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The Sequence Space $BV_{\sigma}^{I}(p)$

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Abstract. In this article we introduce the sequence space $BV_{\sigma}^{I}(p)$ for $p = (p_k)$, a sequence of positive real numbers and study the topological properties and some inclusion relations on this space.

1. Introduction

Let N, R and C be the sets of all natural, real and complex numbers respectively. We write

$$\omega = \{x = (x_k) : x_k \in R \text{ or } C\},$$

the space of all real or complex sequences.

Let ℓ_∞, c and c_0 denote the Banach spaces of bounded, convergent and null sequences respectively normed by $||x||_\infty = \sup_k |x_k|$.

The following subspaces of ω were first introduced and discussed by Maddox[6-7].
$\ell(p) = \{x \in \omega : \sum_k |x_k|^{p_k} < \infty\}$,
$\ell_\infty(p) = \{x \in \omega : \sup_k |x_k|^{p_k} < \infty\}$,
$c(p) := \{x \in \omega : \lim_k |x_k - l|^{p_k} = 0, \text{ for some } l \in C\}$,
$c_0(p) := \{x \in \omega : \lim_k |x_k|^{p_k} = 0, \}$,
where $p = (p_k)$ is a sequence of striclty positive real numbers.

After then Lascarides[2-3] defined the following sequence spaces
$\ell_\infty\{p\} = \{x \in \omega : \text{ there exists } r > 0 \text{ such that } \sup_k |x_k r|^{p_k} t_k < \infty\}$,
$c_0\{p\} = \{x \in \omega : \text{ there exists } r > 0 \text{ such that } \lim_k |x_k r|^{p_k} t_k = 0, \}$,
$\ell\{p\} = \{x \in \omega : \text{ there exists } r > 0 \text{ such that } \sum_{k=1}^{\infty} |x_k r|^{p_k} t_k < \infty\}$,
Where $t_k = p_k^{-1}$, for all $k \in N$.

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Let v denote the space of sequences of bounded variation, that is

$$v = \{x = (x_k) : \sum_{k=0}^{\infty} |x_k - x_{k-1}| < \infty, x_{-1} = 0\}.$$

v is a Banach space normed by

$$\|x\| = \sum_{k=0}^{\infty} |x_k - x_{k-1}| (See[15]).$$

Let σ be a mapping of the set of the positive integers into itself having no finite orbits. A continuous linear functional ϕ on ℓ_∞ is said to be an invariant mean or σ-mean if and only if

(1) $\phi(x) \geq 0$ where the sequence $x = (x_k)$ has $x_k \geq 0$ for all k,
(2) $\phi(e) = 1$, where $e = \{1, 1, 1, ...\}$, and
(3) $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in \ell_\infty$.

In case σ is the translation mapping n→n+1, a σ-mean is often called a Banach limit(See[20]) and V_σ, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences. If $x = (x_k)$, set $Tx = \{Tx_k\} = \{x_{\sigma(n)}\}$.

It can be shown that

$$V_\sigma = \{x = (x_k) : \lim_{m\to\infty} t_{m,k}(x) = L \text{ uniformly in k, L} = \sigma - \lim x\} \quad [1]$$

where $m \geq 0, k > 0$

$$t_{m,k}(x) = \frac{x_k + x_{\sigma(k)} + ... + x_{\sigma^m(k)}}{m+1}, \text{and } t_{-1,k} = 0.(See[4]).$$

where $\sigma^m(k)$ denotes the mth iterate of σ at k. The special case of [1] in which σ(n)=n+1 was given by (Lorentz[4,Theorem.1.]), and that the general result can be proved in a similar way. It is familiar that a Banach limit extends the limit functional on c.

