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Approximate Generalized Additive Mappings in Proper Multi-CQ^*-Algebras

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Abstract. In this paper, we approximate the following additive functional inequality

$$\left\| \left(\sum_{i=1}^{d+1} f(x_{1i}), \cdots, \sum_{i=1}^{d+1} f(x_{ki})\right) \right\|_k \leq \left\| \left(mf\left(\frac{\sum_{i=1}^{d+1} x_{1i}}{m}\right), \cdots, mf\left(\frac{\sum_{i=1}^{d+1} x_{ki}}{m}\right) \right) \right\|_k$$

for all $x_{11}, \cdots, x_{k\,d+1} \in \mathcal{X}$. We investigate homomorphisms in proper multi-CQ^*-algebras and derivations on proper multi-CQ^*-algebras associated with the above additive functional inequality.

1. Introduction

Let X and Y be Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Consider $f : X \to Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Assume that there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x + y) - f(x) - f(y)\|_Y \leq \theta(\|x\|_X^p + \|y\|_X^p)$$

for all $x, y \in X$. Th.M. Rassias [1] showed that there exists a unique $\mathbb{R}$-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\|_Y \leq \frac{2\theta}{2 - 2^p}\|x\|_X^p$$

for all $x \in X$. Găvruta [2] generalized the Rassias' result: Let G be an Abelian group and Y a Banach space. Denote by $\varphi : G \times G \to [0, \infty)$ a function such that

$$\widetilde{\varphi}(x, y) = \sum_{j=0}^{\infty} \frac{1}{2^j}\varphi(2^j x, 2^j y) < \infty$$

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for all $x, y \in G$. Suppose that $f : G \to Y$ is a mapping satisfying

$$\|f(x+y) - f(x) - f(y)\| \le \varphi(x, y)$$

for all $x, y \in G$. Then there exists a unique additive mapping $T : G \to Y$ such that

$$\|f(x) - T(x)\| \le \frac{1}{2}\widetilde{\varphi}(x, x)$$

for all $x \in G$. C. Park [3] applied the Găvruta's result to linear functional equations in Banach modules over a C^*-algebra. Several functional equations have been investigated in [4]–[20].

In this paper, we prove the Hyers-Ulam stability of the multi–additive functional inequality

$$\left\|(\textstyle\sum_{i=1}^{d+1} f(x_{1i}), \cdots, \sum_{i=1}^{d+1} f(x_{ki}))\right\|_k \le \left\|\left(mf\left(\frac{\sum_{i=1}^{d+1} x_{1i}}{m}\right), \cdots, mf\left(\frac{\sum_{i=1}^{d+1} x_{ki}}{m}\right)\right)\right\|_k, \tag{1}$$

where $d \ge 2$ is a fixed integer.

2. Multi-normed spaces

The notion of multi-normed space was introduced by H.G. Dales and M.E. Polyakov in [21]. This concept is somewhat similar to operator sequence space and has some connections with operator spaces and Banach lattices. Motivations for the study of multi-normed spaces and many examples are given in [21–23].

Let $(\mathcal{E}, \|.\|)$ be a complex normed space, and let $k \in \mathbb{N}$. We denote by $\mathcal{E}^k$ the linear space $\mathcal{E} \oplus ... \oplus \mathcal{E}$ consisting of k-tuples $(x_1, ..., x_k)$, where $x_1, ..., x_k \in \mathcal{E}$. The linear operations on $\mathcal{E}^k$ are defined coordinatewise. The zero element of either $\mathcal{E}$ or $\mathcal{E}^k$ is denoted by 0. We denote by $\mathbb{N}_k$ the set $\{1, 2, ..., k\}$ and by Σ_k the group of permutations on k symbols.

Definition 2.1. *A multi-norm on $\{\mathcal{E}^k : k \in \mathbb{N}\}$ is a sequence*

$$(\|.\|_k) = (\|.\|_k : k \in \mathbb{N})$$

such that $\|.\|_k$ is a norm on $\mathcal{E}^k$ for each $k \in \mathbb{N}$:

(A1) $\|(x_{\sigma(1)}, ..., x_{\sigma(k)})\|_k = \|(x_1, ..., x_k)\|_k \qquad (\sigma \in \Sigma_k, x_1, ..., x_k \in \mathcal{E})$;

(A2) $\|(\alpha_1 x_1, ..., \alpha_k x_k)\|_k \le (\max_{i\in\mathbb{N}_k} |\alpha_i|) \|x_1, ..., x_k\|_k$
$(\alpha_1, ..., \alpha_k \in \mathbb{C},\ x_1, ..., x_k \in \mathcal{E})$;

(A3) $\|(x_1, ..., x_{k-1}, 0)\|_k = \|(x_1, ..., x_{k-1})\|_{k-1} \qquad (x_1, ..., x_{k-1} \in \mathcal{E})$;

(A4) $\|(x_1, ..., x_{k-1}, x_{k-1})\|_k = \|(x_1, ..., x_{k-1})\|_{k-1} \qquad (x_1, ..., x_{k-1} \in \mathcal{E})$.

In this case, we say that $((\mathcal{E}^k, \|.\|_k) : k \in \mathbb{N})$ is a multi-normed space.

Lemma 2.2. [23] *Suppose that $((\mathcal{E}^k, \|.\|_k) : k \in \mathbb{N})$ is a multi-normed space, and take $k \in \mathbb{N}$. Then*

(a) $\|(x, ..., x)\|_k = \|x\| \ (x \in \mathcal{E})$;

(b) $\max_{i\in\mathbb{N}_k} \|x_i\| \le \|(x_1, ..., x_k)\|_k \le \sum_{i=1}^k \|x_i\| \le k \max_{i\in\mathbb{N}_k} \|x_i\| \ (x_1, ..., x_k \in \mathcal{E})$.

It follows from (b) that, if $(\mathcal{E}, \|.\|)$ is a Banach space, then $(\mathcal{E}^k, \|.\|_k)$ is a Banach space for each $k \in \mathbb{N}$; in this case $((\mathcal{E}^k, \|.\|_k) : k \in \mathbb{N})$ is a multi-Banach space.

Now we state two important examples of multi-norms for an arbitrary normed space $\mathcal{E}$; cf. [21].

Example 2.3. *The sequence $(\|.\|_k : k \in \mathbb{N})$ on $\{\mathcal{E}^k : k \in \mathbb{N}\}$ defined by*

$$\|x_1, ..., x_k\|_k := \max_{i\in\mathbb{N}_k} \|x_i\| \qquad (x_1, ..., x_k \in \mathcal{E})$$

is a multi-norm called the minimum multi-norm. The terminology 'minimum' is justified by property (b).

