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Exponential Inequality for $\tilde{\rho}$-Mixing Sequences and its Applications

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Abstract. Exponential inequality and complete convergence for $\tilde{\rho}$-mixing sequence are given. By using the exponential inequality, we study the asymptotic approximation of inverse moments for $\tilde{\rho}$-mixing sequences, which generalizes the corresponding one for independent sequence.

1. Introduction

Let $\{Z_n, n \geq 1\}$ be a sequence of independent nonnegative random variables with finite second moments. Denote

$$X_n = \sum_{i=1}^{n} Z_i / B_n \quad \text{and} \quad B_n^2 = \sum_{i=1}^{n} VarZ_i. \tag{1.1}$$

We will show that under suitable conditions the following equivalence relation holds, namely,

$$E(a + X_n)^{-r} \sim (a + EX_n)^{-r}, \quad n \to \infty, \tag{1.2}$$

where $a > 0$ and $r > 0$ are arbitrary real numbers. Here and below, $c_n \sim d_n$ means $c_n d_n^{-1} \to 1$ as $n \to \infty$.

The inverse moments can be applied in many practical applications. For example, they may be applied in Stein estimation and post-stratification (see Wooff [1] and Pittenger [2]), evaluating risks of estimators and powers of tests (see Marciniak and Wesolowski [3] and Fujioka [4]). In addition, they also appear in the reliability (see Gupta and Akman [5]) and life testing (see Mendenhall and Lehman [6]), insruance and financial mathematics (see Ramsay [7]), complex systems (see Jurlewicz and Weron [8]), and so on.

Under certain asymptotic-normality condition on X_n, relation (1.2) is established in Theorem 2.1 of Garcia and Palacios [9]. But, unfortunately, that theorem is not true under the suggested assumptions, as pointed out by Kaluszka and Okolewski [10]. The latter authors established (1.2) by modifying the assumptions, as follows:

(i) $r < 3$ ($r < 4$, in the *i.i.d.* case);

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(ii) $EX_n \to \infty$, $EZ_n^3 < \infty$;

(iii) (L_c condition) $\sum_{i=1}^{n} E|Z_i - EZ_i|^c / B_n^c \to 0$ ($c = 3$).

Hu et al. [11] considered weaker conditions: $EZ_n^{2+\delta} < \infty$, where Z_n satisfies $L_{2+\delta}$ condition and $0 < \delta \leq 1$. For more details about the inverse moment, one can refer to Wu et al. [12], Wang et al. [13], Sung [14], Shen [15], Shen et al. [16], and so forth. The main purpose of the paper is to extend the asymptotic approximation of inverse moment for independent sequence to the case of $\tilde{\rho}$-mixing sequence. It is easily seen that the key to the proof of asymptotic approximation of inverse moment is the exponential inequality. So in Section 2, we first give the exponential inequality for $\tilde{\rho}$-mixing sequence and complete convergence. In Section 3, we study the asymptotic approximation of inverse moments for $\tilde{\rho}$-mixing sequence by using the exponential inequality.

Firstly, we will give the definition of $\tilde{\rho}$-mixing sequence and some useful lemmas.

Let $\{X_n, n \geq 1\}$ be a random variable sequence defined on a fixed probability space $(\Omega, \mathcal{F}, P)$. Let n and m be positive integers. Write $\mathcal{F}_n^m = \sigma(X_i, n \leq i \leq m)$ and $\mathcal{F}_S = \sigma(X_i, i \in S \subset \mathbf{N})$. Given σ-algebras $\mathcal{B}, \mathcal{R}$ in $\mathcal{F}$, let

$$\rho(\mathcal{B}, \mathcal{R}) = \sup_{X \in L_2(\mathcal{B}), Y \in L_2(\mathcal{R})} \frac{|EXY - EXEY|}{\sqrt{Var(X) \cdot Var(Y)}}. \tag{1.3}$$

Define the ρ-mixing coefficients and $\tilde{\rho}$-mixing coefficients by

$$\rho(n) = \sup_{k \geq 1} \rho(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty),\ n \geq 0, \tag{1.4}$$

$$\tilde{\rho}(n) = \sup\{\rho(\mathcal{F}_S, \mathcal{F}_T) : \text{finite subsets } S, T \subset \mathbf{N},\ \text{such that dist}(S, T) \geq n\},\ n \geq 0. \tag{1.5}$$

Definition 1.1. *A sequence $\{X_n, n \geq 1\}$ of random variables is said to be ρ-mixing if $\rho(n) \downarrow 0$ as $n \to \infty$. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be $\tilde{\rho}$-mixing if there exists $k \in \mathbf{N}$ such that $\tilde{\rho}(k) < 1$.*

Remark 1.1. We point out that $\tilde{\rho}$-mixing is similar to ρ-mixing, but both are quite different. In fact, $\tilde{\rho}$-mixing coefficient (1.5) resembles the definition of the so-called maximal correlation coefficient (1.4), which is defined by (1.5) with index sets restricted to subsets S of $[1, k]$ and subsets T of $[n + k, \infty)$, $n, k \in \mathbf{N}$. In addition, in the definition of $\tilde{\rho}$-mixing, $\tilde{\rho}(k) < 1$ for some $k \in \mathbf{N}$ is needed. While in the definition of ρ-mixing, $\rho(n) \downarrow 0$ as $n \to \infty$ is needed. Bryc and Smolenski [17] pointed out that even in the stationary case, it may happen that $\tilde{\rho}(1) < 1$ while $\lim_{k\to\infty} \tilde{\rho}(k) \neq 0$. In this case, $\tilde{\rho}$-mixing is more general than ρ-mixing. For more details about the difference between ρ-mixing and $\tilde{\rho}$-mixing, one can refer to Bradley [18], Utev and Peligrad [19], and so on.