Theorem.1.1 (See[15,Theorem.1.1]) A σ-mean extends the limit functional on c in the sense that $\phi(x) = \lim x$ for all $x \in c$ if and only if σ has no finite orbits, that is to say, if and only if, for all $k \geq 0, j \geq 1$.

$$\sigma^j(k) \neq k$$

. Put

$$\phi_{m,k}(x) = t_{m,k}(x) - t_{m-1,k}(x),$$

assuming that $t_{-1,k} = 0$. A straight forward calculation shows (See[14]) that

$$\phi_{m,k}(x) = \begin{cases} \frac{1}{m(m+1)} \sum_{j=1}^{m} J(x_{\sigma^j(k)} - x_{\sigma^{j-1}(k)}) & (m\geq 1) \\ x_k, & (m = 0) \end{cases}$$

For any sequence x, y and scalar λ we have

$$\phi_{m,k}(x + y) = \phi_{m,k}(x) + \phi_{m,k}(y) \text{ and } \phi_{m,k}(\lambda x) = \lambda \phi_{m,k}(x).$$

Definition.1.2. A sequence $x \in \ell_\infty$ is of σ- bounded variation if and only if
(1) $\sum_{m=0}^{\infty} |\phi_{m,k}(x)|$ converges uniformly in k, and

(2) $\lim_{m\to\infty} t_{m,k}(x)$, which must exist, should take the same value for all k.

Mursaleen [15] defined the sequence space BV_σ, the space of all sequences of σ-bounded variation

$$BV_\sigma = \{x \in \ell_\infty : \sum_m |\phi_{m,k}(x)| < \infty, \text{uniformly in k}\}.$$

Theorem.1.3.(See[15,Theorem.2.2]). BV_σ is a Banach space normed by

$$||x|| = \sup_k \sum_{m=0}^{\infty} |\phi_{m,k}(x)|.$$

The concept of statistical convergence was first introduced by Fast[5] and also independently by Buck[19] and Schoenberg [8] for real and complex sequences.Further this concept was studied by Connor[9-10], Connor, Fridy and Kline [11]and many others. Statistical convergence is a generalization of the usual notion of convergence that parallels the usual theory of convergence.

A sequence $x = (x_k)$ is said to be Statistically convergent to L if for a given $\epsilon > 0$

$$\lim_k \frac{1}{k}|\{i : |x_i - L| \geq \epsilon, i \leq k\}| = 0.$$

The notion of I-convergence is a generalization of the statistical convergence. At the initial stage it was studied by Kostyrko, Šalát, Wilczyński[16]. Later on it was studied by Šalát, Tripathy, Ziman[22-23] and Demirci[13]. Recentlly further it was studied by Tripathy and Hazarika[1], Khan and Ebadullah[25-26].

Here we give some preliminaries about the notion of I-convergence.

Let X be a non empty set. Then a family of sets $I \subseteq 2^X$(2^X denoting the power set of X) is said to be an ideal if I is additive i.e $A, B \in I \Rightarrow A \cup B \in I$ and hereditary i.e $A \in I, B \subseteq A \Rightarrow B \in I$.

A non-empty family of sets $f(I) \subseteq 2^X$ is said to be filter on X if and only if $\phi \notin f(I)$, for $A, B \in f(I)$ we have $A \cap B \in f(I)$ and for each $A \in f(I)$ and $A \subseteq B$ implies $B \in f(I)$.

An Ideal $I \subseteq 2^X$ is called non-trivial if $I \neq 2^X$.
A non-trivial ideal $I \subseteq 2^X$ is called admissible if

$$\{\{x\} : x \in X\} \subseteq I.$$

A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset. For each ideal I, there is a filter $f(I)$ corresponding to I. i.e

$$f(I) = \{K \subseteq N : K^c \in I\},$$

where

$$K^c = N - K.$$

Definition 1.4. A sequence $(x_k) \in \omega$ is said to be I-convergent to a number L if for every $\epsilon > 0$,

$$\{k \in N : |x_k - L| \geq \epsilon\} \in I.$$

In this case we write $I - \lim x_k = L$.

The space c^I of all I-convergent sequences to L is given by

$$c^I = \{(x_k) \in \omega : \{k \in N : |x_k - L| \geq \epsilon\} \in I, \text{for some} L \in C\ \}.$$

Definition 1.5. A sequence $(x_k) \in \omega$ is said to be I-null if $L = 0$ .In this case we write $I - \lim x_k = 0$.

