Filomat 28:4 (2014), 871–878
DOI 10.2298/FIL1404871W

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Local Polynomial Estimation of Time-Dependent Diffusion Parameter for Discretely Observed SDE Models

Ji-xia Wang[a], Qing-xian Xiao[b]

[a] *College of Mathematics and Information Science, Henan Normal University, Xinxiang, 453007, P.R. China*
[b] *Business School, University of Shanghai for Science and Technology, Shanghai, 200093, P.R. China*

Abstract. Extending the results of Yu, Yu, Wang and Lin [10], we study the local polynomial estimation of the time-dependent diffusion parameter for time-inhomogeneous diffusion models. Considering the diffusion parameter being positive, we obtain the local polynomial estimation of the diffusion parameter by taking the diffusion parameter to be local log-polynomial fitting. The asymptotic bias, asymptotic variance and asymptotic normal distribution of the volatility function are discussed. A real data analysis is conducted to show the performance of the estimations proposed.

1. Introduction

The time-dependent parametric models are important tools to explore the dynamic pattern in many scientific fields, such as finance, economics, medical science and so on. In this paper, we consider the time-dependent parametric diffusion models on a filtered probability space $(\Omega, F, (F_t)_{t\geq 0}, P)$,

$$dX(t) = [\alpha(t) + \beta(t)g(X_t)]\,dt + \sigma(t)[F(X_t)]^{\gamma} dW_t, \tag{1}$$

where $\alpha(t)$ and $\beta(t)$ are time-dependent parameters of the drift function, $\sigma(t)$ is time-dependent parameter of diffusion function, γ is a scalar parameter independent of time t, $g(\cdot)$ and $F(\cdot)$ are known second-differentiable functions and W_t is the standard Brownian motion. The model (1) includes these of Ho and Lee [6] (HL), Hull and White [7] (HW), Black and Karasinski [1] (BK), Black et al. [2] (BDT), Fan et al. [3] (FJZZ) and Yu et al. [10] (YYWL). The techniques that we employ here are based on the local polynomial fitting. Considering the diffusion parameter $\sigma(t)$ being local positive, we take the diffusion parameter to be local log-polynomial. That is, for a given time point t_0, we use the approximation

$$\log \sigma^2(t) \approx \nu_0 + \nu_1(t - t_0) + \cdots + \nu_p(t - t_0)^p, \tag{2}$$

for t in a small neighborhood of t_0. Our techniques require selection of the bandwidth. Our bandwidth selection method is a rather simple rule of thumb (ROT) method similar to that in [9].

2010 *Mathematics Subject Classification*. Primary 62M05; Secondary 62F12
Keywords. Diffusion model, Time-dependent parameter, Local polynomial estimation, Kernel weighted, Asymptotic normality
Received: 25 April 2013; Revised: 28 February 2014; Accepted: 25 May 2014
Communicated by Miroslav M. Ristić
Research supported by the National Natural Science Foundation of China (No.11171221), the Natural Science Foundation of Henan Educational Committee (2011B110018) and Shanghai Leading Academic Discipline Project (No.XTKX2012)
Email addresses: jixiawang@163.com (Ji-xia Wang), qxxiao@163.com (Qing-xian Xiao)

The rest of this paper is organized as follows. In Section 2, we propose the local polynomial estimation for the time-dependent parameter and study its asymptotic bias, variance and normality. Proof of the results are given in Section 3. In Section 4, a real data analysis is conducted.

Remark 1.1. When $g(X_t) = X_t$ and $F(X_t) = X_t$, our model (1) yields to the YYWL model. Therefore, our model extends that of Yu, Yu, Wang and Lin [10]. A more general model with γ dependent on time t has been considered by FJZZ.

Remark 1.2. In particular, we obtain the results in Section 3 of Yu, Yu, Wang and Lin [10] by letting $p = 1$ in (2). Therefore, our results in this paper extends that in Yu, Yu, Wang and Lin [10].