The concept of $\tilde{\rho}$-mixing was introduced by Bradley [20]. It is easily seen that $\tilde{\rho}$-mixing sequence contains independent sequence as a special case. Hence, studying the limiting behavior of $\tilde{\rho}$-mixing is of great interest. For more details about $\tilde{\rho}$-mixing random variables, one can refer to Utev and Peligrad [19], Zhu [21], Wu and Jiang [22-24], Wang et al. [25], Zhou et al. [26], Wu [27], and so forth.

The following lemmas are useful.

The first one is the moment inequality for $\tilde{\rho}$-mixing random variables with exponent 2.

Lemma 1.1. *Let $\{X_n, n \geq 1\}$ be a sequence of $\tilde{\rho}$-mixing random variables with $EX_n = 0$ and $EX_n^2 < \infty$ for each $n \geq 1$. Then for any $a \geq 0$ and $n \geq 1$,*

$$E\left(\sum_{i=a+1}^{a+n} X_i\right)^2 \leq \left(1 + 2\sum_{k=1}^{n} \tilde{\rho}(k)\right) \sum_{i=a+1}^{a+n} EX_i^2. \tag{1.6}$$

Proof. It follows from the definition of $\tilde{\rho}$-mixing sequence that

$$\begin{aligned} E\left(\sum_{i=a+1}^{a+n} X_i\right)^2 &= \sum_{i=a+1}^{a+n} EX_i^2 + 2\sum_{a+1\le i<j\le a+n} E(X_iX_j) \\ &\le \sum_{i=a+1}^{a+n} EX_i^2 + 2\sum_{a+1\le i<j\le a+n} \tilde{\rho}(j-i)(EX_i^2)^{1/2}(EX_j^2)^{1/2} \\ &\le \sum_{i=a+1}^{a+n} EX_i^2 + \sum_{k=1}^{n-1}\sum_{i=a+1}^{a+n-k} \tilde{\rho}(k)(EX_i^2 + EX_{k+i}^2) \\ &\le \sum_{i=a+1}^{a+n} EX_i^2 + 2\sum_{k=1}^{n} \tilde{\rho}(k)\sum_{i=a+1}^{a+n} EX_i^2 \\ &= \left(1+2\sum_{k=1}^{n}\tilde{\rho}(k)\right)\sum_{i=a+1}^{a+n} EX_i^2. \end{aligned}$$

This completes the proof of the lemma. ♯

The next one is the Rosenthal type maximal inequality for $\tilde{\rho}$-mixing random variables, which was obtained by Utev and Peligrad [18] as follows.

Lemma 1.2. *Let $\{X_n, n \ge 1\}$ be a sequence of $\tilde{\rho}$-mixing random variables, $EX_i = 0$, $E|X_i|^p < \infty$ for some $p \ge 2$ and for every $i \ge 1$. Then there exists a positive constant C depending only on p such that*

$$E\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} X_i\right|^p\right) \le C\left\{\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} EX_i^2\right)^{p/2}\right\}.$$

Throughout the paper, let $\{X_n, n \ge 1\}$ and $\{Z_n, n \ge 1\}$ be sequences of random variables defined on a fixed probability space $(\Omega, \mathcal{F}, P)$. For random variable X, denote $\| X \|_r = (E|X|^r)^{1/r}$, $r > 0$. C denotes a positive constant which may be different in various places.

2. Exponential Inequality and Complete Convergence for $\tilde{\rho}$-Mixing Sequence

In this section, denote $S_n = \sum_{i=1}^{n} X_i$ and $\Delta_n^2 = \sum_{i=1}^{n} EX_i^2$ for each $n \ge 1$.

Theorem 2.1. *Let $\{X_n, n \ge 1\}$ be a sequence of $\tilde{\rho}$-mixing random variables with $EX_n = 0$ and $|X_n| \le d < \infty$ a.s. for each $n \ge 1$. Then for any $\varepsilon > 0$ and $n \ge 1$,*

$$P(S_n > \varepsilon) \le C_1 \exp\left\{-\frac{\varepsilon^2}{C_2(4\Delta_n^2 + nd\varepsilon)}\right\}, \tag{2.1}$$

$$P(S_n < -\varepsilon) \le C_1 \exp\left\{-\frac{\varepsilon^2}{C_2(4\Delta_n^2 + nd\varepsilon)}\right\}, \tag{2.2}$$

$$P(|S_n| > \varepsilon) \le 2C_1 \exp\left\{-\frac{\varepsilon^2}{C_2(4\Delta_n^2 + nd\varepsilon)}\right\}, \tag{2.3}$$

where $C_1 = \exp\left\{1 + \tilde{\rho}(m+1) + \frac{n-4m}{4m}\right\}$, $C_2 = 8\left(1 + 2\sum_{k=1}^{m}\tilde{\rho}(k)\right)$ and $1 \le m \le n$ is some positive integer.

Proof. For fixed $n \ge 1$, by $1 \le m \le n$ we can see that there exists a nonnegative integer $l \le n$ such that

$$2lm \le n < 2(l+1)m. \tag{2.4}$$

For random variables $X_1, X_2, \cdots, X_n$, we construct the following random variable sequences

$$\begin{aligned} Y_1 &= X_1 + X_2 + \cdots + X_m, & Z_1 &= X_{m+1} + X_{m+2} + \cdots + X_{2m}, \\ Y_2 &= X_{2m+1} + X_{2m+2} + \cdots + X_{3m}, & Z_2 &= X_{3m+1} + X_{3m+2} + \cdots + X_{4m}, \\ &\cdots & &\cdots \\ Y_l &= X_{2(l-1)m+1} + \cdots + X_{(2l-1)m}, & Z_l &= X_{(2l-1)m+1} + \cdots + X_{2lm}. \\ Y_{l+1} &= \begin{cases} 0, & if \ \ 2lm \geq n, \\ X_{2lm} + \cdots + X_n, & if \ \ 2lm < n. \end{cases} \end{aligned}$$

If $2lm > n$, we assume that $X_{n+1}, X_{n+2}, \cdots, X_{2lm}$ above are all zero. Obviously,

$$S_n = \sum_{i=1}^{n} X_i = \sum_{i=1}^{l+1} Y_i + \sum_{i=1}^{l} Z_i. \tag{2.5}$$