Example 2.4. *Let* $\{(\|.\|_k^\alpha : k \in \mathbb{N}) : \alpha \in A\}$ *be the (non-empty) family of all multi-norms on* $\{\mathcal{E}^k : k \in \mathbb{N}\}$*. For* $k \in \mathbb{N}$*, set*

$$|||x_1, ..., x_k|||_k := \sup_{\alpha \in A} \|(x_1, ...x_k)\|_k^\alpha \quad (x_1, ..., x_k \in \mathcal{E}).$$

Then $(|||.|||_k : k \in \mathbb{N})$ *is a multi-norm on* $\{\mathcal{E}^k : k \in \mathbb{N}\}$*, called the maximum multi-norm.*

We need the following observation which can be easily deduced from the triangle inequality for the norm $\|.\|_k$ and the property (b) of multi-norms.

Lemma 2.5. *Suppose that* $k \in \mathbb{N}$ *and* $(x_1, ..., x_k) \in \mathcal{E}^k$*. For each* $j \in \{1, ..., k\}$*, let* $(x_n^j)_{n=1,2,...}$ *be a sequence in* $\mathcal{E}$ *such that* $\lim_{n\to\infty} x_n^j = x_j$*. Then for each* $(y_1, ..., y_k) \in \mathcal{E}^k$ *we have*

$$\lim_{n\to\infty}(x_n^1 - y_1, ..., x_n^k - y_k) = (x_1 - y_1, ..., x_k - y_k).$$

Definition 2.6. *Let* $((\mathcal{E}^k, \|.\|_k) : k \in \mathbb{N})$ *be a multi-normed space. A sequence* (x_n) *in* $\mathcal{E}$ *is a* multi-null *sequence if, for each* $\varepsilon > 0$*, there exists* $n_0 \in \mathbb{N}$ *such that*

$$\sup_{k\in\mathbb{N}} \|(x_n, ..., x_{n+k-1})\|_k < \varepsilon \quad (n \geq n_0).$$

Let $x \in \mathcal{E}$*. We say that the sequence* (x_n) *is multi-convergent to* $x \in \mathcal{E}$ *and write*

$$\lim_{n\to\infty} x_n = x$$

if $(x_n - x)$ *is a multi-null sequence.*

Definition 2.7. *[21, 24] Let* $(\mathcal{A}, \|.\|)$ *be a normed algebra such that* $((\mathcal{A}^k, \|.\|_k) : k \in \mathbb{N})$ *is a multi-normed space. Then* $((\mathcal{A}^k, \|.\|_k) : k \in \mathbb{N})$ *is a multi-normed algebra if*

$$\|(a_1b_1, ..., a_kb_k)\|_k \leq \|(a_1, ..., a_k)\|_k \cdot \|(b_1, ..., b_k)\|_k$$

for all $k \in \mathbb{N}$ *and all* $a_1, ..., a_k, b_1, ..., b_k \in \mathcal{A}$*. Further, the multi-normed algebra* $((\mathcal{A}^k, \|.\|_k) : k \in \mathbb{N})$ *is a multi-Banach algebra if* $((\mathcal{A}^k, \|.\|_k) : k \in \mathbb{N})$ *is a multi-Banach space.*

Example 2.8. *[21, 24] Let* p, q *with* $1 \leq p \leq q < \infty$*, and* $\mathcal{A} = \ell^p$*. The algebra* $\mathcal{A}$ *is a Banach sequence algebra with respect to coordinatewise multiplication of sequences. Let* $(\|.\|_k : k \in \mathbb{N})$ *be the standard* (p, q)*-multi-norm on* $\{\mathcal{A}^k : k \in \mathbb{N}\}$*. Then* $((\mathcal{A}^k, \|.\|_k) : k \in \mathbb{N})$ *is a multi-Banach algebra.*

Definition 2.9. *Let* $(\mathcal{A}, \|.\|)$ *be a Banach star algebra with involution* $*$*. A Multi-*C^**-algebra is multi-Banach algebra such that*

$$\|(a_1a_1^*, \cdots, a_ka_k^*)\| = \|(a_1, \cdots, a_k)\|^2$$

In a series of papers [25]–[32] and [17]–[19], many authors have considered a special class of quasi $*$-algebras, called *proper* CQ^**-algebras*, which arise as completions of C^*-algebras. They can be introduced in the following way:

Let $\mathfrak{A}$ be a linear space and $\mathcal{A}$ a $*$-algebra contained in $\mathfrak{A}$. We say that $\mathfrak{A}$ is a quasi $*$-algebra over $\mathcal{A}$ if the right and left multiplications of an element of $\mathfrak{A}$ and an element of $\mathcal{A}$ are always defined and linear, and an involution $*$ which extends the involution of $\mathcal{A}$ is defined in $\mathfrak{A}$ with the property $(ab)^* = b^*a^*$ whenever the multiplication is defined.

A quasi $*$-algebra $(\mathfrak{A}, \mathcal{A})$ is called topological if a locally convex topology τ on $\mathfrak{A}$ such that :
(Q1) the involution $a \mapsto a^*$ is continuous
(Q2) the maps $a \mapsto ab$ and $a \mapsto ba$ are continuous for each $b \in \mathcal{A}$
(Q3)$\mathcal{A}$ is dense in $\mathfrak{A}$ with topology τ.

In a topology quasi $*$-algebra the associative law holds in the following two formulations

$$a(bc) = (ab)c; \quad b(ac) = (ba)c \qquad (b, c \in \mathcal{A}, a \in \mathfrak{A}).$$

A CQ^*-algebra is a topological quasi$*$-algebra $(\mathfrak{A}, \mathcal{A})$ with the following properties:
(CQ1) $(\mathcal{A}, \|.\|_*)$ is a C^*-algebra with respect to the norm $\|.\|_*$ and the involution $*$.
(CQ2) $(\mathfrak{A}, \|.\|)$ is a Banach space and $\|a^*\| = \|a\|$ for every $a \in \mathfrak{A}$.
(CQ3) for every $b \in \mathcal{A}$ we have

$$\|b\|_* = \max\{\sup_{\|a\|\leq 1} \|ab\|, \quad \sup_{\|a\|\leq 1} \|ba\|\}.$$

F. Bagarello and C. Trapani [33] showed that both $(L^p(X, \mu), C_0(X))$ and $(L^p(X, \mu), L^\infty(X))$ are CQ^*-algebras. Now, we define the multi-CQ^*-algebra.

Let $(\mathfrak{A}, \mathcal{A})$ be a CQ^*-algebra. We say that $\{(\mathfrak{A}^k, \mathcal{A}^k) : k \in \mathbb{N}\}$ is a multi-CQ^*-algebra if for every $k \in \mathbb{N}$ the couple $(\mathfrak{A}^k, \mathcal{A}^k)$ is a CQ^*-algebra where $\{\mathfrak{A}^k : k \in \mathbb{N}\}$ and $\{\mathcal{A}^k : k \in \mathbb{N}\}$ are multi-Banach and multi-C^*-algebra respectively.