Definition 1.6. A sequence $(x_k) \in \omega$ is said to be I-cauchy if for every $\epsilon > 0$ there exists a number $m = m(\epsilon)$ such that

$$\{k \in N : |x_k - x_m| \geq \epsilon\} \in I.$$

Definition 1.7. A sequence $(x_k) \in \omega$ is said to be I-bounded if there exists $M > 0$ such that

$$\{k \in N : |x_k| > M\} \in I.$$

Definition 1.8. For any set E of sequences the space of multipliers of E, denoted by $M(E)$ is given by

$$M(E) = \{a \in \omega : ax \in E \text{ for all } x \in E\}(\text{see}[21]).$$

Definition 1.9. A map $\hbar$ defined on a domain $D \subset X$ i.e $\hbar : D \subset X \to R$ is said to satisfy Lipschitz condition if $|\hbar(x) - \hbar(y)| \leq K|x - y|$ where $K$is known as the Lipschitz constant.The class of K-Lipschitz functions defined on D is denoted by $\hbar \in (D, K)$.(see[25]).

Definition 1.10. A convergence field of I-covergence is a set

$$F(I) = \{x = (x_k) \in \ell_\infty : \text{there exists } I - \lim x \in R\}.$$

The convergence field $F(I)$ is a closed linear subspace of ℓ_∞ with respect to the supremum norm, $F(I) = \ell_\infty \cap c^I$(See[22,23]).

Define a function $\hbar : F(I) \to R$ such that $\hbar(x) = I - \lim x$, for all $x \in F(I)$, then the function $\hbar : F(I) \to R$ is a Lipschitz function (see[22,23,25])(c.f.[9],[12],[17],[18],[21],[24],[27]).

Recently khan, Ebadullah and Suantai[26] defined the following sequence space

$$BV_\sigma^I = \{(x_k) \in \omega : \{k \in N : |\phi_{m,k}(x) - L| \geq \epsilon\} \in I, \text{for some L} \in C\ \}.$$

Main Results.

In this article we introduce the sequence space.

$$BV_\sigma^I(p) = \{(x_k) \in \omega : \{k \in N : |\phi_{m,k}(x) - L|^{p_k} \geq \epsilon\} \in I, \text{for some L} \in C\ \}.$$

Theorem 2.1. $BV_\sigma^I(p)$ is a linear space.

Proof. Let $(x_k), (y_k) \in BV_\sigma^I(p)$ and let α, β be scalars. Then for a given $\epsilon > 0$.
we have

$$\{k \in N : |\phi_{m,k}(x) - L_1|^{p_k} \geq \frac{\epsilon}{2M_1}, \text{ for some } L_1 \in \text{C}\} \in I$$

$$\{k \in N : |\phi_{m,k}(y) - L_2|^{p_k} \geq \frac{\epsilon}{2M_2}, \text{ for some } L_2 \in \text{C}\} \in I$$

where

$$M_1 = D.max\{1, \sup_k |\alpha|^{p_k}\}$$

$$M_2 = D.max\{1, \sup_k |\beta|^{p_k}\}$$

and

$$D = max\{1, 2^{H-1}\} \text{ where } H = \sup_k p_k \geq 0.$$

Let

$$A_1 = \{k \in N : |\phi_{m,k}(x) - L_1|^{p_k} < \frac{\epsilon}{2M_1}, \text{ for some } L_1 \in \mathbb{C}\} \in f(I)$$

$$A_2 = \{k \in N : |\phi_{m,k}(y) - L_2|^{p_k} < \frac{\epsilon}{2M_2}, \text{ for some } L_2 \in \mathbb{C}\} \in f(I)$$

be such that $A_1^c, A_2^c \in I$. Then

$$A_3 = \{k \in N : |(\alpha\phi_{m,k}(x) + \beta\phi_{m,k}(y) - (\alpha L_1 + \beta L_2)|^{p_k}) < \epsilon\}$$

$$\supseteq \{k \in N : |\alpha|^{p_k}|\phi_{m,k}(x) - L_1|^{p_k} < \frac{\epsilon}{2M_1}|\alpha|^{p_k}.D\}$$

$$\cap\{k \in N : |\beta|^{p_k}|\phi_{m,k}(y) - L_2|^{p_k} < \frac{\epsilon}{2M_2}|\beta|^{p_k}.D\}.$$