For any $0 < t \leq \frac{1}{4md}$, it follows from (2.4) that

$$|tY_{l+1}| \leq t(n - 2lm)d \leq 2tmd < 1 \ \ a.s..$$

By (2.5), Markov's inequality and Hölder's inequality, we have

$$\begin{aligned} P(S_n > \varepsilon) &= P(tS_n > t\varepsilon) \ \leq \ \exp\{-t\varepsilon\} E \exp\{tS_n\} \\ &= \exp\{-t\varepsilon\} E \left[\exp\{tY_{l+1}\} \exp\left\{ t \sum_{i=1}^{l} Y_i \right\} \exp\left\{ t \sum_{i=1}^{l} Z_i \right\} \right] \\ &\leq \exp\{1 - t\varepsilon\} E \left[\exp\left\{ t \sum_{i=1}^{l} Y_i \right\} \exp\left\{ t \sum_{i=1}^{l} Z_i \right\} \right] \\ &\leq \exp\{1 - t\varepsilon\} \left(E \exp\left\{ 2t \sum_{i=1}^{l} Y_i \right\} \right)^{1/2} \left(E \exp\left\{ 2t \sum_{i=1}^{l} Z_i \right\} \right)^{1/2}. \end{aligned} \tag{2.6}$$

Denote $t_{i1} = 2(i-1)m + 1$, $t_{i2} = (2i-1)m$ and $\Delta(i) = \sum_{j=t_{i1}}^{t_{i2}} EX_j^2$ for $i = 1, 2, \cdots, l$. It follows from Lemma 1.1 that

$$EY_i^2 \leq \left(1 + 2 \sum_{k=1}^{m} \tilde{\rho}(k) \right) \Delta(i). \tag{2.7}$$

By $EY_i = 0$, $|4tY_i| \leq 4tmd \leq 1$ $a.s.$ and $1 + x \leq e^x$ $(x \geq 0)$, we can see that

$$\begin{aligned} E\left(e^{2tY_i}\right)^2 &= Ee^{4tY_i} \ = \ 1 + \sum_{j=2}^{\infty} \frac{E(4tY_i)^j}{j!} \\ &\leq 1 + \frac{E(4tY_i)^2}{2!} \left(1 + \frac{1}{3} + \frac{1}{4 \times 3} + \frac{1}{5 \times 4 \times 3} + \frac{1}{6 \times 5 \times 4 \times 3} + \cdots \right) \\ &\leq 1 + \frac{E(4tY_i)^2}{2!} \left(1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \cdots \right) \\ &= 1 + \frac{E(4tY_i)^2}{2!} \cdot \frac{1}{1 - \frac{1}{3}} \ \leq \ 1 + 16t^2 EY_i^2 \\ &\leq \exp\left\{ 16t^2 EY_i^2 \right\} \ \leq \ \exp\left\{ 16t^2 \left(1 + 2 \sum_{k=1}^{m} \tilde{\rho}(k) \right) \Delta(i) \right\}, \end{aligned}$$

which implies that

$$\left\|\exp\{2tY_i\}\right\|_2 = \left(E\left|e^{2tY_i}\right|^2\right)^{1/2} \le \exp\left\{C_2 t^2 \Delta(i)\right\},\ \ i = 1, 2, \cdots, l. \tag{2.8}$$

Together with the definition of $\tilde{\rho}$-mixing sequence and $1 + x \le e^x$ $(x \ge 0)$, it follows that

$$\begin{aligned}
E\exp\left\{2t\sum_{i=1}^{l} Y_i\right\} &= E\left(\exp\left\{2t\sum_{i=1}^{l-1} Y_i\right\}\exp\{2tY_l\}\right) \\
&\le E\exp\left\{2t\sum_{i=1}^{l-1} Y_i\right\}E\exp\{2tY_l\} + \tilde{\rho}(m+1)\left\|\exp\left\{2t\sum_{i=1}^{l-1} Y_i\right\}\right\|_2 \left\|\exp\{2tY_l\}\right\|_2 \\
&\le \left\|\exp\left\{2t\sum_{i=1}^{l-1} Y_i\right\}\right\|_2 \left\|\exp\{2tY_l\}\right\|_2 + \tilde{\rho}(m+1)\left\|\exp\left\{2t\sum_{i=1}^{l-1} Y_i\right\}\right\|_2 \left\|\exp\{2tY_l\}\right\|_2 \\
&= (1+\tilde{\rho}(m+1))\left\|\exp\left\{2t\sum_{i=1}^{l-1} Y_i\right\}\right\|_2 \left\|\exp\{2tY_l\}\right\|_2 \\
&\le (1+\tilde{\rho}(m+1))\exp\left\{C_2 t^2\Delta(l)\right\}\left\|\exp\left\{2t\sum_{i=1}^{l-1} Y_i\right\}\right\|_2 \\
&\le \exp\left\{\tilde{\rho}(m+1) + C_2 t^2\Delta(l)\right\}\left\|\exp\left\{2t\sum_{i=1}^{l-1} Y_i\right\}\right\|_2 .
\end{aligned}$$