Example 2.10. *In [33], the authors showed that the couple $(\mathfrak{A}, \mathcal{A})$ is CQ^*-algebra where $\mathfrak{A} = \ell^p$ and $\mathcal{A} = c_0$. Now, consider Example 2.8 then $\{(\mathfrak{A}^k, \mathcal{A}^k) : k \in \mathbb{N}\}$ is a multi-CQ^*-algebra.*

The purpose of this paper is to investigate the Hyers-Ulam stability of homomorphisms in proper multi-CQ^*-algebras and of derivations on proper multi-CQ^*-algebras associated with the additive functional inequality (1). We denote that $\mathbb{T}^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$.

3. Stability of $\mathbb{C}$-linear mappings in multi-Banach spaces

We investigate the Hyers-Ulam stability of $\mathbb{C}$-linear mappings in multi-Banach spaces associated with the multi–additive functional inequality

$$\left\|(\textstyle\sum_{i=1}^{d+1} f(x_{1i}), \cdots, \sum_{i=1}^{d+1} f(x_{ki}))\right\|_k \leq \left\|\left(mf\left(\frac{\sum_{i=1}^{d+1} x_{1i}}{m}\right), \cdots, mf\left(\frac{\sum_{i=1}^{d+1} x_{ki}}{m}\right)\right)\right\|_k,$$

where $d \geq 2$ is a fixed integer. In this section, we assume that $(\mathcal{X}, \|.\|)$ and $(\mathcal{Y}, \|.\|)$ are Banach spaces such that $(\mathcal{X}^k, \|.\|_k)$ and $(\mathcal{Y}^k, \|.\|_k)$ are multi-Banach spaces.

Lemma 3.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying* (1) *in which $f(0) = 0$. Then f is additive.*

Proof. Letting $x_3 = \cdots = x_{d+1} = 0$ and replacing x_1 by x and x_2 by $-x$ in (1), we get

$$\|f(x) + f(-x)\| \leq \|mf(0)\| = 0$$

for all $x \in \mathcal{X}$. Hence $f(-x) = -f(x)$ for all $x \in \mathcal{X}$.

Replacing x_1 by x, x_2 by y and x_3 by $-x-y$ and putting $x_4 = \cdots = x_{d+1} = 0$ in (1), we get

$$\begin{aligned}\|f(x) + f(y) - f(x+y)\| &= \|f(x) + f(y) + f(-x-y)\| \\ &\leq \|mf(0)\|_Y = 0, \quad \forall\, x, y \in \mathcal{X}.\end{aligned}$$

Thus we have

$$f(x+y) = f(x) + f(y), \quad \forall\, x, y \in \mathcal{X}.$$

This completes the proof. $\square$

Theorem 3.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping. If there exists a function $\varphi : \mathcal{X}^{kd+k} \to [0,\infty)$ satisfying*

$$\begin{aligned}&\left\|\left(\sum_{i=1}^{d+1} f(x_{1i}),\cdots,\sum_{i=1}^{d+1} f(x_{ki})\right)\right\|_k \\ &\leq \left\|\left(mf\left(\frac{\sum_{i=1}^{d+1} x_{1i}}{m}\right),\cdots,mf\left(\frac{\sum_{i=1}^{d+1} x_{ki}}{m}\right)\right)\right\|_k + \varphi(x_{11},\cdots,x_{1\,d+1},\cdots,x_{k1},\cdots,x_{k\,d+1}),\end{aligned} \tag{2}$$

$$\begin{aligned}&\widetilde{\varphi}(x_{11},\cdots,x_{1\,d+1},\cdots,x_{k1},\cdots,x_{k\,d+1}) : \\ &= \sum_{j=0}^{\infty}\sup_{k\in\mathbb{N}} d^j\varphi\left(d^{-j-1}x_{11},\cdots,d^{-j-1}x_{1\,d+1},\cdots,d^{-j-1}x_{k1},\cdots,d^{-j-1}x_{k\,d+1}\right) < \infty,\end{aligned} \tag{3}$$

for all $x_{11},\cdots,x_{k\,d+1} \in \mathcal{X}$, then there exists a unique additive mapping $L : \mathcal{X} \to \mathcal{Y}$ satisfying

$$\begin{aligned}&\sup_{k\in\mathbb{N}} \|(f(x_1)-L(x_1),\cdots,f(x_k)-L(x_k))\|_k \\ &\qquad \leq \sup_{k\in\mathbb{N}} \widetilde{\varphi}(x_1,x_1,\cdots,-dx_1,\cdots,x_k,x_k,\cdots,-dx_k)\end{aligned} \tag{4}$$

for all $x_1,...,x_k \in \mathcal{X}$.

Proof. Since $\widetilde{\varphi}(0,...,0) < \infty$ in (3), we have $\varphi(0,...,0) = 0$ and so $f(0) = 0$. Replacing $x_{i1},...,x_{id}$ by x_i and x_{id+1} by $-dx_i$ in which $1 \leq i \leq k$, respectively in (2). Since $f(0) = 0$, we get

$$\begin{aligned}&\left\|(df(x_1)-f(dx_1),\cdots,df(x_k)-f(dx_k))\right\|_k \\ &\qquad = \left\|(df(x_1)+f(-dx_1),\cdots,df(x_k)+f(-dx_k))\right\|_k \\ &\qquad \leq \|(mf(0),\cdots,mf(0))\|_k \\ &\qquad + \varphi(x_1,x_1,\cdots,-dx_1,\cdots,x_k,x_k,\cdots,-dx_k)\end{aligned}$$

for all $x_1,...,x_k \in \mathcal{X}$. From the above inequality, we have

$$\begin{aligned}&\left\|\left(f(x_1)-df\left(\frac{x_1}{d}\right),\cdots,f(x_k)-df\left(\frac{x_k}{d}\right)\right)\right\|_k \\ &\quad \leq \varphi\left(\frac{x_1}{d},\frac{x_1}{d},\cdots,-x_1,\cdots,\frac{x_k}{d},\frac{x_k}{d},\cdots,-x_k\right)\end{aligned}$$

for all $x_1,...,x_k \in \mathcal{X}$. Replacing x_i by $d^{-n}x_i$, $1 \leq i \leq k$, in the above inequality, we have

$$\begin{aligned}&\left\|\left(d^nf\left(\frac{x_1}{d^n}\right)-d^{n+1}f\left(\frac{x_1}{d^{n+1}}\right),\cdots,d^nf\left(\frac{x_k}{d^n}\right)-d^{n+1}f\left(\frac{x_k}{d^{n+1}}\right)\right)\right\|_k \\ &\leq d^n\varphi\left(\frac{x_1}{d^{n+1}},\frac{x_1}{d^{n+1}},\cdots,-\frac{x_1}{d^n},\cdots,\frac{x_k}{d^{n+1}},\frac{x_k}{d^{n+1}},\cdots,-\frac{x_k}{d^n}\right)\end{aligned}$$