Thus

$$A_3^c = A_1^c \cap A_2^c \in I.$$

Hence

$$(\alpha\phi_{m,k}(x) + \beta\phi_{m,k}(y)) \in BV_\sigma^I(p).$$

Hence $BV_\sigma^I(p)$ is a linear space.

Theorem 2.2. Let $(p_k) \in \ell_\infty$. Then $BV_\sigma^I(p)$ is a paranormed space, paranormed by $\|x\|_* = \sup_k |\phi_{m,k}(x)|^{\frac{p_k}{M}}$ where $M = max\{1, \sup_k p_k\}$.

Proof. Let $x = (x_k), \ y = (y_k) \in BV_\sigma^I(p)$.
(1) Clearly, $\|x\|_* = 0$ if and only if $x = 0$.
(2) $\|x\|_* = \|-x\|_*$ is obvious.
(3) Since $\frac{p_k}{M} \leq 1$ and $M > 1$, using Minkowski's inequality we have

$$\sup_k |\phi_{m,k}(x) + \phi_{m,k}(y)|^{\frac{p_k}{M}} \leq \sup_k |\phi_{m,k}(x)|^{\frac{p_k}{M}} + \sup_k |\phi_{m,k}(y)|^{\frac{p_k}{M}}.$$

(4) Now for any complex λ we have (λ_k) such that $\lambda_k \to \lambda$, $(k \to \infty)$.
Let $(x_k) \in BV_\sigma^I(p)$ such that $|\phi_{m,k}(x) - L|^{p_k} \geq \epsilon$.
Therefore, $\|\phi_{m,k}(x) - L\|_* = \sup_k |\phi_{m,k}(x) - L|^{\frac{p_k}{M}} \leq \sup_k |\phi_{m,k}(x)|^{\frac{p_k}{M}} + \sup_k |L|^{\frac{p_k}{M}}$.
Hence $\|\lambda_n\phi_{m,k}(x) - \lambda L)\|_* \leq \|\lambda_n\phi_{m,k}(x)\|_* + \|\lambda L\|_* = \lambda_n\|\phi_{m,k}(x)\|_* + \lambda\|L\|_*$ as $(k \to \infty)$.
Hence $BV_\sigma^I(p)$ is a paranormed space.

Theorem 2.3. $BV_\sigma^I(p)$ is a closed subspace of $\ell_\infty(p)$.
Proof. Let $(x_k^{(n)})$ be a cauchy sequence in $BV_\sigma^I(p)$ such that $x^{(n)} \to x$.
We show that $x \in BV_\sigma^I(p)$.
Since $(x_k^{(n)}) \in BV_\sigma^I(p)$, then there exists a_n such that

$$\{k \in N : |\phi_{m,k}(x^{(n)}) - a_n|^{p_k} \geq \epsilon\} \in I.$$

We need to show that
(1) (a_n) converges to a.

(2) If $U = \{k \in N : |x_k - a|^{p_k} < \epsilon\}$, then $U^c \in I$.

(1) Since $(x_k^{(n)})$ is a cauchy sequence in $BV_\sigma^I(p)$ then for a given $\epsilon > 0$, there exists $k_0 \in N$ such that

$$\sup_k |\phi_{m,k}(x_k^{(n)}) - \phi_{m,k}(x_k^{(i)})|^{\frac{p_k}{M}} < \frac{\epsilon}{3}, \text{for all n,i} \geq k_0$$