By the generalized C-S inequality (Kuang [28, p.6]), we can get that

$$\begin{aligned}
\left\|\exp\left\{2t\sum_{i=1}^{l-1} Y_i\right\}\right\|_2 &\le \prod_{i=1}^{l-1}\left\|\exp 2tY_i\right\|_{2(l-1)} = \prod_{i=1}^{l-1}\left[E\exp\{4(l-1)tY_i\}\right]^{\frac{1}{2(l-1)}} \\
&= \prod_{i=1}^{l-1}\left[E\exp\{4tY_i\}\exp\{4tY_i(l-2)\}\right]^{\frac{1}{2(l-1)}} \\
&\le \prod_{i=1}^{l-1}\left[\exp\{l-2\}E\left(e^{2tY_i}\right)^2\right]^{\frac{1}{2(l-1)}} \\
&\le \prod_{i=1}^{l-1}\left[\exp\{l-2\}\exp\left\{16t^2\left(1+2\sum_{k=1}^{m}\tilde{\rho}(k)\right)\Delta(i)\right\}\right]^{\frac{1}{2(l-1)}} \\
&= \prod_{i=1}^{l-1}\exp\left\{\frac{l-2}{2(l-1)}\right\}\exp\left\{\frac{8}{l-1}t^2\left(1+2\sum_{k=1}^{m}\tilde{\rho}(k)\right)\Delta(i)\right\} \\
&= \exp\left\{\frac{l-2}{2}\right\}\exp\left\{\frac{1}{l-1}C_2 t^2\sum_{i=1}^{l-1}\Delta(i)\right\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
E\exp\left\{2t\sum_{i=1}^{l} Y_i\right\} &\le \exp\left\{\tilde{\rho}(m+1) + C_2 t^2\Delta(l)\right\}\left\|\exp\left\{2t\sum_{i=1}^{l-1} Y_i\right\}\right\|_2 \\
&\le \exp\left\{\tilde{\rho}(m+1) + \frac{l-2}{2} + C_2 t^2\Delta_n^2\right\} \\
&\le \exp\left\{\tilde{\rho}(m+1) + \frac{n-4m}{4m} + C_2 t^2\Delta_n^2\right\}.
\end{aligned} \tag{2.9}$$

Similarly, we also have

$$E\exp\left\{2t\sum_{i=1}^{l}Z_i\right\}\leq\exp\left\{\tilde{\rho}(m+1)+\frac{n-4m}{4m}+C_2t^2\Delta_n^2\right\}. \tag{2.10}$$

It follows from (2.6), (2.9) and (2.10) that

$$P(S_n>\varepsilon)\leq C_1\exp\left\{-t\varepsilon+C_2t^2\Delta_n^2\right\}. \tag{2.11}$$

Since $\{-X_n, n\geq 1\}$ is also a sequence of $\tilde{\rho}$-mixing random variables with $E(-X_n)=0$ and $|-X_n|\leq d<\infty$ *a.s.* for each $n\geq 1$, it follows from (2.11) that

$$P(S_n<-\varepsilon)=P(-S_n>\varepsilon)\leq C_1\exp\left\{-t\varepsilon+C_2t^2\Delta_n^2\right\}. \tag{2.12}$$

(2.11) and (2.12) yield that

$$P(|S_n|>\varepsilon)=P(S_n>\varepsilon)+P(S_n<-\varepsilon)\leq 2C_1\exp\left\{-t\varepsilon+C_2t^2\Delta_n^2\right\}. \tag{2.13}$$

Take $t=\dfrac{2\varepsilon}{C_2(4\Delta_n^2+nd\varepsilon)}$. It is easy to check that

$$C_2=8\left(1+2\sum_{k=1}^{m}\tilde{\rho}(k)\right)\geq 8,\ \ tmd\leq\frac{2\varepsilon}{C_2(4\Delta_n^2+nd\varepsilon)}nd\leq\frac{1}{4}.$$

Therefore, (2.11) implies that

$$\begin{aligned}P(S_n>\varepsilon) &\leq C_1\exp\left\{-\frac{2\varepsilon^2}{C_2(4\Delta_n^2+nd\varepsilon)}+\frac{2\varepsilon}{C_2(4\Delta_n^2+nd\varepsilon)}\cdot\frac{2C_2\Delta_n^2\varepsilon}{C_2(4\Delta_n^2+nd\varepsilon)}\right\}\\ &\leq C_1\exp\left\{-\frac{2\varepsilon^2}{C_2(4\Delta_n^2+nd\varepsilon)}\left[1-\frac{2\Delta_n^2}{4\Delta_n^2+nd\varepsilon}\right]\right\}\\ &\leq C_1\exp\left\{-\frac{\varepsilon^2}{C_2(4\Delta_n^2+nd\varepsilon)}\right\},\end{aligned}$$

which implies (2.1). Similarly, we can get inequality (2.2) and (2.3) from (2.12) and (2.13), respectively. We complete the proof of the theorem. ♯

Theorem 2.2 *Let $\{X_n, n\geq 1\}$ be a sequence of $\tilde{\rho}$-mixing random variables with $EX_n=0$ and $|X_n|\leq d<\infty$ a.s. for each $n\geq 1$. Assume that $\sum_{n=1}^{\infty}\tilde{\rho}(n)<\infty$ and $\sum_{n=1}^{\infty}EX_n^2<\infty$. Then for any $r>1$,*

$$n^{-r}S_n\to 0,\ \ \textit{completely}, \tag{2.14}$$

and in consequence $n^{-r}S_n\to 0$ a.s..

Proof. For any $n\geq 1$, we can choose a positive integer m such that $n-4m\leq 0$, which implies that $C_1<\infty$. Thus, by Theorem 2.1, for any $\varepsilon>0$, we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}P(|S_n|>n^r\varepsilon) &\leq 2C_1\sum_{n=1}^{\infty}\exp\left\{-\frac{n^{2r}\varepsilon^2}{C_2(4\sum_{i=1}^{n}EX_i^2+ndn^r\varepsilon)}\right\}\\ &\leq 2C_1\sum_{n=1}^{\infty}\exp\left\{-\frac{n^{2r}\varepsilon^2}{C_2(4\sum_{i=1}^{\infty}EX_i^2+n^{1+r}d\varepsilon)}\right\}\\ &\leq C+C\sum_{n=1}^{\infty}[\exp(-C)]^{n^{r-1}}<\infty.\end{aligned}$$

This completes the proof of the theorem. ♯

3. Asymptotic Approximation of Inverse Moments for Nonnegative $\tilde{\rho}$-Mixing Sequence

In this section, we will study the asymptotic approximation of inverse moments for nonnegative $\tilde{\rho}$-mixing random variables with non-identical distribution. The first one is based on the exponential inequality that we established in Section 2.