¿From the above inequality, we have

$$\begin{aligned}&\sup_{k\in\mathbb{N}}\left\|\left(d^nf\left(\frac{x_1}{d^n}\right)-d^qf\left(\frac{x_1}{d^q}\right),\cdots,d^nf\left(\frac{x_k}{d^n}\right)-d^qf\left(\frac{x_k}{d^q}\right)\right)\right\|_k \\ &\leq \sum_{j=q}^{n-1}\sup_{k\in\mathbb{N}}\left\|\left(d^{j+1}f\left(\frac{x_1}{d^{j+1}}\right)-d^jf\left(\frac{x_1}{d^j}\right),\cdots,d^{j+1}f\left(\frac{x_k}{d^{j+1}}\right)-d^jf\left(\frac{x_k}{d^j}\right)\right)\right\|_k \\ &\leq \sum_{j=q}^{n-1}\sup_{k\in\mathbb{N}} d^j\varphi\left(\frac{x_1}{d^{j+1}},\frac{x_1}{d^{j+1}},\cdots,-\frac{dx_1}{d^{j+1}},\cdots,\frac{x_k}{d^{j+1}},\frac{x_k}{d^{j+1}},\cdots,-\frac{dx_k}{d^{j+1}}\right)\end{aligned}$$

for all $x_1, ..., x_k \in \mathcal{X}$ and all non-negative integers q, n with $q < n$. ¿From (3), the sequence $\left\{d^n f\left(\frac{x}{d^n}\right)\right\}$ is a Cauchy sequence for all $x \in \mathcal{X}$ and convergent in the complete multi-norm $\mathcal{Y}$. So we can define a mapping $L : \mathcal{X} \to \mathcal{Y}$ by

$$L(x) := \lim_{n\to\infty} d^n f\left(\frac{x}{d^n}\right)$$

for all $x \in \mathcal{X}$.

In order to prove that L satisfies (4), if we put $q = 0$ and let $n \to \infty$ in the previous inequality then we obtain

$$\begin{aligned}
&\sup_{k\in\mathbb{N}} \|(f(x_1) - L(x_1), \cdots, f(x_k) - L(x_k))\|_k \\
&\leq \sum_{j=0}^{n-1} \sup_{k\in\mathbb{N}} d^j \varphi\left(\frac{x_1}{d^{j+1}}, \frac{x_1}{d^{j+1}}, \cdots, -\frac{dx_1}{d^{j+1}}, \cdots, \frac{x_k}{d^{j+1}}, \frac{x_k}{d^{j+1}}, \cdots, -\frac{dx_k}{d^{j+1}}\right) \\
&= \sup_{k\in\mathbb{N}} \widetilde{\varphi}(x_1, x_1, \cdots, -dx_1, \cdots, x_k, x_k, \cdots, -dx_k)
\end{aligned}$$

for all $x_1, ..., x_k \in \mathcal{X}$.

Replacing x_{ij} by $\frac{x_{1j}}{d^n}$, $(1 \leq i \leq k$ and $1 \leq j \leq d+1)$ respectively, and product by d^{n+1} in (2), we get

$$\begin{aligned}
&\left\|\sum_{i=1}^{d+1} d^n f\left(\frac{x_{1i}}{d^{n+1}}\right)\right\| \\
&\leq \left\|md^n f\left(\frac{\sum_{i=1}^{d+1} x_{1i}}{md^{n+1}}\right)\right\| + d^n \varphi\left(\frac{x_{11}}{d^{n+1}}, \cdots, \frac{x_{1d+1}}{d^{n+1}}, \cdots \frac{x_{11}}{d^{n+1}}, \cdots, \frac{x_{1d+1}}{d^{n+1}}\right),
\end{aligned}$$

for all $x_{1j} \in \mathcal{X}$ $(1 \leq j \leq d+1)$. Since (3) gives that

$$\lim_{n\to\infty} d^n \sup_{k\in\mathbb{N}} \varphi\left(\frac{x_{11}}{d^{n+1}}, \cdots, \frac{x_{1d+1}}{d^{n+1}}, \cdots \frac{x_{11}}{d^{n+1}}, \cdots, \frac{x_{1d+1}}{d^{n+1}}\right) = 0$$

for all $x_{1j} \in \mathcal{X}$ $(1 \leq j \leq d+1)$, if we let $n \to \infty$ in the above inequality then we get

$$\left\|\sum_{i=1}^{d+1} L(x_{1i})\right\| \quad \leq \quad \left\|mL\left(\frac{\sum_{i=1}^{d+1} x_{1i}}{m}\right)\right\|, \tag{5}$$

and so L is additive by Lemma 3.1.

Now to prove the uniqueness of L, let $L' : \mathcal{X} \to \mathcal{Y}$ be another additive mapping satisfying (4). Since L and L' are additive, we have

$$\begin{aligned}
&\|L(x) - L'(x)\| \\
&= d^n \left\| L\left(\frac{x}{d^n}\right) - L'\left(\frac{x}{d^n}\right) \right\| \\
&\leq d^n \left(\left\| L\left(\frac{x}{d^n}\right) - f\left(\frac{x}{d^n}\right) \right\| + \left\| L'\left(\frac{x}{d^n}\right) - f\left(\frac{x}{d^n}\right) \right\| \right) \\
&\leq d^n \cdot 2\widetilde{\varphi}\left(\frac{x}{d^n}, \cdots, \frac{x}{d^n}, \frac{-dx}{d^n}, \cdots, \frac{x}{d^n}, \cdots, \frac{x}{d^n}, \frac{-dx}{d^n}\right) \\
&= 2\sum_{j=0}^{\infty} \sup_{k\in\mathbb{N}} d^{n+j}\varphi\left(\overbrace{\overbrace{\frac{x}{d^{n+j+1}}, \cdots, \frac{x}{d^{n+j+1}}, \frac{-dx}{d^{n+j+1}}}^{d+1}, \cdots, \overbrace{\frac{x}{d^{n+j+1}}, \cdots, \frac{x}{d^{n+j+1}}, \frac{-dx}{d^{n+j+1}}}^{d+1}}^{k} \right)
\end{aligned}$$

which goes to zero as $n \to \infty$ for all $x \in \mathcal{X}$ by (3). Consequently, L is the unique additive mapping satisfying (4), as desired. □

