Filomat 28:4 (2014), 821–828
DOI 10.2298/FIL1404821G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some Properties of Functions Related to Certain Classes of Completely Monotonic Functions and Logarithmically Completely Monotonic Functions

Senlin Guo[a], H. M. Srivastava[b], Wing-Sum Cheung[c]

[a] *Department of Mathematics, Zhongyuan University of Technology, Zhengzhou, Henan* 450007*, People's Republic of China*
[b] *Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W* 3*R*4*, Canada*
[c] *Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong*

Abstract. In this article, we establish several properties of the composition of functions which are related to certain classes of completely monotonic functions and logarithmically completely monotonic functions.

1. Introduction, Preliminaries and the Main Results

Throughout this paper, we denote by $\mathbb{N}$ the set of all positive integers,

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} \qquad \text{and} \qquad \mathbb{R}^+ := (0, \infty).$$

Furthermore, I^+ is an open interval contained in $\mathbb{R}^+$, I^o is the interior of the interval $I \subset \mathbb{R}$, $\mathbb{R}$ is the set of all real numbers, $\mathcal{R}(f)$ denotes the range of the function f and $C(I)$ is the class of all continuous functions on the interval I.

We first recall some definitions which we shall use and some basic results which are related to them.

Definition A (see [27]). A function f is said to be absolutely monotonic on an interval I, if $f \in C(I)$ has derivatives of all orders on I^o and

$$f^{(n)}(x) \geqq 0 \qquad (x \in I^o)$$

for all $n \in \mathbb{N}_0$.

The class of all absolutely monotonic functions on I is denoted by $AM(I)$.

Definition B (see [27]). A function f is said to be completely monotonic on an interval I, if $f \in C(I)$ has derivatives of all orders on I^o and

$$(-1)^n f^{(n)}(x) \geqq 0 \qquad (x \in I^o)$$

for all $n \in \mathbb{N}_0$.

2010 *Mathematics Subject Classification*. Primary 26A48; Secondary 26E40
Keywords. Composition of functions; Leibniz's rule; Completely monotonic functions; Strongly completely monotonic functions; Logarithmically completely monotonic functions; Strongly logarithmically completely monotonic functions.
Received: 03 December 2013; Accepted: 22 December 2013
Communicated by Dragan S. Djordjević
Email addresses: sguo@zut.edu.cn, sguo@hotmail.com (Senlin Guo), harimsri@math.uvic.ca (H. M. Srivastava), wscheung@hku.hk (Wing-Sum Cheung)

Remark 1. In the existing literature on the subject of our investigation, the terminology *completely monotone* is also used instead of the terminology *completely monotonic* which we have used in our present investigation.

The class of all completely monotonic functions on I is denoted by $CM(I)$.

By Leibniz's rule for the derivative of order n of the function fg, we can easily prove that, if $f, g \in CM(I) \quad \big(AM(I)\big)$, then $fg \in CM(I) \quad \big(AM(I)\big)$.

Definition C (see [2] and [19]). A function f is said to be logarithmically completely monotonic on an interval I if $f > 0$, $f \in C(I)$ has derivatives of all orders on I^o and

$$(-1)^n [\ln f(x)]^{(n)} \geqq 0 \qquad (x \in I^o)$$

for all $n \in \mathbb{N}_0$.

The set of all logarithmically completely monotonic functions on I is denoted by $LCM(I)$.

The result below was proved recently (see, for example, [19]).

Theorem A. *The following assertion holds true*: $LCM(I) \subset CM(I)$.

Definition D (see [26]). A function f is said to be strongly completely monotonic on I^+ if, for all $n \in \mathbb{N}_0$, the functions $(-1)^n x^{n+1} f^{(n)}(x)$ are nonnegative and decreasing on I^+.

The class of strongly completely monotonic functions on I^+ is denoted by $SCM(I^+)$.

Definition E (see [12]). A function f is said to be strongly logarithmically completely monotonic on I^+ if $f > 0$ and

$$(-1)^n x^{n+1} [\ln f(x)]^{(n)} \qquad (n \in \mathbb{N})$$

are nonnegative and decreasing on I^+.

The class of strongly logarithmically completely monotonic on I^+ is denoted by $SLCM(I^+)$.