For a given $\epsilon > 0$, we have

$$B_{ni} = \{k \in N : |\phi_{m,k}(x_k^{(n)}) - \phi_{m,k}(x_k^{(i)})|^{p_k} < (\frac{\epsilon}{3})^M\}$$

$$B_i = \{k \in N : |\phi_{m,k}(x_k^{(i)}) - a_i|^{p_k} < (\frac{\epsilon}{3})^M\}$$

$$B_n = \{k \in N : |\phi_{m,k}(x_k^{(n)}) - a_n|^{p_k} < (\frac{\epsilon}{3})^M\}$$

Then $B_{ni}^c, B_i^c, B_n^c \in I$. Let

$$B^c = B_{ni}^c \cap B_i^c \cap B_n^c,$$

where

$$B = \{k \in N : |a_i - a_n|^{p_k} < \epsilon\} \in f(I).$$

Then $B^c \in I$. We choose $k_0 \in B^c$,then for each $n, i \geq k_0$, we have

$$\{k \in N : |a_i - a_n|^{p_k} < \epsilon\} \supseteq \{k \in N : |\phi_{m,k}(x_k^{(i)}) - a_i|^{p_k} < (\frac{\epsilon}{3})^M\}$$

$$\cap \{k \in N : |\phi_{m,k}(x_k^{(n)}) - \phi_{m,k}(x_k^{(i)})|^{p_k} < (\frac{\epsilon}{3})^M\}$$

$$\cap \{k \in N : |\phi_{m,k}(x_k^{(n)}) - a_n|^{p_k} < (\frac{\epsilon}{3})^M\}$$

Then (a_n) is a cauchy sequence of scalars in C, so there exists a scalar $a \in C$ such that $(a_n) \to a$, as $n \to \infty$.

(2) Let $0 < \delta < 1$ be given. Then we show that if

$$U = \{k \in N : |\phi_{m,k}(x) - a|^{p_k} < \delta\},$$

then $U^c \in I$. Since $\phi_{m,k}(x^{(n)}) \to \phi_{m,k}(x)$, then there exists $q_0 \in N$ such that

$$P = \{k \in N : |\phi_{m,k}(x^{(q_0)} - \phi_{m,k}(x)|^{p_k} < (\frac{\delta}{3D})^M\} \qquad [2]$$

which implies that $P^c \in I$.
The number q_0 can be so choosen that together with [2], we have

$$Q = \{k \in N : |a_{q_0} - a|^{p_k} < \frac{\delta}{3D}\}$$

such that $Q^c \in I$. Since

$$\{k \in N : |\phi_{m,k}(x_k^{(q_0)}) - a_{q_0}|^{p_k} \geq \delta\} \in I.$$

Then we have a subset S of N such that $S^c \in I$, where

$$S = \{k \in N : |\phi_{m,k}(x_k^{(q_0)}) - a_{q_0}|^{p_k} < (\frac{\delta}{3D})^M\}.$$

Let $U^c = P^c \cap Q^c \cap S^c$, where

$$U = \{k \in N : |\phi_{m,k}(x) - a|^{p_k} < \delta\}.$$

Therefore for each $k \in U^c$, we have

$$\{k \in N : |\phi_{m,k}(x) - a|^{p_k} < \delta\} \supseteq \{k \in N : |\phi_{m,k}(x^{(q_0)} - \phi_{m,k}(x)|^{p_k} < (\frac{\delta}{3D})^M\}$$

$$\cap \{k \in N : |\phi_{m,k}(x_k^{(q_0)}) - a_{q_0}|^{p_k} < (\frac{\delta}{3D})^M\}$$

$$\cap \{k \in N : |a_{q_0} - a|^{p_k} < (\frac{\delta}{3})^M\}.$$

Then the result follows.
Since the inclusion $BV_\sigma^I(p) \subset \ell_\infty^I(p)$ is strict so in view of Theorem 2.3 we have the following result.

Theorem 2.4. The space $BV_\sigma^I(p)$ is nowhere dense subset of $\ell_\infty(p)$.

Theorem 2.5. The space $BV_\sigma^I(p)$ is not seperable.