Corollary 3.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping with $f(0) = 0$. If there exists a function $\varphi : \mathcal{X}^{kd+k} \to [0, \infty)$ satisfying* (2) *and*

$$\widetilde{\varphi}(x_{11}, \cdots, x_{kd+k}) := \sum_{j=0}^{\infty} \sup_{k\in\mathbb{N}} \frac{1}{d^{j+1}} \varphi\left(d^j x_{11}, \cdots, d^j x_{k\,d+1}\right) < \infty, \tag{6}$$

for all $x_{11}, \cdots, x_{k\,d+1} \in \mathcal{X}$, then there exists a unique additive mapping $L : \mathcal{X} \to \mathcal{Y}$ satisfying

$$\begin{aligned}
&\sup_{k\in\mathbb{N}} \|(f(x_1) - L(x_1), \cdots, f(x_k) - L(x_k))\|_k \\
&\leq \sup_{k\in\mathbb{N}} \widetilde{\varphi}(x_1, x_1, \cdots, -dx_1, \cdots, x_k, x_k, \cdots, -dx_k)
\end{aligned} \tag{7}$$

for all $x_1, ..., x_k \in \mathcal{X}$.

Proof. The proof is the same as in the corresponding part of the proof of Theorem 3.2, as desired. □

Lemma 3.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying*

$$\left\| \sum_{i=1}^{d} f(x_i) + \mu f(x_{d+1}) \right\| \quad \leq \quad \left\| mf\left(\frac{\sum_{i=1}^{d} x_i + \mu x_{d+1}}{m} \right) \right\|, \tag{8}$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, ..., x_{d+1} \in \mathcal{X}$. Then f is $\mathbb{C}$-linear.

Proof. If we put $\mu = 1$ in , then f is additive by Lemma 3.1.

Putting $x_1 = x, x_i = 0, 2 \leq i \leq d$ and $x_{d+1} = -x$, respectively, we get $f(\mu x) + \mu f(-x) = 0$ and so $f(\mu x) = \mu f(x)$ for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{X}$. Thus we have $f(\mu x + \bar{\mu} x) = f(\mu x) + f(\bar{\mu} x) = \mu f(x) + \bar{\mu} f(x)$ for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{X}$, and so $f(tx) = tf(x)$ for any real number t with $|t| \leq 1$ and all $x \in \mathcal{X}$.

On the other hand, since $f(mx) = mf(x)$, we get $f(m^n x) = m^n f(x)$ for all $n \in \mathbb{N}$. So for any real number t, there is a natural number n with $|t| \leq m^n$. Thus we have

$$f(tx) = f\left(m^n \cdot \frac{t}{m^n} x\right) = m^n f\left(\frac{t}{m^n} x\right) = m^n \cdot \frac{t}{m^n} f(x) = tf(x).$$

Now we consider any $\alpha \in \mathbb{C}$ with $\alpha = t + si$ for some real numbers t, s. Since $f(ix) = if(x)$ holds, we have

$$f(\alpha x) = f(tx) + f(six) = tf(x) + sf(ix) = tf(x) + sif(x) = \alpha f(x)$$

and so f is $\mathbb{C}$-linear, as desired. □

Theorem 3.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping. If there exists a function $\varphi : \mathcal{X}^{kd+k} \to [0, \infty)$ satisfying* (3) *and*

$$\begin{aligned} &\left\|\left(\sum_{i=1}^{d} f(x_{1i}) + \mu f(x_{1d+1}), \cdots, \sum_{i=1}^{d} f(x_{ki}) + \mu f(x_{kd+1})\right)\right\|_k \\ \leq &\left\|\left(mf\left(\frac{\sum_{i=1}^{d} x_{1i} + \mu x_{1d+1}}{m}\right), \cdots, mf\left(\frac{\sum_{i=1}^{d} x_{ki} + \mu x_{kd+1}}{m}\right)\right)\right\|_k \\ + &\varphi(x_{11}, \cdots, x_{kd+1}), \end{aligned} \tag{9}$$

for all $\mu \in \mathbb{T}^1$ and all $x_{11}, \cdots, x_{kd+1} \in \mathcal{X}$, then there exists a unique $\mathbb{C}$-linear mapping $L : \mathcal{X} \to \mathcal{Y}$ satisfying (4)*.*

Proof. If we put $\mu = 1$ in (9), then by Theorem 3.2 there exists a unique additive mapping $L : \mathcal{X} \to \mathcal{Y}$ defined by

$$L(x) := \lim_{n\to\infty} d^n f\left(\frac{x}{d^n}\right)$$

for all $x \in \mathcal{X}$ which satisfies (4). By a similar method to the corresponding part of the proof of Theorem 3.2, L satisfies

$$\left\|\sum_{i=1}^{d} L(x_i) + \mu L(x_{d+1})\right\| \quad \leq \quad \left\|mL\left(\frac{\sum_{i=1}^{d} x_i + \mu x_{d+1}}{m}\right)\right\|,$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, ..., x_{d+1} \in \mathcal{X}$. Thus Lemma 3.4 gives that L is $\mathbb{C}$-linear. □

Corollary 3.6. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping with $f(0) = 0$. If there exists a function $\varphi : \mathcal{X}^{kd+k} \to [0, \infty)$ satisfying* (6) *and* (9)*, then there exists a unique $\mathbb{C}$-linear mapping $L : \mathcal{X} \to \mathcal{Y}$ satisfying* (7)*.*

Proof. The rest of the proof is the same as in the corresponding part of the proof of Theorem 3.5, as desired. □

4. Stability of homomorphisms in proper multi-CQ^*-algebras

We investigate the Hyers-Ulam stability of isomorphisms in proper multi-CQ^*-algebras associated with the additive functional inequality. In this section, we assume that $(\mathcal{A}, \|.\|)$ and $(\mathcal{B}, \|.\|)$ are Banach algebras such that $(\mathcal{A}^k, \|.\|_k)$ and $(\mathcal{B}^k, \|.\|_k)$ are multi-Banach algebras.