2. Methods and Results of Estimations

Let the data $\{X_{t_i}, i = 1, 2, \cdots, n+1\}$ be sampled at discrete time points, $t_1 < t_2 < \cdots < t_{n+1}$. Suppose the time points are equally spaced. For example, weekly data are sampled at $t_i = t_0 + i/52, i = 1, 2, \cdots, n+1$ when the time unit is a year, where t_0 is the initial time point. Denote $Y_{t_i} = X_{t_{i+1}} - X_{t_i}$, $Z_{t_i} = W_{t_{i+1}} - W_{t_i}$ and $\Delta_i = t_{i+1} - t_i$. Due to the independent increment property of Brownian motion W_{t_i}, the Z_{t_i} are independent and normally distributed with mean zero and variance Δ_i. Thus the discretized version of the model (1) can be expressed as

$$Y_{t_i} = [\alpha(t_i) + \beta(t_i) g(X_{t_i})]\, \Delta_i + \sigma(t_i)[F(X_t)]^{\gamma} \sqrt{\Delta_i} \epsilon_{t_i}, \tag{3}$$

where $\{\epsilon_{t_i}\}$ are independent and standard normal. Based on the findings in [8] and [5], the first-order discretized approximation error to the continuous-time model is extremely small, as long as data are sampled monthly or more frequently. Their findings simplify the estimation procedure.

For estimations of the drift parameters, following the local regression technique (see [4]), $\alpha(t)$ and $\beta(t)$ can be estimated via minimizing the locally weighted least-squares function

$$\sum_{i=1}^{n} \left\{ \frac{Y_{t_i}}{\Delta_i} - a - b g(X_{t_i}) \right\}^2 K_{h_1}(t_i - t_0) \tag{4}$$

with respect to a and b, where $K(\cdot)$ is kernel function, $K_{h_1}(t_i - t_0) = \frac{1}{h_1} K(\frac{t_i - t_0}{h_1})$ and h_1 is a properly selected bandwidth.

Let $\hat{a}$ and $\hat{b}$ be the minimizers of the locally weighted function (4). Denote $\theta = (a, b)^T$, $Y = \left(\frac{Y_{t_1}}{\Delta_1}, \frac{Y_{t_1}}{\Delta_1}, \cdots, \frac{Y_{t_n}}{\Delta_n} \right)^T$, $W = diag(K_{h_1}(t_i - t_0))$ and

$$X = \begin{pmatrix} 1 & g(X_{t_1}) \\ 1 & g(X_{t_2}) \\ \vdots & \vdots \\ 1 & g(X_{t_n}) \end{pmatrix}.$$

Then, the local estimation of θ is

$$\hat{\theta} = (\hat{a}, \hat{b})^T = (X^T W X)^{-1} X^T W Y.$$

Let $\hat{\alpha}(t_0) = \hat{a}$, $\hat{\beta}(t_0) = \hat{b}$, for a given time point t_0. To obtain the estimated functions, $\hat{\alpha}(t)$ and $\hat{\beta}(t)$, we can evaluate the estimations at hundreds of grid points.

Now we discuss the estimations of the diffusion parameters. An appropriate localized normal log-likelihood function for model (3) is given by -1 times

$$\sum_{i=0}^{n} \left\{ \frac{1}{\Delta_i} \frac{[Y_{t_i} - (\alpha(t_i) + \beta(t_i) g(X_{t_i}))\Delta_i]^2}{\sigma^2(t_i)[F(X_{t_i})]^{2\gamma}} + \log(\sigma^2(t_i)[F(X_{t_i})]^{2\gamma}) \right\} K_h(t_i - t_0). \tag{5}$$

Due to the diffusion parameter $\sigma(t)$ being positive, we take $\sigma^2(t)$ to be locally log-polynomial fitting. That is, for a given time point t_0, we use the approximation

$$\log \sigma^2(t) \approx \nu_0 + \nu_1(t-t_0) + \cdots + \nu_p(t-t_0)^p,$$

for t in a small neighborhood of t_0.