It is apparent that if each of the functions f and g belongs to

$$SLCM(I^+) \quad \big(LCM(I)\big),$$

then

$$fg \in SLCM(I^+) \quad \big(LCM(I)\big).$$

Guo and Srivastava [12] proved an important relationship between $SLCM(\mathbb{R}^+)$ and $SCM(\mathbb{R}^+)$ as follows.

Theorem B (see [12]). *The following assertion holds true*:

$$SLCM(\mathbb{R}^+) \cap SCM(\mathbb{R}^+) = \emptyset.$$

The following result (see [12]) also reveals a relationship between $SLCM(I^+)$ and $SCM(I^+)$.

Theorem C (see [12]). *Suppose that*

$$f \in C(I^+), \quad f > 0 \quad \textit{and} \quad f' \in SCM(I^+).$$

If

$$x f'(x) \geqq f(x) \qquad (x \in I^+),$$

then

$$\frac{1}{f} \in SLCM(I^+).$$

Remark 2. The condition:

$$xf'(x) \geqq f(x) \qquad (x \in I^+)$$

in Theorem C can not be dropped. See a counterexample given by Guo and Srivastava [12].

In order to simplify the statements of our results, we also use the following two terminologies.

Definition F (see [12]). A function f is said to be almost strongly completely monotonic on I^+ if, for all $n \in \mathbb{N}$, the functions $(-1)^n x^{n+1} f^{(n)}(x)$ are nonnegative and decreasing on I^+.

The class of all almost strongly completely monotonic functions on the interval I^+ is denoted by $ASCM(I^+)$.

Definition G (see [25]). A function f is said to be almost completely monotonic on an interval I, if the function $f \in C(I)$ has derivatives of all orders on I^o and

$$(-1)^n f^{(n)}(x) \geqq 0 \qquad (x \in I^o)$$

for all $n \in \mathbb{N}$.

The class of almost completely monotonic functions on the interval $I$is denoted by $ACM(I^+)$.

The following result was established by Guo and Srivastava [12].

Theorem D (see [12]). *The following assertion holds true*: $SLCM(I^+) \subset ASCM(I^+)$.

For compositions of completely monotonic and related functions, the following two results were given in [27, Chapter IV]

Theorem E. *Suppose that*

$$f \in AM(I_1), \quad g \in AM(I) \qquad \text{and} \qquad \mathcal{R}(g) \subset I_1.$$

Then $f \circ g \in AM(I)$.

Theorem F. *Suppose that*

$$f \in AM(I_1), \quad g \in CM(I) \qquad \text{and} \qquad \mathcal{R}(g) \subset I_1.$$

Then $f \circ g \in CM(I)$.

The result below is a converse of Theorem F (see [17, Theorem 5]).

Theorem G. *Let the function* f *be defined on* $[0, \infty)$. *If, for each* $g \in CM(\mathbb{R}^+)$, $f \circ g \in CM(\mathbb{R}^+)$, *then* $f \in AM(\mathbb{R}^+)$.

Recently, Srivastava *et al.* [25] proved a number of interesting results including (for example)the following theorem.

Theorem H (see [25]). *Suppose that*

$$f \in AM(I), \quad g \in ASCM(I^+) \qquad \text{and} \qquad \mathcal{R}(g) \subset I.$$

Then $f \circ g \in ASCM(I^+)$.

There is a rich literature on completely monotonic and related functions. For several recent works, see (for example) [1], [3], [5] to [15], [16] and [18] to [25].

In this article, we further investigate the properties of the composition of functions which are related to the above-defined classes of completely monotonic functions and logarithmically completely monotonic functions. We begin by stating our main results as follows.

Theorem 1. *Suppose that*

$$f \in ACM(I_1), \quad g \in C(I), \quad g' \in CM(I^o) \qquad \text{and} \qquad \mathcal{R}(g) \subset I_1.$$

Then $f \circ g \in ACM(I)$.

Corollary 1. *Suppose that*

$$f \in ACM(I_1), \quad -g \in ACM(I) \qquad \text{and} \qquad \mathcal{R}(g) \subset I_1.$$

Then $f \circ g \in ACM(I)$.

Theorem 2. *Suppose that*

$$f \in LCM(I_1), \quad g \in C(I), \quad g' \in CM(I^o) \qquad \text{and} \qquad \mathcal{R}(g) \subset I_1.$$

Then $f \circ g \in LCM(I)$.