Theorem 4.1. *Let $f : \mathcal{A} \to \mathcal{B}$ be a mapping. If there exists a function $\varphi : \mathcal{X}^{kd+k} \to [0, \infty)$ satisfying* (3) *and*

$$\begin{aligned} &\left\|\left(\sum_{i=1}^{d} f(x_{1i}) + \mu f(x_{1d+1}), \cdots, \sum_{i=1}^{d} f(x_{ki}) + \mu f(x_{kd+1})\right)\right\|_k \\ \leq &\left\|\left(mf\left(\frac{\sum_{i=1}^{d} x_{1i} + \mu x_{1d+1}}{m}\right), \cdots, mf\left(\frac{\sum_{i=1}^{d} x_{ki} + \mu x_{kd+1}}{m}\right)\right)\right\|_k \\ + &\varphi(x_{11}, \cdots, x_{kd+1}), \end{aligned} \tag{10}$$

for all $\mu \in \mathbb{T}^1$ and all $x_{11}, \cdots, x_{kd+1} \in \mathcal{A}$. If, in addition, there exists a function $\phi : \mathcal{A}^{2k} \to [0, \infty)$ satisfying

$$\begin{aligned} &\|(f(x_1y_1) - f(x_1)f(y_1), \cdots, f(x_ky_k) - f(x_k)f(y_k))\|_k \\ \leq &\phi(x_1, y_1, \cdots, x_k, y_k), \end{aligned} \tag{11}$$

$$\lim_{n\to\infty}\sup_{k\in\mathbb{N}} d^{2n}\phi(d^{-n}x_1, d^{-n}y_1, \cdots, d^{-n}x_k, d^{-n}y_k) = 0 \tag{12}$$

for all $x_1, ..., x_k, y_1, ..., y_k \in \mathcal{A}$ whenever the multiplication is defined. Then there exists a unique proper CQ^*-algebra homomorphism $h : \mathcal{A} \to \mathcal{B}$ satisfying

$$\begin{aligned}&\sup_{k\in\mathbb{N}} \|(f(x_1) - h(x_1), \cdots, f(x_k) - h(x_k))\|_k \\ \leq &\sup_{k\in\mathbb{N}} \widetilde{\varphi}(x_1, x_1, \cdots, -dx_1, \cdots, x_k, x_k, \cdots, -dx_k)\end{aligned} \tag{13}$$

for all $x_1, ..., x_k \in \mathcal{A}$.

Proof. By Theorem 3.5, we have a unique $\mathbb{C}$-linear mapping $h : \mathcal{A} \to \mathcal{B}$ defined by

$$h(x) := \lim_{n\to\infty} d^n f\left(\frac{x}{d^n}\right)$$

for all $x \in \mathcal{A}$ which satisfies (4).

Now we show that $h(xy) = h(x)h(y)$ for all $x, y \in \mathcal{A}$ whenever the multiplication is defined.

Replacing x_i, y_i by $d^{-n}x_i, d^{-n}y_i$, $1 \leq i \leq k$, respectively, and multiplying by d^{2n} in (11), we get

$$\begin{aligned}&\Big\|(d^{2n}[f(d^{-n}x_1 d^{-n}y_1) - f(d^{-n}x_1)f(d^{-n}y_1)] \\ &, \cdots, d^{2n}[f(d^{-n}x_k d^{-n}y_k) - f(d^{-n}x_k)f(d^{-n}y_k)])\Big\|_k \\ \leq & d^{2n}\phi(d^{-n}x_1, d^{-n}y_1, \cdots, d^{-n}x_k, d^{-n}y_k)\end{aligned} \tag{14}$$

for all $x_1, ..., x_k, y_1, ..., y_k \in \mathcal{A}$ whenever the multiplication is defined. We have

$$\lim_{n\to\infty} d^{2n} f(d^{-n}xd^{-n}y) = \lim_{n\to\infty} d^{2n} f(d^{-2n}xy) = h(xy)$$

$$\lim_{n\to\infty} d^{2n} f(d^{-n}x)f(d^{-n}y) = \lim_{n\to\infty} d^n f(d^{-n}x) \cdot \lim_{n\to\infty} d^n f(d^{-n}y) = h(x)h(y)$$

for all $x, y \in \mathcal{A}$ whenever the multiplication is defined. If we let $n \to \infty$ in the above inequality then (12) gives $h(xy) = h(x)h(y)$ for all $x, y \in \mathcal{A}$, whenever the multiplication is defined. □

Corollary 4.2. *Let θ, p be nonnegative real numbers with $p > 1$ and $f : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying*

$$\begin{aligned}&\left\|\left(\sum_{i=1}^{d} f(x_{1i}) + \mu f(x_{1d+1}), \cdots, \sum_{i=1}^{d} f(x_{ki}) + \mu f(x_{kd+1})\right)\right\|_k \\ \leq &\left\|\left(mf\left(\frac{\sum_{i=1}^{d} x_{1i} + \mu x_{1d+1}}{m}\right), \cdots, mf\left(\frac{\sum_{i=1}^{d} x_{ki} + \mu x_{kd+1}}{m}\right)\right)\right\|_k \\ &+ \theta \cdot \sum_{l=1}^{k}\sum_{i=1}^{d+1} \|x_{li}\|^p,\end{aligned} \tag{15}$$

for all $\mu \in \mathbb{T}^1$ and all $x_{11}, \cdots, x_{kd+1} \in \mathcal{A}$. If, in addition,

$$\begin{aligned}&\|(f(x_1y_1) - f(x_1)f(y_1), \cdots, f(x_ky_k) - f(x_k)f(y_k))\|_k \\ &\leq \theta \cdot \sum_{i=1}^{d+1} (\|x_i\|^{2p} + \|y_i\|^{2p})\end{aligned} \tag{16}$$

for all $x_1, ..., x_k, y_1, ..., y_k \in \mathcal{A}$ whenever the multiplication is defined. Then there exists a unique proper CQ^*-algebra homomorphism $h : \mathcal{A} \to \mathcal{B}$ satisfying

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - h(x_1), \cdots, f(x_k) - h(x_k))\|_k \leq \sup_{k\in\mathbb{N}} \frac{d^{p-1}+1}{d^{p-1}-1}\theta \sum_{l=1}^{k} \|x_l\|^p$$

for all $x_1, ..., x_k \in \mathcal{A}$.

Proof. Let $\varphi : \mathcal{A}^{kd+k} \to [0, \infty)$ be

$$\varphi(x_{11}, \cdots, x_{1\,d+1}, \cdots, x_{k1}, \cdots, x_{k\,d+1}) = \theta \cdot \sum_{l=1}^{k} \sum_{i=1}^{d+1} \|x_{li}\|^p.$$

When $p > 1$, we get

$$\begin{aligned}
&\widetilde{\varphi}(x_{11}, \cdots, x_{1\,d+1}, \cdots, x_{k1}, \cdots, x_{k\,d+1}) : \\
&= \sum_{j=0}^{\infty} \sup_{k\in\mathbb{N}} d^j \varphi\left(d^{-j-1}x_{11}, \cdots, d^{-j-1}x_{1\,d+1}, \cdots, d^{-j-1}x_{k1}, \cdots, d^{-j-1}x_{k\,d+1}\right) \\
&= \frac{1}{d} \sum_{j=0}^{\infty} \frac{d^{j+1}}{d^{(j+1)p}} \sup_{k\in\mathbb{N}} \theta \cdot \sum_{l=1}^{k} \sum_{i=1}^{d+1} \|x_{li}\|^p \\
&= \frac{\theta}{d^p - d} \sup_{k\in\mathbb{N}} \sum_{l=1}^{k} \sum_{i=1}^{d+1} \|x_{li}\|^p.
\end{aligned}$$