Theorem 3.1. *Let $\{Z_n, n \geq 1\}$ be a nonnegative $\tilde{\rho}$-mixing sequence with $\sum_{n=1}^{\infty} \tilde{\rho}(n) < \infty$. Suppose that*

(i) $EZ_n^2 < \infty, \ \forall\, n \geq 1$;

(ii) $EX_n \to \infty$, *where* X_n *is defined by* (1.1);

(iii) *for some* $\eta > 0$,

$$R_n(\eta) := B_n^{-2} \sum_{i=1}^{n} E\{Z_i^2 I(Z_i > \eta B_n)\} \to 0,\ n \to \infty;$$

(iv) *for some* $t \in (0, 1)$ *and any positive constants* $a,\ r,\ C$,

$$\lim_{n\to\infty} (a + EX_n)^r \cdot \exp\left\{-C \cdot \frac{(EX_n)^t}{n}\right\} = 0.$$

Then for any $a > 0$ and $r > 0$, (1.2) holds

Proof. Firstly, let us decompose X_n as

$$X_n = U_n + V_n, \tag{3.1}$$

where

$$U_n = B_n^{-1} \sum_{i=1}^{n} Z_i I(Z_i \leq \eta B_n),\ \ V_n = B_n^{-1} \sum_{i=1}^{n} Z_i I(Z_i > \eta B_n), \tag{3.2}$$

and denote

$$\tilde{\mu}_n = EU_n, \quad \tilde{B}_n^2 = \sum_{i=1}^{n} \mathrm{Var}\{Z_i I(Z_i \leq \eta B_n)\}. \tag{3.3}$$

¿From (3.2) and condition (*iii*), it can be seen that

$$EV_n \leq \frac{1}{\eta B_n^2} \sum_{i=1}^{n} E\{Z_i^2 I(Z_i > \eta B_n)\} \to 0,\ \text{ as }\ n \to \infty. \tag{3.4}$$

Thus, $EX_n = EU_n + EV_n \sim \tilde{\mu}_n$ following from condition (*ii*). Therefore, (1.2) will be proved if we show that

$$E(a + X_n)^{-r} \sim (a + \tilde{\mu}_n)^{-r}. \tag{3.5}$$

By Jensen's inequality, we have

$$E(a + X_n)^{-r} \geq (a + EX_n)^{-r}. \tag{3.6}$$

Therefore

$$\liminf_{n\to\infty} (a + \tilde{\mu}_n)^r E(a + X_n)^{-r} \geq \liminf_{n\to\infty} (a + \tilde{\mu}_n)^r (a + EX_n)^{-r} = 1. \tag{3.7}$$

It is easily seen that

$$\begin{aligned}\tilde{B}_n^2 &= \sum_{i=1}^{n}\{E[Z_iI(Z_i \le \eta B_n)]^2 - [EZ_iI(Z_i \le \eta B_n)]^2\} \\ &= \sum_{i=1}^{n}\{E[Z_i - Z_iI(Z_i > \eta B_n)]^2 - [EZ_i - Z_iI(Z_i > \eta B_n)]^2\} \\ &= B_n^2 + 2\sum_{i=1}^{n} EZ_i \cdot EZ_iI(Z_i > \eta B_n) - B_n^2R_n(\eta) - \sum_{i=1}^{n}[EZ_iI(Z_i > \eta B_n)]^2,\end{aligned}$$

hence

$$|\tilde{B}_n^2B_n^{-2} - 1| \le 2B_n^{-2}\sum_{i=1}^{n} EZ_i \cdot EZ_iI(Z_i > \eta B_n) + R_n(\eta) + B_n^{-2}\sum_{i=1}^{n}[EZ_iI(Z_i > \eta B_n)]^2. \tag{3.8}$$

By Jensen's inequality and condition (*iii*), we have

$$B_n^{-2}\sum_{i=1}^{n}[EZ_iI(Z_i > \eta B_n)]^2 \le B_n^{-2}\sum_{i=1}^{n} EZ_i^2I(Z_i > \eta B_n) = R_n(\eta) \to 0. \tag{3.9}$$

By condition (*iii*) again and (3.4),

$$\begin{aligned}B_n^{-2}\sum_{i=1}^{n} EZ_i \cdot EZ_iI(Z_i > \eta B_n) &\le B_n^{-2}\sum_{i=1}^{n} EZ_iI(Z_i \le \eta B_n)\cdot EZ_iI(Z_i > \eta B_n) + B_n^{-2}\sum_{i=1}^{n}[EZ_iI(Z_i > \eta B_n)]^2 \\ &\le \eta B_n^{-1}\sum_{i=1}^{n} EZ_iI(Z_i > \eta B_n) + B_n^{-2}\sum_{i=1}^{n} EZ_i^2I(Z_i > \eta B_n) \\ &= \eta EV_n + R_n(\eta) \to 0, \ \text{as} \ n \to \infty. \end{aligned}\tag{3.10}$$

Therefore, $\tilde{B}_n^2 \sim B_n^2$ follows from (3.8)-(3.10) immediately, which implies that $\tilde{B}_n \sim B_n$. For $t \in (0, 1)$, where t is defined in condition (*iv*), denote

$$E(a + X_n)^{-r} = Q_1 + Q_2, \tag{3.11}$$

where

$$Q_1 = E(a + X_n)^{-r}I(U_n < \tilde{\mu}_n - \tilde{\mu}_n^t), \tag{3.12}$$
$$Q_2 = E(a + X_n)^{-r}I(U_n \ge \tilde{\mu}_n - \tilde{\mu}_n^t). \tag{3.13}$$

Since $X_n \ge U_n$, it follows that

$$Q_2 \le E(a + X_n)^{-r}I(X_n \ge \tilde{\mu}_n - \tilde{\mu}_n^t) \le (a + \tilde{\mu}_n - \tilde{\mu}_n^t)^{-r}.$$