Theorem 3. *Suppose that*

$$f \in SLCM(I_1^+), \quad g' \in SCM(I^+) \qquad \text{and} \qquad \mathcal{R}(g) \subset I_1^+.$$

If

$$2xg'(x) \geqq g(x) \qquad (x \in I^+),$$

then $f \circ g \in SLCM(I^+)$.

Remark 3. The condition:

$$2xg'(x) \geqq g(x) \qquad (x \in I^+)$$

in Theorem 3 cannot be waived. For example, we let

$$f(x) := e^{1/x} \qquad \text{and} \qquad g(x) := \ln x$$

and suppose that

$$I^+ := (e^2, \infty).$$

Then it is easy to verify that

$$f \in SLCM(\mathbb{R}^+) \qquad \text{and} \qquad g' \in SCM(I^+).$$

Moreover, the following condition:

$$2xg'(x) \geqq g(x) \qquad (x \in I^+)$$

is not satisfied. We can show that

$$h(x) := f \circ g(x) = \exp\left(\frac{1}{\ln x}\right) \notin SLCM(I^+).$$

In fact, we have

$$(-1)^1 x^2 [\ln h(x)]' = \frac{x}{\ln^2 x} \to \infty \qquad (x \to \infty).$$

Therefore, the function $(-1)^1 x^2 [\ln h(x)]'$ cannot be decreasing on I^+. Consequently, we find that

$$f \circ g \notin SLCM(I^+).$$

Theorem 4. *Suppose that*

$$f \in LCM(I_1), \quad -g \in ACM(I) \qquad \text{and} \qquad \mathcal{R}(g) \subset I_1.$$

Then $f \circ g \in LCM(I)$.

Theorem 5. *Let* I_1 *and* I *be open intervals. Also let* f *and* g *be defined on* I_1 *and* I*, respectively. If*

$$f' \in LCM(I_1), \quad g' \in LCM(I) \qquad \text{and} \qquad \mathcal{R}(g) \subset I_1,$$

then $(f \circ g)' \in LCM(I)$.

2. A Set of Lemmas

We need each of the following lemmas to prove our main results which are stated already in Section 1.

Lemma 1 (see [4, p. 21]). *Suppose that the functions* $y = y(x)$ $(x \in I_1)$ *and* $x = \varphi(t)$ $(t \in I)$ *are* n *times differentiable and that* $\mathcal{R}(\varphi) \subset I_1$. *Then, for* $t \in I$,

$$\frac{d^n y}{dt^n} = \sum_{(i_1,\ldots,i_n)\in\Lambda_n} \left(\frac{n!}{i_1!\cdots i_n!}\right) \frac{d^m y(\varphi(t))}{dx^m} \prod_{j=1}^{n} \left\{\left(\frac{\varphi^{(j)}(t)}{j!}\right)^{i_j}\right\},$$

where

$$m = i_1 + \cdots + i_n$$

and

$$\Lambda_n := \left\{(i_1,\ldots,i_n): \ i_1,\ldots,i_n \in \mathbb{N}_0 \quad \text{and} \quad \sum_{\nu=1}^{n} \nu i_\nu = n\right\}. \tag{1}$$

Lemma 2 (see [25, Theorem 3]). *Suppose that*

$$f \in ASCM(I_1^+), \quad g' \in SCM(I^+) \qquad \text{and} \qquad \mathcal{R}(g) \subset I_1^+.$$

If

$$2xg'(x) \geqq g(x) \qquad (x \in I^+),$$

then $f \circ g \in ASCM(I^+)$.

3. Proofs of the Main Results

Proof. **[Proof of Theorem** 1**]**

Let

$$h(x) := f \circ g(x) = f\big(g(x)\big) \qquad (x \in I).$$

By Lemma 1, for $n \in \mathbb{N}$, we find that

$$(-1)^n h^{(n)}(x) = (-1)^n \sum_{(i_1,\ldots,i_n)\in\Lambda_n} \left(\frac{n!}{i_1!\cdots i_n!}\right) f^{(m)}(g(x)) \prod_{j=1}^{n} \left\{\left(\frac{g^{(j)}(x)}{j!}\right)^{i_j}\right\}$$

$$= \sum_{(i_1,\ldots,i_n)\in\Lambda_n} \left(\frac{n!}{i_1!\cdots i_n!}\right) (-1)^m f^{(m)}(g(x)) \prod_{j=1}^{n} \left\{\left(\frac{(-1)^{j-1} g^{(j)}(x)}{j!}\right)^{i_j}\right\}, \tag{2}$$

where

$$m = i_1 + \cdots + i_n \geqq 1$$

and Λ_n is defined by (1).