In addition, let $\phi : \mathcal{A}^{2k} \to [0, \infty)$ be

$$\phi(x_1, y_1, \cdots, x_k, y_k) = \theta \cdot \sum_{i=1}^{d+1} (\|x_i\|^{2p} + \|y_i\|^{2p}).$$

When $p > 1$, we have

$$\lim_{n\to\infty} d^{2n}\phi(d^{-n}x_1, d^{-n}y_1, \cdots, d^{-n}x_k, d^{-n}y_k) = \lim_{n\to\infty} \frac{d^{2n}}{d^{2pn}} \theta \cdot \sum_{i=1}^{d+1} (\|x_i\|^{2p} + \|y_i\|^{2p}) = 0$$

for all $x_1, ..., x_k, y_1, ..., y_k \in \mathcal{A}$. By applying Theorem 4.1, there exists a unique proper CQ^*-algebra homomorphism $h : \mathcal{A} \to \mathcal{B}$ such that

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - h(x_1), \cdots, f(x_k) - h(x_k))\|_k \leq \sup_{k\in\mathbb{N}} \frac{d^p + d}{d^p - d} \theta \sum_{l=1}^{k} \|x_l\|^p$$

for all $x_1, ..., x_k \in \mathcal{A}$. □

Corollary 4.3. *Let θ, p be nonnegative real numbers with $p > 1$ and $f : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying*

$$\begin{aligned}
&\left\|\left(\sum_{i=1}^{d} f(x_{1i}) + \mu f(x_{1d+1}), \cdots, \sum_{i=1}^{d} f(x_{ki}) + \mu f(x_{k\,d+1})\right)\right\|_k \\
&\leq \left\|\left(mf\left(\frac{\sum_{i=1}^{d} x_{1i} + \mu x_{1d+1}}{m}\right), \cdots, mf\left(\frac{\sum_{i=1}^{d} x_{ki} + \mu x_{k\,d+1}}{m}\right)\right)\right\|_k \\
&\quad + \theta \cdot \sum_{l=1}^{k} \sum_{i=1}^{d+1} \|x_{li}\|^p,
\end{aligned} \tag{17}$$

for all $\mu \in \mathbb{T}^1$ and all $x_{11}, \cdots, x_{k\,d+1} \in \mathcal{A}$. If, in addition,

$$\|(f(x_1y_1) - f(x_1)f(y_1), \cdots, f(x_ky_k) - f(x_k)f(y_k))\|_k \leq \theta \cdot \sum_{i=1}^{d+1} (\|x_i\|^p \cdot \|y_i\|^p) \tag{18}$$

for all $x_1, ..., x_k, y_1, ..., y_k \in \mathcal{A}$ whenever the multiplication is defined. Then there exists a unique proper CQ^*-algebra homomorphism $h : \mathcal{A} \to \mathcal{B}$ satisfying

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - h(x_1), \cdots, f(x_k) - h(x_k))\|_k \le \sup_{k\in\mathbb{N}} \frac{d^{p-1}+1}{d^{p-1}-1}\theta \sum_{l=1}^{k} \|x_l\|^p$$

for all $x_1, ..., x_k \in \mathcal{A}$.

Proof. Let $\varphi : \mathcal{A}^{kd+k} \to [0, \infty)$ be

$$\varphi(x_{11}, \cdots, x_{1\,d+1}, \cdots, x_{k1}, \cdots, x_{k\,d+1}) = \theta \cdot \sum_{l=1}^{k}\sum_{i=1}^{d+1} \|x_{li}\|^p.$$

When $p > 1$, we get

$$\begin{aligned}
&\widetilde{\varphi}(x_{11}, \cdots, x_{1\,d+1}, \cdots, x_{k1}, \cdots, x_{k\,d+1}) :\\
&= \sum_{j=0}^{\infty} \sup_{k\in\mathbb{N}} d^j \varphi\left(d^{-j-1}x_{11}, \cdots, d^{-j-1}x_{1\,d+1}, \cdots, d^{-j-1}x_{k1}, \cdots, d^{-j-1}x_{k\,d+1}\right)\\
&= \frac{1}{d}\sum_{j=0}^{\infty} \frac{d^{j+1}}{d^{(j+1)p}} \sup_{k\in\mathbb{N}} \theta \cdot \sum_{l=1}^{k}\sum_{i=1}^{d+1} \|x_{li}\|^p\\
&= \frac{\theta}{d^p - d} \sup_{k\in\mathbb{N}} \sum_{l=1}^{k}\sum_{i=1}^{d+1} \|x_{li}\|^p.
\end{aligned}$$

In addition, let $\phi : \mathcal{A}^{2k} \to [0, \infty)$ be

$$\phi(x_1, y_1, \cdots, x_k, y_k) = \theta \cdot \sum_{i=1}^{d+1} (\|x_i\|^p \cdot \|y_i\|^p).$$

When $p > 1$, we have

$$\lim_{n\to\infty} d^{2n}\phi(d^{-n}x_1, d^{-n}y_1, \cdots, d^{-n}x_k, d^{-n}y_k) = \lim_{n\to\infty} \frac{d^{2n}}{d^{2pn}}\theta \cdot \sum_{i=1}^{d+1}(\|x_i\|^p \cdot \|y_i\|^p) = 0$$

for all $x_1, ..., x_k, y_1, ..., y_k \in \mathcal{A}$. By applying Theorem 4.1, there exists a unique proper CQ^*-algebra homomorphism $h : \mathcal{A} \to \mathcal{B}$ such that

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - h(x_1), \cdots, f(x_k) - h(x_k))\|_k \le \sup_{k\in\mathbb{N}} \frac{d^p + d}{d^p - d}\theta \sum_{l=1}^{k} \|x_l\|^p$$

for all $x_1, ..., x_k \in \mathcal{A}$. □

Theorem 4.4. *Let $f : \mathcal{A} \to \mathcal{B}$ be a mapping with $f(0) = 0$. If there exists a function $\varphi : \mathcal{X}^{kd+k} \to [0, \infty)$ satisfying* (6) *and*

$$\begin{aligned}
&\left\|\left(\sum_{i=1}^{d} f(x_{1i}) + \mu f(x_{1d+1}), \cdots, \sum_{i=1}^{d} f(x_{ki}) + \mu f(x_{k\,d+1})\right)\right\|_k \qquad (19)\\
&\le \left\|\left(mf\left(\frac{\sum_{i=1}^{d} x_{1i} + \mu x_{1d+1}}{m}\right), \cdots, mf\left(\frac{\sum_{i=1}^{d} x_{ki} + \mu x_{k\,d+1}}{m}\right)\right)\right\|_k\\
&+\varphi(x_{11}, \cdots, x_{k\,d+1}),
\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x_{11}, \cdots, x_{kd+1} \in \mathcal{A}$. If, in addition, there exists a function $\phi : \mathcal{A}^{2k} \to [0, \infty)$ satisfying