Let $\hat{\mu}(t, X_t) = \hat{\alpha}(t) + \hat{\beta}(t)g(X_t)$ stand for the estimated mean function from above, and denote $\hat{E}_{t_i} = (Y_{t_i} - \hat{\mu}(t_i, X_{t_i})\Delta_i)/\sqrt{\Delta_i}$. Then, the local kernel weighted log-likelihood function (5) can be expressed as

$$\sum_{i=0}^{n}\left[\hat{E}_{t_i}^2 \exp\left\{-\sum_{j=0}^{p}\nu_j(t_i-t_0)^j\right\}[F(X_{t_i})]^{-2\gamma} + \sum_{j=0}^{p}\nu_j(t_i-t_0)^j + \gamma\log[F(X_{t_i})]^2\right]K_{h_2}(t_i-t_0), \tag{6}$$

where $K_{h_2}(t_i - t_0) = \frac{1}{h_2}K(\frac{t_i-t_0}{h_2})$ and h_2 is a properly selected bandwidth which will be discussed at the end of this section. We can minimize (6) with respect to $\nu_0, \nu_1, \cdots, \nu_p$ and γ as following.

In Step 1, the parameter γ is given. Taking the partial derivatives of (6) with respect to $\nu_0, \nu_1, \cdots, \nu_p$ respectively, and setting the partial derivatives to zero, we obtain the estimations of $\nu_0, \nu_1, \cdots, \nu_p$.

In Step 2, let $\hat{\nu}_0, \hat{\nu}_1, \cdots, \hat{\nu}_p$ be the estimates of the local parameters $\nu_0, \nu_1, \cdots, \nu_p$ from Step 1, respectively. We can find the estimate of γ via the global minimization of the following function

$$\sum_{i=0}^{n}\left[\hat{E}_{t_i}^2 \exp\left\{-\sum_{j=0}^{p}\hat{\nu}_j(t_i-t_0)^j\right\}[F(X_{t_i})]^{-2\gamma} + \gamma\log[F(X_{t_i})]^2\right], \tag{7}$$

with respect to the parameter γ. Let $\hat{\gamma}$ be the minimizer of (7). Then, we obtain the estimation of γ.

Once we have estimations $\hat{\nu}_0, \hat{\nu}_1, \cdots, \hat{\nu}_p$ and $\hat{\gamma}$, for a given time point t_0, we can estimate $\sigma(t)$ and the volatility by

$$\hat{\sigma}(t) = \exp\left(\frac{\hat{\nu}_0(t)}{2}\right) \quad \text{and} \quad \hat{\sigma}(t, X_t) = \exp\left(\frac{\hat{\nu}_0(t)}{2}\right)[F(X_t)]^{\hat{\gamma}},$$

respectively.

As the following, we discuss the asymptotic properties of the estimating squared volatility. Denote the drift function $\mu(t, X_t) = \alpha(t) + \beta(t)g(X_t)$, the squared volatility $\sigma^2(t, X_t) = \sigma^2(t)[F(X_t)]^{2\gamma}$ and $\sigma(t, X_t) = \sigma(t)[F(X_t)]^{\gamma}$. Let $f(\cdot)$ be the density function of time, usually a uniform distribution on time interval $[a, b]$.

Now we need the following technical assumptions. Throughout this paper, C denotes a positive generic constant independent of all other variables.

(A1) The parametric functions $\alpha(t)$, $\beta(t)$ and $\sigma(t)$ in model (1) are $(p+1)th$ continuously differentiable in t.

(A2) The drift $\mu(t, X_t)$ and the volatility $\sigma(t, X_t)$ are second-differentiable functions.

(A3) The kernel function $K(\cdot)$ is a Lipschitz continuous symmetric density on $[-1, 1]$.

(A4) The bandwidths $h_i = h_i(n) \to 0$ and $nh_i^{2+\delta} \to \infty$ for some $\delta > 0, i = 1, 2$.