Proof. Let $M = \{m_1 < m_2 < m_3 < ....\}$ be an infinite subset of N such that $M \in I$.
Let

$$p_k = \begin{cases} 1, \text{if k} \in M, \\ 2, otherwise. \end{cases}$$

Let

$$P_0 = \{(x_k) : \phi_{m,k}(x) = 0 \text{ or } 1, \text{ for } \; k = m_j, \; j \in N \;\; and \;\; \phi_{m,k}(x) = 0, \text{otherwise}\}.$$

Since M is infinite, so P_0 is uncountable. Consider the class of open balls

$$B_1 = \{B(z, \frac{1}{2}) : z \in P_0\}.$$

Let C_1 be an open cover of $BV_\sigma^I(p)$ containing B_1. Since B_1 is uncountable, so C_1 cannot be reduced to a countable subcover for $BV_\sigma^I(p)$.Thus $BV_\sigma^I(p)$ is not seperable.

Theorem 2.6. The function $\hbar : BV_\sigma^I(p) \to R$ is the Lipschitz function and is uniformly continuous.

Proof. Let $x, y \in BV_\sigma^I(p)$ and $x \neq y$.Then the sets

$$A_x = \{k \in N : |\phi_{m,k}(x) - \hbar(x)|^{p_k} \geq ||x - y||_*\} \in I,$$

$$A_y = \{k \in N : |\phi_{m,k}(y) - \hbar(y)|^{p_k} \geq ||x - y||_*\} \in I.$$

Thus the sets,

$$B_x = \{k \in N : |\phi_{m,k}(x) - \hbar(x)|^{p_k} < ||x - y||_*\} \in f(I),$$

$$B_y = \{k \in N : |\phi_{m,k}(y) - \hbar(y)|^{p_k} < ||x - y||_*\} \in f(I).$$

Hence also

$$B = B_x \cap B_y \in f(I),$$

so that $B \neq \phi$.Now taking $k \in B$,

$$|\hbar(x) - \hbar(y)|^{p_k} \leq |\hbar(x) - \phi_{m,k}(x)|^{p_k} + |\phi_{m,k}(x) - \phi_{m,k}(y)|^{p_k} + |\phi_{m,k}(y) - \hbar(y)|^{p_k}$$

$$\leq 3||x - y||_*.$$

Thus $\hbar$ is a Lipschitz function.

Theorem 2.7. $c_0^I(p) \subset BV_\sigma^I(p) \subset \ell_\infty^I(p)$.

Proof. Let $(x_k) \in c_0^I(p)$.
Then we have

$$\{k \in N : |x_k|^{p_k} \geq \epsilon\} \in I.$$

Since $c_0 \subset BV_\sigma$. $(x_k) \in BV_\sigma^I(p)$ implies

$$\{k \in N : |\phi_{m,k}(x)|^{p_k} \geq \epsilon\} \in I.$$

Now let

$$A_1 = \{k \in N : |x_k|^{p_k} < \epsilon\} \in f(I).$$

$$A_2 = \{k \in N : |\phi_{m,k}(x)|^{p_k} < \epsilon\} \in f(I).$$

be such that $A_1^c, A_2^c \in I$. As

$$\ell_\infty(p) = \{x = (x_k) : \sup_k |x_k|^{p_k} < \infty\},$$

taking supremum over k we get $A_1^c \subset A_2^c$. Hence

$$c_0^I(p) \subset BV_\sigma^I(p) \subset \ell_\infty^I(p).$$

Theorem 2.8. $c^I(p) \subset BV_\sigma^I(p) \subset \ell_\infty^I(p)$.

