequently, it follows that

$$\lim_{m\to\infty} \frac{i_{m+1}}{i_m} = \lim_{m\to\infty} \frac{i_{m+1}}{m+1} . \frac{m+1}{m} . \frac{m}{i_m} = 1. \tag{4}$$

By the definition of (i_m), we get

$$\lim_{m\to\infty} x_{i_m} = \lim_{m\to\infty} y_{i_m} = x_0. \tag{5}$$

By (4) and (5) we conclude that for each closed τ-neighbourhood V of zero, there exists a positive integer n_0 such that if $m > n_0$ then $(x_k - x_{i_m}) \in V$ whenever $i_m < k < i_{m+1}$. Since V is arbitrary, it follows that

$$\lim_{m\to\infty} (x_m - x_{i_m}) = 0.$$

By (5.3), we have (x_m) is convergent to x_0. This completes the proof of the theorem.

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