Theorem 2.1. *Under Assumptions (A1)-(A4), the estimation $\hat{\sigma}^2(t, X_t)$ from (6) satisfies*

$$E\left(\hat{\sigma}^2(t, X_t)\right) - \sigma^2(t, X_t) = \frac{1}{2}a_2(K)b(t)h_2^2(1 + O(h_2)),$$

$$Var\left(\hat{\sigma}^2(t, X_t)\right) = \frac{2[\sigma^2(t, X_t)]^2}{nh_2 f(t)}R(K)(1 + o(1))$$

and as $n \to \infty$,

$$\sqrt{nh_2}\left(s(\sigma^2(t, X_t))\right)^{-1}\left[\hat{\sigma}^2(t, X_t) - \sigma^2(t, X_t) - \frac{1}{2}a_2(K)b(t)h_2^2(1 + O(h_2))\right] \to_D N(0, 1),$$

where $a_2(K) = \int_{-1}^{1} z^2K(z)dz$, $b(t) = \sigma^2(t, X_t)\frac{\partial^2 \log \sigma^2(t, X_t)}{\partial t^2}$, $R(K) = \int_{-1}^{1} K^2(z)dz$ and $s^2(\sigma^2(t, X_t)) = \frac{2[\sigma^2(t, X_t)]^2}{f(t)}R(K)$. The symbol $\to_D$ means convergence in distribution.

By using the Taylor expansion, we have

$$\sqrt{\hat{\sigma}^2(t,X_t)} - \sqrt{\sigma^2(t,X_t)} \sim \left[\frac{1}{2\sigma(t,X_t)}\right]\left(\hat{\sigma}^2(t,X_t) - \sigma^2(t,X_t)\right).$$

Thus, we obtain the following corollary.

Corollary 2.2. *If Assumptions* $(A1)$-$(A4)$ *hold, as* $n \to \infty$*, we have*

$$\sqrt{nh_2}\left(s^*(\sigma^2(t,X_t))\right)^{-1}\left[\hat{\sigma}(t,X_t) - \sigma(t,X_t) - \frac{1}{2}a_2(K)b(t)h_2^2(1+O(h_2))\right] \to_D N(0,1),$$

where $\left[s^*(\sigma^2(t,X_t))\right]^2 = \frac{\sigma^2(t,X_t)}{2f(t)}R(K)$.

By applying the likelihood estimation property to the log-likelihood function (7) over parameter γ, we have the asymptotic properties of the estimation $\hat{\gamma}$.

Theorem 2.3. *When (7) is a second continuous differentiable function on* $(0,\infty)$ *over* γ *and* $n \to \infty$*, we have*

$$\hat{\gamma} \to_P \gamma$$

and

$$\sqrt{n}[I(\gamma)]^{1/2}(\hat{\gamma} - \gamma) \to_D N(0,1),$$

where

$$I(\gamma) = E\left[-\gamma\sum_{i=1}^{n}\left(\frac{(1/\Delta_i)(Y_{t_i}/\Delta_i - \mu(t_i,X_{t_i}))^2}{\sigma^2(t_i)[F^2(X_{t_i})]^{\gamma+1}} + \log[F^2(X_{t_i})]\right)\right]^2.$$

At the end of this section, we select the bandwidth h_2. It is well-known that the choice of the bandwidth parameter is rather crucial. We will select the bandwidth by the method similar to that in [9] and [10]. It is based on minimizing the asymptotic mean integrated squared errors (MISE), using the results from Theorem in this section. Based on this idea, the bandwidth selector is

$$h_2 = \left[\frac{2R(K)V_1}{a_2^2(K)Bn}\right]^{1/5},$$

where $a_2(K)$ and $R(K)$ are defined in Theorem 2.1,

$$B = \left(\frac{4}{n}\right)\sum_{i=1}^{n}(\hat{c}_2 + 3\hat{c}_3t_i)^2\exp\left\{2\left(\hat{c}_0 + \hat{c}_1t + \hat{c}_2t^2 + \hat{c}_3t^3\right)\right\}$$

and

$$V_1 = \int_a^b \exp\left\{2\left(\hat{c}_0 + \hat{c}_1t + \hat{c}_2t^2 + \hat{c}_3t^3\right)\right\}dt,$$

the latter being obtained numerically, $\hat{c}_i, (i = 1,2,3)$ is obtained via fitting a cubic function to the log-squared residuals arising from an initial fitting of drift (see [9] for details).