Since

$$f \in ACM(I_1) \qquad \text{and} \qquad \mathcal{R}(g) \subset I_1,$$

we get, for $i \in \mathbb{N}$,

$$(-1)^i f^{(i)}\big(g(x)\big) \geqq 0 \qquad (x \in I). \tag{3}$$

Since $g' \in CM(I)$, we find for $j \in \mathbb{N}_0$ that

$$(-1)^j\Big(g'(x)\Big)^{(j)} = (-1)^j g^{(j+1)}(x) \geqq 0 \qquad (x \in I)$$

or, equivalently, that

$$(-1)^{i-1} g^{(i)}(x) \geqq 0 \qquad (x \in I;\ i \in \mathbb{N}). \tag{4}$$

By (3) and (4), we find from (2) that

$$(-1)^n h^{(n)}(x) \geqq 0 \qquad (x \in I)$$

for $n \in \mathbb{N}$.

The proof of Theorem 1 is thus completed. □

Proof. **[Proof of Corollary** 1**]**

Since

$$-g \in ACM(I),$$

we get

$$g \in C(I) \qquad \text{and} \qquad g' \in CM(I^o).$$

Then, by Theorem 1, we find that

$$f \circ g \in ACM(I).$$

This evidently completes the proof of Corollary 1. □

Proof. **[Proof of Theorem** 2**]**

Since

$$f \in LCM(I_1),$$

we get

$$\ln f \in ACM(I_1).$$

Then, by Theorem 1, we have

$$(\ln f) \circ g \in ACM(I). \tag{5}$$

Since

$$(\ln f) \circ g = \ln(f \circ g),$$

wev find from (5) that

$$\ln(f \circ g) \in ACM(I). \tag{6}$$

Also, from (6) we observe that

$$f \circ g \in LCM(I).$$

The proof of Theorem 2 is thus completed. □

Proof. **[Proof of Theorem** 3**]**

Since
$$f \in SLCM(I_1^+),$$
we get
$$\ln f \in ASCM(I_1^+).$$
Then, by Lemma 2, we have
$$(\ln f) \circ g \in ASCM(I^+). \tag{7}$$
Since
$$\ln(f \circ g) = (\ln f) \circ g, \tag{8}$$
we find from (7) that
$$\ln(f \circ g) \in ASCM(I^+). \tag{9}$$
Furthermore, from (9), we see that
$$f \circ g \in SLCM(I^+).$$
The proof of Theorem 3 is evidently completed. □

Proof. **[Proof of Theorem** 4**]**

Since
$$f \in LCM(I_1),$$
we get
$$\ln f \in ACM(I_1). \tag{10}$$
Then, by Corollary 1, we have
$$(\ln f) \circ g \in ACM(I). \tag{11}$$
Since
$$(\ln f) \circ g = \ln(f \circ g), \tag{12}$$
we find from (11) that
$$\ln(f \circ g) \in ACM(I). \tag{13}$$
Moreover, from (13), we observe that
$$f \circ g \in LCM(I).$$
The proof of Theorem 4 is thus completed. □

Proof. **[Proof of Theorem** 5**]**

First of all, we know that
$$(f \circ g)'(x) = f'\big(g(x)\big) \cdot g'(x) \tag{14}$$
By Theorem 2 and in view of the fact that
$$LCM(I) \subset CM(I),$$

we have

$$f' \circ g \in LCM(I). \tag{15}$$

Since

$$g' \in LCM(I),$$

we find from (14) and (15) that

$$(f \circ g)' \in LCM(I).$$

This evidently completes the proof of Theorem 5. □

References

[1] H. Alzer and N. Batir, Monotonicity properties of the gamma function, *Appl. Math. Lett.* **20** (2007), 778–781.

[2] R. D. Atanassov and U. V. Tsoukrovski, Some properties of a class of logarithmically completely monotonic functions, *C. R. Acad. Bulgare Sci.* **41** (1988), 21–23.