Therefore

$$\limsup_{n\to\infty}(a + \tilde{\mu}_n)^{-r}Q_2 \le 1 \tag{3.14}$$

from the fact that $\tilde{\mu}_n \to \infty$ as $n \to \infty$. By $X_n \ge 0$, we have

$$Q_1 = E(a + X_n)^{-r}I(U_n < \tilde{\mu}_n - \tilde{\mu}_n^t) \le a^{-r}EI(U_n < \tilde{\mu}_n - \tilde{\mu}_n^t) = a^{-r}P(U_n < \tilde{\mu}_n - \tilde{\mu}_n^t). \tag{3.15}$$

In the following, we will estimate the probability $P(U_n < \tilde{\mu}_n - \tilde{\mu}_n^t)$. For fixed $n \ge 1$, denote

$$W_i = -Z_iI(Z_i \le \eta B_n) + EZ_iI(Z_i \le \eta B_n), \ \ 1 \le i \le n,$$

then $\left\{\frac{W_1}{B_n}, \frac{W_2}{B_n}, \cdots, \frac{W_n}{B_n}\right\}$ are $\tilde{\rho}$-mixing random variables and

$$P(U_n < \tilde{\mu}_n - \tilde{\mu}_n^t) = P\left(\sum_{i=1}^{n} \frac{W_i}{B_n} > \tilde{\mu}_n^t\right).$$

Obviously

$$C_2 = 8\left(1 + 2\sum_{k=1}^{m} \tilde{\rho}(k)\right) \le 8\left(1 + 2\sum_{k=1}^{\infty} \tilde{\rho}(k)\right) < \infty.$$

For any $n \ge 1$, we can choose a positive integer m such that $n - 4m \le 0$, which implies that $C_1 = \exp\left\{1 + \tilde{\rho}(m+1) + \frac{n-4m}{4m}\right\} < \infty$.

It is easy to check that

$$\sum_{i=1}^{n} \frac{EW_i^2}{B_n^2} = \frac{\tilde{B}_n^2}{B_n^2} \to 1, \quad n \to \infty, \qquad \left|\frac{W_i}{B_n}\right| \le 2\eta, \ 1 \le i \le n.$$

By Theorem 2.1, we can get

$$\begin{aligned} P(U_n < \tilde{\mu}_n - \tilde{\mu}_n^t) &= P\left(\sum_{i=1}^{n} \frac{W_i}{B_n} > \tilde{\mu}_n^t\right) \\ &\le C_1 \exp\left\{-\frac{\tilde{\mu}_n^{2t}}{C_2\left(4\sum_{i=1}^{n} EW_i^2/B_n^2 + n \cdot 2\eta \cdot \tilde{\mu}_n^t\right)}\right\} \\ &\le C_1 \exp\left\{-C \cdot \frac{\tilde{\mu}_n^t}{n}\right\} \end{aligned}$$

for all n sufficiently large.

By condition (iv) and $EX_n \sim \tilde{\mu}_n$, we have

$$\lim_{n\to\infty} (a + \tilde{\mu}_n)^r Q_1 \le \lim_{n\to\infty} C(a + EX_n)^r \exp\left\{-C \cdot \frac{(EX_n)^t}{n}\right\} = 0. \tag{3.16}$$

Together with (3.11), (3.14) and (3.16), we obtain

$$\limsup_{n\to\infty} (a + \tilde{\mu}_n)^r E(a + X_n)^{-r} \le 1. \tag{3.17}$$

Combining (3.7) and (3.17), we get (3.5), which implies (1.2). The desired result is obtained. ♯

Remark 3.1. If $\{Z_n^2, n \ge 1\}$ is a nonnegative and uniformly integrable random variables sequence with $Z_n \ge 0$ and $B_n^2 \ge Cn$, then for any $\eta > 0$, $R_n(\eta) \to 0$ as $n \to \infty$. In fact,

$$\begin{aligned} R_n(\eta) &= B_n^{-2} \sum_{i=1}^{n} E\{Z_i^2 I(Z_i > \eta B_n)\} \le \frac{n}{B_n^2} \sup_{i \ge 1} EZ_i^2 I(Z_i > \eta B_n) \\ &\le C \sup_{i\ge 1} EZ_i^2 I(Z_i > \eta B_n) \to 0, \ \text{ as } n \to \infty. \end{aligned}$$

Remark 3.2. The result of Theorem 3.1 for nonnegative ρ-mixing random variables with non-identical distribution has been obtained by Shen et al. [16, Theorem 3.1]. Just as Remark 1.1 stated that ρ-mixing and $\tilde{\rho}$-mixing are similar, but different and $\tilde{\rho}$-mixing is more general than ρ-mixing. Hence, Theorem 3.1 in the paper extends the corresponding one of Shen et al. [16] for ρ-mixing random variables to the case of $\tilde{\rho}$-mixing random variables.

Remark 3.3. We point out that there is no any specific meaning for condition (*iv*) in Theorem 3.1, which is just a technical condition. If the tool exponential inequality used in Theorem 3.1 is replaced by Rosenthal type maximal inequality, we will show that (1.2) holds under very mild conditions and the condition (*iv*) in Theorem 3.1 can be deleted. The result is as follows.

Theorem 3.2. *Let $0 < s < 1$ and $\{Z_n, n \geq 1\}$ be a sequence of nonnegative $\tilde{\rho}$-mixing random variables. Let $\{M_n, n \geq 1\}$ and $\{a_n, n \geq 1\}$ be sequences of positive constants such that $a_n \geq C$ for all n sufficiently large, where C is a positive constant. Denote $X_n = M_n^{-1} \sum_{k=1}^{n} Z_k$ and $\mu_n = EX_n$. Suppose that the following conditions hold:*

(i) $EZ_n < \infty$ for all $n \geq 1$;

(ii) $\mu_n \to \infty$ as $n \to \infty$;

(iii) For some positive number $\eta > 0$,

$$\frac{\sum_{k=1}^{n} EZ_k I(Z_k > \eta M_n \mu_n^s / a_n)}{\sum_{k=1}^{n} EZ_k} \to 0 \text{ as } n \to \infty.$$

Then (1.2) holds for all real numbers $a > 0$ and $r > 0$.