3. Proof of Results

Proof of Theorem 2.1. We prove this theorem by using Taylor series expansion of a normalized function of (6) and the Cramer-Wold rule. Let vector $\nu = (\nu_0, \nu_1, \cdots, \nu_p)^T$ and vector $\hat{\nu} = (\hat{\nu}_0, \hat{\nu}_1, \cdots, \hat{\nu}_p)^T$, where $\hat{\nu}$ minimizes

$$l(\nu) = \sum_{i=0}^{n} \left[\hat{E}_{t_i}^2 \exp\left\{ -\sum_{j=0}^{p} \nu_j (t_i - t_0)^j \right\} [F(X_{t_i})]^{-2\gamma} + \sum_{j=0}^{p} \nu_j (t_i - t_0)^j \right] K_{h_2}(t_i - t_0).$$

Let

$$\tilde{\sigma}(t, x) = \sigma(t) + \sigma'(t)(x - t) + \cdots + \sigma^{(p)}(t)(x - t)^p / p!,$$

$$\nu^* = \sqrt{nh_2} \left[\nu_0 - \sigma(t), h_2(\nu_1 - \sigma'(t)), \cdots, h^p(p!\nu_p - \sigma^{(p)}(t)) \right]^T$$

and

$$z_{pi} = (1, (t_i - t)/h_2, \cdot, (t_i - t)^p / h^p p!)^T.$$

Then,

$$\nu_0 + \nu_1(t_i - t) + \cdots + \nu_p(t_i - t)^p = \tilde{\sigma}(t, t_i) + a_n(\nu^*)^T z_{pi},$$

where $a_n = (nh_2)^{-\frac{1}{2}}$.

If $\hat{\nu}$ minimizes $l(\nu)$, then $\hat{\nu}^*$ minimizes

$$\sum_{i=0}^{n} \left[\hat{E}_{t_i}^2 \exp\{-\tilde{\sigma}(t, t_i) - a_n(\nu^*)^T z_{pi}\} [F(X_{t_i})]^{-2\gamma} + (\tilde{\sigma}(t, t_i) + a_n(\nu^*)^T z_{pi}) \right] K_{h_2}(t_i - t_0)$$

as a function of ν^*.

To study the asymptotic properties of $\hat{\nu}^*$, consider the normalized function

$$\begin{aligned} l(\nu^*) \quad = \quad & \sum_{i=0}^{n} \left[\hat{E}_{t_i}^2 \exp\{-\tilde{\sigma}(t, t_i) - a_n(\nu^*)^T z_{pi}\} \right] [F(X_{t_i})]^{-2\gamma} K_{h_2} - \sum_{i=0}^{n} \left[\hat{E}_{t_i}^2 \exp\{-\tilde{\sigma}(t, t_i)\} \right] [F(X_{t_i})]^{-2\gamma} K_{h_2} \\ & + \sum_{i=0}^{n} (\tilde{\sigma}(t, t_i) + a_n(\nu^*)^T z_{pi}) K_{h_2} - \sum_{i=0}^{n} (\tilde{\sigma}(t, t_i)) K_{h_2}. \end{aligned}$$

Then $\hat{\nu}^*$ minimizes $l(\nu^*)$.