Proof. Let $(x_k) \in c^I(p)$. Then we have

$$\{k \in N : |x_k - L|^{p_k} \geq \epsilon\} \in I.$$

Since $c \subset BV_\sigma \subset \ell_\infty$
$(x_k) \in BV_\sigma^I(p)$ implies

$$\{k \in N : |\phi_{m,k}(x) - L|^{p_k} \geq \epsilon\} \in I.$$

Now let

$$B_1 = \{k \in N : |\phi_k - L|^{p_k} < \epsilon\} \in f(I).$$

$$B_2 = \{k \in N : |\phi_{m,k}(x) - L|^{p_k} < \epsilon\} \in f(I).$$

be such that $B_1^c, B_2^c \in I$. As

$$\ell_\infty(p) = \{x = (x_k) : \sup_k |x_k|^{p_k} < \infty\},$$

taking supremum over k we get $B_1^c \subset B_2^c$. Hence

$$c^I(p) \subset BV_\sigma^I(p) \subset \ell_\infty^I(p).$$

Theorem 2.9. If $H = \sup_k p_k < \infty$, then we have $\ell_\infty^I \subset M(BV_\sigma^I(p))$, where the inclusion may be proper.

Proof. Let $a \in \ell_\infty^I$.This implies that $\sup_k |a_k| < 1 + K$. for some $K > 0$ and all k.

Therefore $x \in BV_\sigma^I(p)$ implies

$$\sup_k(|a_k\phi_{m,k}(x)|^{p_k}) \leq (1+K)^H \sup_k(|\phi_{m,k}(x)|^{p_k}) < \infty.$$

which gives $\ell_{\infty}^{I} \subset M(BV_{\sigma}^{I}(p))$.
To show that the inclusion may be proper, consider the case when $p_k = \frac{1}{k}$ for all k. Take $a_k = k$ for all k.Therefore $x \in BV_{\sigma}^{I}(p)$ implies

$$\sup_k(|a_k\phi_{m,k}(x)|^{p_k}) \leq \sup_k(|k|^{\frac{1}{k}}) \sup_k(|\phi_{m,k}(x)|^{p_k}) < \infty.$$

Thus in this case $a = (a_k) \in M(BV_{\sigma}^{I}(p))$ while $a \notin \ell_{\infty}^{I}$.

Theorem 2.10. Let (p_k) and (q_k) be two sequences of positive real numbers. Then $BV_{\sigma}^{I}(p) \supseteq BV_{\sigma}^{I}(q)$ if and only if $\liminf_{k \in K} \frac{p_k}{q_k} > 0$, where $K^c \subseteq N$ such that $K \in I$.

Proof. Let $\liminf_{k \in K} \frac{p_k}{q_k} > 0$ and $(x_k) \in BV_{\sigma}^{I}(q)$.
Then there exists $\beta > 0$ such that $p_k > \beta q_k$, for all sufficiently large $k \in K$.
Since $(x_k) \in BV_{\sigma}^{I}(q)$ for a given $\epsilon > 0$, we have

$$B_0 = \{k \in N : |\phi_{m,k}(x) - L|^{q_k} \geq \epsilon\} \in I$$

Let $G_0 = K^c \cup B_0$ Then $G_0 \in I$.
Then for all sufficiently large $k \in G_0$,

$$\{k \in N : |\phi_{m,k}(x) - L|^{p_k}) \geq \epsilon\} \subseteq \{k \in N : |\phi_{m,k}(x) - L|^{\beta q_k}) \geq \epsilon\} \in I.$$

Therefore $(x_k) \in BV_{\sigma}^{I}(p)$.
The converse part of the result follows obviously.

Theorem 2.11. Let (p_k) and (q_k) be two sequences of positive real numbers. Then $BV_{\sigma}^{I}(q) \supseteq BV_{\sigma}^{I}(p)$ if and only if $\liminf_{k \in K} \frac{q_k}{p_k} > 0$, where $K^c \subseteq N$ such that $K \in I$.

Proof. The proof follows similarly as the proof of Theorem 2.8.

Theorem 2.12. Let (p_k) and (q_k) be two sequences of positive real numbers. Then $BV_{\sigma}^{I}(p) = BV_{\sigma}^{I}(q)$ if and only if $\liminf_{k \in K} \frac{p_k}{q_k} > 0$, and $\liminf_{k \in K} \frac{q_k}{p_k} > 0$, where $K \subseteq N$ such that $K^c \in I$.

Proof. On combining Theorem 2.10 and 2.11 we get the required result.

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