Proof. Noting that $f(x) = (a + x)^{-\alpha}$ is a convex function of x on $[0, \infty)$, by Jensen's inequality, we have

$$E(a + X_n)^{-\alpha} \geq (a + EX_n)^{-\alpha}, \tag{3.18}$$

which implies that

$$\liminf_{n\to\infty} (a + EX_n)^{\alpha} E(a + X_n)^{-\alpha} \geq 1. \tag{3.19}$$

To prove (1.2), it is enough to prove that

$$\limsup_{n\to\infty} (a + EX_n)^{\alpha} E(a + X_n)^{-\alpha} \leq 1. \tag{3.20}$$

In order to prove (3.20), we only need to show that for all $\delta \in (0, 1)$,

$$\limsup_{n\to\infty} (a + EX_n)^{\alpha} E(a + X_n)^{-\alpha} \leq (1 - \delta)^{-\alpha}. \tag{3.21}$$

By the condition (*iii*), we can see that for all $\delta \in (0, 1)$, there exists positive integer $n(\delta) > 0$ such that

$$\sum_{k=1}^{n} EZ_k I(Z_k > \eta M_n \mu_n^s / a_n) \leq \frac{\delta}{2} \sum_{k=1}^{n} EZ_k, \quad n \geq n(\delta). \tag{3.22}$$

Let

$$U_n = M_n^{-1} \sum_{k=1}^{n} Z_k I(Z_k \leq \eta M_n \mu_n^s / a_n) \doteq M_n^{-1} \sum_{k=1}^{n} Z'_{nk}$$

and

$$\begin{aligned} E(a + X_n)^{-\alpha} &= E(a + X_n)^{-\alpha} I(U_n \geq \mu_n - \delta\mu_n) + E(a + X_n)^{-\alpha} I(U_n < \mu_n - \delta\mu_n) \\ &\doteq Q_1 + Q_2. \end{aligned} \tag{3.23}$$

For Q_1, we have by the fact $X_n \geq U_n$ that

$$Q_1 \leq E(a + X_n)^{-\alpha} I(X_n \geq \mu_n - \delta\mu_n) \leq (a + \mu_n - \delta\mu_n)^{-\alpha}, \tag{3.24}$$

which implies by condition (*ii*) that

$$\limsup_{n\to\infty} (a + \mu_n)^{\alpha} Q_1 \leq \limsup_{n\to\infty} (a + \mu_n)^{\alpha} (a + \mu_n - \delta\mu_n)^{-\alpha} = (1 - \delta)^{-\alpha}. \tag{3.25}$$

For Q_2, we have, by (3.22), that for $n \geq n(\delta)$,

$$\left|\mu_n - EU_n\right| = M_n^{-1} \sum_{k=1}^{n} EZ_k I(Z_k > \eta M_n \mu_n^s / a_n) \leq \delta \mu_n / 2. \tag{3.26}$$

Hence, by (3.26), Markov's inequality, Lemma 1.2 and C_r inequality, we have for any $p \geq 2$ and all $n \geq n(\delta)$ that,

$$\begin{aligned}
Q_2 &\leq a^{-\alpha} P\left(U_n < \mu_n - \delta\mu_n\right) \\
&= a^{-\alpha} P\left(EU_n - U_n > \delta\mu_n - (\mu_n - EU_n)\right) \\
&\leq a^{-\alpha} P\left(EU_n - U_n > \delta\mu_n/2\right) \\
&\leq a^{-\alpha} P\left(|U_n - EU_n| > \delta\mu_n/2\right) \leq C\mu_n^{-p} M_n^{-p} E\left|\sum_{k=1}^{n}\left(Z'_{nk} - EZ'_{nk}\right)\right|^p \\
&\leq C\mu_n^{-p}\left[M_n^{-2}\sum_{k=1}^{n} EZ_k^2 I(Z_k \leq \eta M_n \mu_n^s/a_n)\right]^{p/2} + C\mu_n^{-p} M_n^{-p}\sum_{k=1}^{n} EZ_k^p I(Z_k \leq \eta M_n\mu_n^s/a_n) \\
&\leq C\mu_n^{-p}\left[M_n^{-1}\mu_n^s/a_n \sum_{k=1}^{n} EZ_k I(Z_k \leq \eta M_n\mu_n^s/a_n)\right]^{p/2} \\
&\quad + C\mu_n^{-p} M_n^{-1}\mu_n^{s(p-1)}/a_n^{p-1}\sum_{k=1}^{n} EZ_k I(Z_k \leq \eta M_n \mu_n^s/a_n) \\
&\leq C\mu_n^{-p}\left[(\mu_n^s/a_n \cdot \mu_n)^{p/2} + \mu_n^{s(p-1)}/a_n^{p-1}\cdot\mu_n\right] \\
&= C\left[\mu_n^{-(1-s)p/2}/a_n^{p/2} + \mu_n^{-(1-s)(p-1)}/a_n^{p-1}\right] \\
&\leq C\mu_n^{-(1-s)p/2} + C\mu_n^{-(1-s)(p-1)}.
\end{aligned} \tag{3.27}$$

Taking $p > \max\left\{2, \frac{2\alpha}{1-s}\right\}$ and noting that $p - 1 \geq \frac{p}{2}$, we have by (3.27) that

$$\limsup_{n\to\infty}(a+\mu_n)^\alpha Q_2 \leq \limsup_{n\to\infty}(a+\mu_n)^\alpha\left[C\mu_n^{-(1-s)p/2} + C\mu_n^{-(1-s)(p-1)}\right] = 0. \tag{3.28}$$

Hence, (3.21) follows from (3.25) and (3.28) immediately. This completes the proof of the theorem. ♯

Remark 3.4. When $M_n = B_n$ and $a_n = \mu_n^s$, the condition (*iii*) in Theorem 3.2 is weaker than (*iii*) in Theorem 3.1. Actually, if for some $\eta > 0$,

$$R_n(\eta) := B_n^{-2}\sum_{i=1}^{n} EZ_i^2 I(Z_i > \eta B_n) \to 0,\ n \to \infty,$$

then

$$B_n^{-1}\sum_{i=1}^{n} EZ_i I(Z_i > \eta B_n) \leq \eta^{-1} B_n^{-2}\sum_{i=1}^{n} EZ_i^2 I(Z_i > \eta B_n) \to 0,\ \ n\to\infty,$$

which implies by $\mu_n \to \infty$ that

$$\frac{B_n^{-1}\sum_{i=1}^{n} EZ_i I(Z_i > \eta B_n)}{\mu_n} = \frac{\sum_{i=1}^{n} EZ_i I(Z_i > \eta B_n)}{\sum_{i=1}^{n} EZ_i} \to 0,\ \ n \to \infty,$$

i.e., condition (*iii*) in Theorem 3.2 holds.