Using a Taylor series expansion about $\tilde{\sigma}(t, t_i)$,

$$p(x + h) - p(x_0) \approx h \frac{d}{dx} p(x_0) + \frac{1}{2} h^2 \frac{d^2}{dx^2} p(x_0),$$

we have

$$\begin{aligned} l(\nu^*) \quad \approx \quad & -a_n(\nu^*)^T \left[\sum_{i=0}^{n} \hat{E}_{t_i}^2 \exp\{-\tilde{\sigma}(t, t_i)\} z_{pi} [F(X_{t_i})]^{-2\gamma} K_{h_2} \right] + \frac{a_n^2}{2} (\nu^*)^T \left[\sum_{i=0}^{n} \hat{E}_{t_i}^2 \exp\{-\tilde{\sigma}(t, t_i)\} z_{pi} z_{pi}^T [F(X_{t_i})]^{-2\gamma} K_{h_2} \right] \nu^* \\ & + a_n(\nu^*)^T \left[\sum_{i=0}^{n} z_{pi} K_{h_2} \right] + 3rd \ term, \end{aligned}$$

where the $3rd$ term is the term of smaller order. Firstly, for the term of $\frac{a_n^2}{2}$ of the equation above, let

$$C_n = a_n^2 \sum_{i=0}^{n} \hat{E}_{t_i}^2 \exp\{-\tilde{\sigma}(t, t_i)\} z_{pi} z_{pi}^T K_{h_2}.$$

Then from

$$E\left[\frac{(Y_1-\mu(t,X_t))^2}{[F(X_t)^2]^\gamma}\right] = E\left[\frac{(Y_1-\hat{\mu}(t,X_t))^2}{[F(X_t)^2]^\gamma}\right]+E\left[\frac{(\hat{\mu}(t,X_t)-\mu(t,X_t))^2}{[F(X_t)^2]^\gamma}\right] = \exp\{\sigma(t_1)\}+E\left[\frac{(\hat{\mu}(t,X_t)-\mu(t,X_t))^2}{[F(X_t)^2]^\gamma}\right],$$

we have

$$EC_n = h_2^{-1}E\left[\exp\{\sigma(t_1)-\tilde{\sigma}(t,t_1)\}\right]K_{h_2}(t_1-t)z_{p1}z_{p1}^T+h_2^{-1}E\left[\frac{(\hat{\mu}(t,X_t)-\mu(t,X_t))^2}{[F(X_t)^2]^\gamma}\right]\exp\{-\tilde{\sigma}(t,t_1)\}K_{h_2}(t_1-t)z_{p1}z_{p1}^T+o(h_2),$$

we obtain

$$(i-1)!(j-1)!(EC_n)_{ij} = f(t)\omega_{i+j-2}+h_2f'(t)\omega_{i+j-1}+o(h_2^p)$$

and

$$Var\left\{(C_n)_{ij}\right\} = O\left((nh_2)^{-1}\right),$$

where $i,j=1,2,\cdots,p+1$.

Now let Q_p be the $(p+1)\times(p+1)$ matrices having (i,j)th entry equal to $\omega_{i+j-1}=\int z^{i+j-1}K(z)dz$, and define $D=diag(1,1/1!,\cdots,1/p!)$, $E_t=f(t)DQ_pD$, $F_t=f'(t)DQ_pD$. Then

$$EC_n = E_t+h_2F_t+o(h_2).$$

Thus,

$$l(\nu^*) = \nu_n^T\nu^*-\frac{1}{2}(\nu^*)^T(E_t+h_2F_t)\nu^*+o_p(h_2),$$

where $\nu_n=a_n\sum_{i=0}^n Y_{\nu i}$ with

$$Y_{\nu i} = \left[1-\frac{(Y_i-\mu(t,X_t))^2}{[F(X_t)^2]^\gamma}\exp\{-\tilde{\sigma}(t,t_i)\}\right]z_{pi}K_{h_2}.$$

Since $\hat{\nu}^*$ minimizes $l_n(\nu^*)$ and $l_n(\nu^*)$ is a quadratic form of ν^*, $\hat{\nu}^*$ can be given clearly by

$$\frac{d}{d\nu^*}l_n(\nu^*) = 0.$$

Calculations of $E(\nu_n)$ and $Var(\nu_n)$ are outlined below.