[3] N. Batir, On some properties of the gamma function, *Exposition. Math.* **26** (2008), 187–196.

[4] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, Sixth Edition, Academic Press, New York, 2000.

[5] B.-N. Guo and F. Qi, A completely monotonic function involving the tri-gamma function and with degree one, *Appl. Math. Comput.* **218** (2012), 9890–9897.

[6] S. Guo, Logarithmically completely monotonic functions and applications, *Appl. Math. Comput.* **221** (2013), 169–176.

[7] S. Guo, Some properties of completely monotonic sequences and related interpolation, *Appl. Math. Comput.* **219** (2013), 4958–4962.

[8] S. Guo and F. Qi, A class of logarithmically completely monotonic functions associated with the gamma function, *J. Comput. Appl. Math.* **224** (2009), 127–132.

[9] S. Guo, F. Qi and H. M. Srivastava, A class of logarithmically completely monotonic functions related to the gamma function with applications, *Integral Transforms Spec. Funct.* **23** (2012), 557–566.

[10] S. Guo, F. Qi and H. M. Srivastava, Supplements to a class of logarithmically completely monotonic functions associated with the gamma function, *Appl. Math. Comput.* **197** (2008), 768–774.

[11] S. Guo, F. Qi and H. M. Srivastava, Necessary and sufficient conditions for two classes of functions to be logarithmically completely monotonic, *Integral Transforms Spec. Funct.* **18** (2007), 819–826.

[12] S. Guo and H. M. Srivastava, A certain function class related to the class of logarithmically completely monotonic functions, *Math. Comput. Modelling* **49** (2009), 2073–2079.

[13] S. Guo and H. M. Srivastava, A class of logarithmically completely monotonic functions, *Appl. Math. Lett.* **21** (2008), 1134–1141.

[14] S. Guo, H. M. Srivastava and N. Batir, A certain class of completely monotonic sequences, *Adv. Difference Equations* **2013** (2013), Article ID 294, 1–9.

[15] S. Guo, J.-G. Xu and F. Qi, Some exact constants for the approximation of the quantity in the Wallis' formula, *J. Inequal. Appl.* **2013** (2013), Article ID 67, 1–7.

[16] V. B. Krasniqi, H. M. Srivastava and S. S. Dragomir, Some complete monotonicity properties for the (p, q)-gamma function, *Appl. Math. Comput.* **219** (2013), 10538–10547.

[17] L. Lorch and D. J. Newman, On the composition of completely monotonic functions, completely monotonic sequences and related questions, *J. London Math. Soc.* (*Ser.* 2) **28** (1983), 31–45.

[18] F. Qi, Three classes of logarithmically completely monotonic functions involving gamma and psi functions, *Integral Transforms Spec. Funct.* **18** (2007), 503–509.

[19] F. Qi and C.-P. Chen, A complete monotonicity property of the gamma function, *J. Math. Anal. Appl.* **296** (2004), 603–607.

[20] F. Qi and B.-N. Guo, Necessary and sufficient conditions for functions involving the tri- and tetra-gamma functions to be completely monotonic, *Adv. Appl. Math.* **44** (2010), 71–83.

[21] F. Qi, B.-N. Guo and C.-P. Chen, Some completely monotonic functions involving the gamma and polygamma functions, *J. Austral. Math. Soc.* **80** (2006), 81–88.

[22] F. Qi, S. Guo and B.-N. Guo, Complete monotonicity of some functions involving polygamma functions, *J. Comput. Appl. Math.* **233** (2010), 2149–2160.

[23] A. Salem, A completely monotonic function involving q-gamma and q-digamma functions, *J. Approx. Theory* **164** (2012), 971–980.

[24] H. M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier Science Publishers, Amsterdam, London and New York, 2012.

[25] H. M. Srivastava, S. Guo and F. Qi, Some properties of a class of functions related to completely monotonic functions, *Comput. Math. Appl.* **64** (2012), 1649–1654.

[26] S. Y. Trimble, J. Wells and F. T. Wright, Supperaddative functions and a statistical application, *SIAM J. Math. Anal.* **20** (1989), 1255–1259.

[27] D. V. Widder, *The Laplace Transform*, Seventh Printing, Princeton University Press, Princeton, 1966.