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References

[1] D.A. Wooff, Bounds on reciprocal moments with applications and developments in Stein estimation and post-stratification, Journal of the Royal Statistical Society- Series B, 47 (1985) 362–371.
[2] A.O. Pittenger, Sharp mean-variance bounds for Jensen-type inequalities, Statistics & Probability Letters, 10 (1990) 91–94.
[3] E. Marciniak, J. Wesolowski, Asymptotic Eulerian expansions for binomial and negative binomial reciprocals, Proceedings of the American Mathematical Society, 127 (1999) 3329–3338.
[4] T. Fujioka, Asymptotic approximations of the inverse moment of the non-central chi-squared variable, Journal of The Japan Statistical Society, 31 (2001) 99–109.
[5] R.C. Gupta, O. Akman, Statistical inference based on the length-biased data for the inverse Gaussian distribution, Statistics, 31 (1998) 325–337.
[6] W. Mendenhall, E.H. Lehman, An approximation to the negative moments of the positive binomial useful in life-testing, Technometrics, 2 (1960) 227–242.
[7] C.M. Ramsay, A note on random survivorship group benefits, ASTIN Bulletin, 23 (1993) 149–156.
[8] A. Jurlewicz, K. Weron, Relaxation of dynamically correlated clusters, Journal of Non-Crystalline Solids, 305 (2002) 112–121.
[9] N.L. Garcia, J.L. Palacios, On inverse moments of nonnegative random variables, Statistics & Probability Letters, 53 (2001) 235–239.
[10] M. Kaluszka, A. Okolewski, On Fatou-type lemma for monotone moments of weakly convergent random variables, Statistics & Probability Letters, 66 (2004) 45–50.
[11] S.H. Hu, G.J. Chen, X.J. Wang, E.B. Chen, On inverse moments of nonnegative weakly convergent random variables, Acta Mathematicae Applicatae Sinica, 30 (2007) 361–367.
[12] T.J. Wu, X.P. Shi, B.Q. Miao, Asymptotic approximation of inverse moments of nonnegative random variables, Statistics & Probability Letters, 79 (2009) 1366–1371.
[13] S.H. Sung, On inverse moments for a class of nonnegative random variables. Journal of Inequalities and Applications, 2010 (2010) 13 pages.
[14] X.J. Wang, S.H. Hu, W.Z. Yang, N.X. Ling, Exponential inequalities and inverse moment for NOD sequence, Statistics & Probability Letters, 80 (2010) 452–461.
[15] A.T. Shen, A note on the inverse moments for nonnegative ρ-mixing random variables, Discrete Dynamics in Nature and Society, 2011 (2011) 8 pages.
[16] A.T. Shen, Y. Shi, W.J. Wang, B. Han, Bernstein-type inequality for weakly dependent sequence and its applications, Revista Matemática Complutense, 25 (2012) 97–108.
[17] W. Bryc, W. Smolenski, Moment conditions for almost sure convergence of weakly correlated random variables, Proceedings of the American Mathematical Society, 119 (1993) 629–635.
[18] R.C. Bradley, Equivalent Mixing Conditions for Random Fields, The Annals of Probability, 21 (1993) 1921–1926.
[19] S. Utev, M. Peligrad, Maximal inequalities and an invariance principle for a class of weakly dependent random variables, Journal of Theoretical Probability, 16 (2003) 101–115.
[20] R.C Bradley, On the spectral density and asymptotic normality of weakly dependent random fields, Journal of Theoretical Probability, 5 (1992) 355–374.
[21] M.H. Zhu, Strong laws of large numbers for arrays of rowwise ρ^*-mixing random variables, Discrete Dynamics in Nature and Society, 2007 (2007) 6 pages.
[22] Q.Y. Wu, Y.Y. Jiang, Some strong limit theorems for $\tilde{\rho}$-mixing sequences of random variables, Statistics & Probability Letters, 78(2008), 1017–1023.
[23] Q.Y. Wu, Y.Y. Jiang, Some strong limit theorems for weighted product sums of $\tilde{\rho}$-mixing sequences of random variables, Journal of Inequalities and Applications, 2009 (2009) 10 pages.
[24] Q.Y. Wu, Y.Y. Jiang, Chover-type laws of the k-iterated logarithm for $\tilde{\rho}$-mixing sequences of random variables, Journal of Mathematical Analysis and Applications, 366 (2010) 435–443.
[25] X.J. Wang, S.H. Hu, Y. Shen, W.Z. Yang, Some new results for weakly dependent random variable sequences, Chinese Journal of Applied Probability and Statistics, 26 (2010), 637–648.
[26] X.C. Zhou, C.C. Tan, J.G. Lin, On the strong laws for weighted sums of ρ^*-mixing random variables, Journal of Inequalities and Applications, Volume 2011 (2011) 8 pages.
[27] Q.Y. Wu, Further study strong consistency of estimator in linear model for $\tilde{\rho}$-mixing random sequences, Journal of Systems Science and Complexity, 24 (2011) 969–980.
[28] J.C. Kuang, Applied Inequality, (3nd ed.), Shangdong Science and Technology Press, Jinan, 2003.