$$\begin{aligned} E(Y_{\nu i}) &= \frac{-h_2}{(i-1)!}\int[1-\exp\{\sigma(t+h_2z)\}+(\mu(t+h_2z)-\hat{\mu}(t+h_2z))^2\exp\{-\tilde{\sigma}(t,t+h_2z)\}]z^{i-1}K(z)f(t+h_2z)dz \\ &= \frac{-1}{(i-1)!}\left(h_2\nu_{i-1}f(t)+h_2^2\nu_if'(t)\right)+\frac{h_2}{(i-1)!}\int\left[\exp\left\{\frac{\sigma^{(p+1)}(t)}{(p+1)!}(h_2z)^{p+1}+\frac{\sigma^{(p+2)}(t)}{(p+2)!}(h_2z)^{p+2}\right\}z^{i-1}K(z)f(t+h_2z)\right]dz \\ &= \frac{1}{(i-1)!}\left[h_2^{p+2}\frac{\sigma^{(p+1)}(t)}{(p+1)!}f(t)\nu_{p+i}+h_2^{p+3}\frac{\sigma^{(p+2)}(t)}{(p+2)!}f(t)\nu_{p+i+1}+h_2^{p+3}\frac{\sigma^{(p+1)}(t)}{(p+1)!}f'(t)\nu_{p+i+1}\right]+o(h_2^{p+3}). \end{aligned}$$

Similarly,

$$Cov(Y_{\nu i},Y_{\nu j}) = h_2f(t)\exp\{-2\sigma(t)\}\Sigma^2(t)\int\left[\frac{z^{i+j-2}}{(i-1)!(j-1)!}K^2(z)\right]dz,$$

where $\Sigma^2(t)=E[Y^4|t]$ and it can be approximated by $2(E[Y^2|t])^2$ when $Y|t$ is normal. Last, note that $\hat{\nu}(t)-\nu(t)=\exp\{\sigma(t)\}(\hat{\sigma}(t)-\sigma(t))$ and $\nu(t)=\exp\{\sigma(t)\}$. This completes the proof of Theorem 2.1. □

4. A Real Data Example

As an illustration, we apply the proposed local log-polynomial methodology to the Shanghai Interbank Offered Rate(Shibor) data which is calculated, announced and named on the technological platform of the National Interbank Funding Center in Shanghai. It is a simple, no-guarantee, wholesale interest rate calculated by arithmetically averaging all the interbank RMB lending rates offered by the price quotation group of banks with a high credit rating. Currently, the Shibor consists of eight maturities: overnight, 1-week, 2-week, 1-month, 3-month, 6-month, 9-month and 1-year. Here we apply our local log-polynomial estimations to the 1-year Shibor data, which yields to Hull-White (see [7]) interest rate model with $g(X_t) = X_t$ and $[F(X_t)]^\gamma = \sqrt{X_t}$.

The data set contains 499 observations from January 4, 2011 to December 31, 2012. This data can be downloaded from http://www.shibor.org/shibor/web/DataService.jsp. The 1-year Shibor data and their changes are plotted in Figure 4.1 and Figure 4.2, respectively. The real volatility of 1-year Shibor data is measured by $Y_{t_i}^2 = (X_{t_{i+1}} - X_{t_i})^2$, which is plotted by the real line in Figure 4.3. The real volatility of changes is clearly time-inhomogeneous. We select the kernel function $K(x) = \frac{3}{4}(1 - x^2)_+$, which is the Epanechnikov kernel. By using our local log-polynomial method, the estimation curve of volatility is plotted by the dotted line in Figure 4.3.

Figure 4.1. 1−year Shibor data set

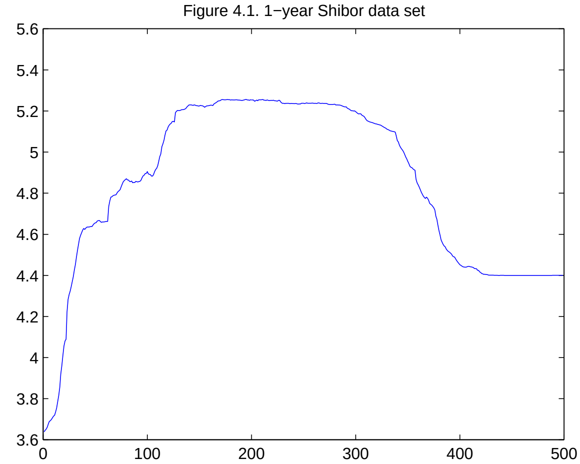
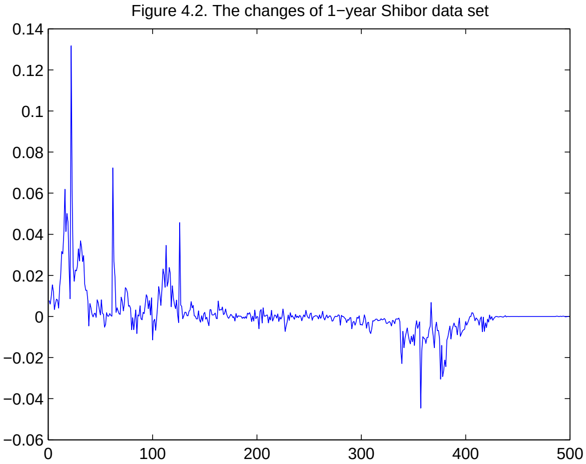
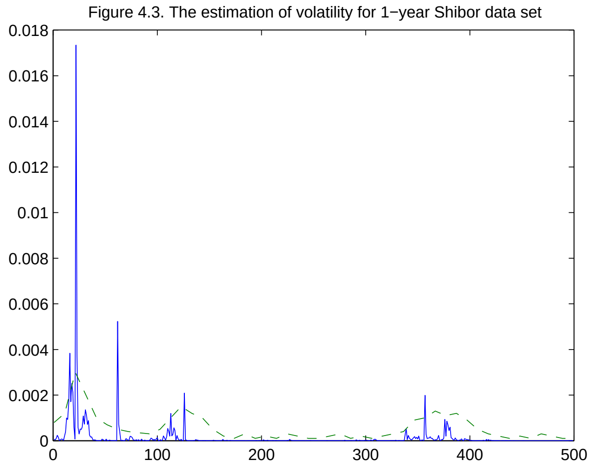


Figure 4.2. The changes of 1−year Shibor data set





From Figure 4.3, we can see that our estimation results catch the major trend of volatility well.

References

[1] F. Black, P. Karasinski, Bond and option pricing when short rates are lognormal, Finan. Analysts J. 47 (1991), 52–59.

[2] F. Black, E. Derman, W. Toy, A one-factor model of interest rate and its application to treasury bond options, Finan. Analysts J. 46 (1990), 33–39.

[3] J. Fan, J. Jiang, C. Zhang, Z. Zhou, Time-dependent diffusion models for term structure dynamics and the stock price volatility, Statistica Sinica 13 (2003), 965–992.

[4] J. Fan, I. Gijbels, Local polynomial modelling and its applications, Chapman and Hall, London, 1996.

[5] J. Fan, C. Zhang, A re-examination of stanton's diffusion estimations with applications to finacial model validation, J. Amer. Statist. Assoc. 98 (2003), 118–134.

[6] T. S. Y. Ho, S. B. Lee, Term structure movements and pricing interest rate contingent claims, J. Finance 41 (1986), 1011–1029.

[7] J. Hull, A. White, Pricing interest-rate derivative securities, Rev. Finan. Stud. 3 (1990), 573–592.

[8] R. Stanton, A nonparametric model of term structure dynamics and the market price of interest rate risk, J. Finance 52 (1997), 1973–2002.

[9] K. Yu, M. C. Jones, Likelihood-based local linear estimation of the conditional variance function, J. Amer. Statist. Assoc. 99 (2004), 139–144.

[10] Y. Yu, K. Yu, H. Wang, M. Li, Semiparametric estimation for a class of time-inhomogeneous diffusion processes Statistica Sinica 19 (2009), 843–867.