$$\begin{aligned}&\|(f(x_1y_1) - f(x_1)f(y_1), \cdots, f(x_ky_k) - f(x_k)f(y_k))\|_k \\ \leq &\phi(x_1, y_1, \cdots, x_k, y_k),\end{aligned} \tag{20}$$

$$\lim_{n\to\infty} \sup_{k\in\mathbb{N}} d^{-2n}\phi(d^n x_1, d^n y_1, \cdots, d^n x_k, d^n y_k) = 0 \tag{21}$$

for all $x_1, ..., x_k, y_1, ..., y_k \in \mathcal{A}$ whenever the multiplication is defined. Then there exists a unique proper CQ^*-algebra homomorphism $h : \mathcal{A} \to \mathcal{B}$ satisfying

$$\begin{aligned}&\sup_{k\in\mathbb{N}} \|(f(x_1) - h(x_1), \cdots, f(x_k) - h(x_k))\|_k \\ \leq &\sup_{k\in\mathbb{N}} \widetilde{\varphi}(x_1, x_1, \cdots, -dx_1, \cdots, x_k, x_k, \cdots, -dx_k)\end{aligned} \tag{22}$$

for all $x_1, ..., x_k \in \mathcal{A}$.

Proof. We omit the proof because it is similar to the proof of Theorem 4.1. □

Corollary 4.5. *Let θ, p be nonnegative real numbers with $p < 1$ and $f : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying*

$$\begin{aligned}&\left\|\left(\sum_{i=1}^{d} f(x_{1i}) + \mu f(x_{1d+1}), \cdots, \sum_{i=1}^{d} f(x_{ki}) + \mu f(x_{kd+1})\right)\right\|_k \\ \leq &\left\|\left(mf\left(\frac{\sum_{i=1}^{d} x_{1i} + \mu x_{1d+1}}{m}\right), \cdots, mf\left(\frac{\sum_{i=1}^{d} x_{ki} + \mu x_{kd+1}}{m}\right)\right)\right\|_k \\ &+\theta \cdot \sum_{l=1}^{k}\sum_{i=1}^{d+1} \|x_{li}\|^p,\end{aligned} \tag{23}$$

for all $\mu \in \mathbb{T}^1$ and all $x_{11}, \cdots, x_{kd+1} \in \mathcal{A}$. If, in addition,

$$\begin{aligned}&\|(f(x_1y_1) - f(x_1)f(y_1), \cdots, f(x_ky_k) - f(x_k)f(y_k))\|_k \\ \leq &\theta \cdot \sum_{i=1}^{d+1} (\|x_i\|^{2p} + \|y_i\|^{2p})\end{aligned} \tag{24}$$

for all $x_1, ..., x_k, y_1, ..., y_k \in \mathcal{A}$ whenever the multiplication is defined. Then there exists a unique proper CQ^*-algebra homomorphism $h : \mathcal{A} \to \mathcal{B}$ satisfying

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - h(x_1), \cdots, f(x_k) - h(x_k))\|_k \leq \sup_{k\in\mathbb{N}} \frac{d + d^p}{d - d^p}\theta \sum_{l=1}^{k} \|x_l\|^p$$

for all $x_1, ..., x_k \in \mathcal{A}$.

Proof. Let $\phi : \mathcal{A}^{2k} \to [0, \infty)$ be

$$\phi(x_1, y_1, \cdots, x_k, y_k) = \theta \cdot \sum_{i=1}^{d+1} (\|x_i\|^{2p} + \|y_i\|^{2p}).$$

When $p < 1$, we have

$$\lim_{n\to\infty} d^{-2n}\phi(d^n x_1, d^n y_1, \cdots, d^n x_k, d^n y_k) = \lim_{n\to\infty} \frac{d^{2np}}{d^{2n}}\theta \cdot \sum_{i=1}^{d+1} (\|x_i\|^{2p} + \|y_i\|^{2p}) = 0$$

for all $x_1, ..., x_k, y_1, ..., y_k \in \mathcal{A}$. By applying Theorem 4.4, there exists a unique proper CQ^*-algebra homomorphism $h : \mathcal{A} \to \mathcal{B}$ such that

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - h(x_1), \cdots, f(x_k) - h(x_k))\|_k \leq \sup_{k\in\mathbb{N}} \frac{d + d^p}{d - d^p} \theta \sum_{l=1}^{k} \|x_l\|^p$$

for all $x_1, ..., x_k \in \mathcal{A}$. □

Corollary 4.6. *Let θ, p be nonnegative real numbers with $p < 1$ and $f : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying*

$$\begin{aligned} &\left\|\left(\sum_{i=1}^{d} f(x_{1i}) + \mu f(x_{1d+1}), \cdots, \sum_{i=1}^{d} f(x_{ki}) + \mu f(x_{kd+1})\right)\right\|_k \\ \leq &\left\|\left(mf\left(\frac{\sum_{i=1}^{d} x_{1i} + \mu x_{1d+1}}{m}\right), \cdots, mf\left(\frac{\sum_{i=1}^{d} x_{ki} + \mu x_{kd+1}}{m}\right)\right)\right\|_k \\ &+ \theta \cdot \sum_{l=1}^{k}\sum_{i=1}^{d+1} \|x_{li}\|^p, \end{aligned} \tag{25}$$

for all $\mu \in \mathbb{T}^1$ and all $x_{11}, \cdots, x_{kd+1} \in \mathcal{A}$. If, in addition,

$$\begin{aligned} &\|(f(x_1y_1) - f(x_1)f(y_1), \cdots, f(x_ky_k) - f(x_k)f(y_k))\|_k \\ \leq &\theta \cdot \sum_{i=1}^{d+1} (\|x_i\|^p \cdot \|y_i\|^p) \end{aligned} \tag{26}$$

for all $x_1, ..., x_k, y_1, ..., y_k \in \mathcal{A}$ whenever the multiplication is defined. Then there exists a unique proper CQ^*-algebra homomorphism $h : \mathcal{A} \to \mathcal{B}$ satisfying

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - h(x_1), \cdots, f(x_k) - h(x_k))\|_k \leq \sup_{k\in\mathbb{N}} \frac{d + d^p}{d - d^p} \theta \sum_{l=1}^{k} \|x_l\|^p$$

for all $x_1, ..., x_k \in \mathcal{A}$.

Proof. Let $\phi : \mathcal{A}^{2k} \to [0, \infty)$ be

$$\phi(x_1, y_1, \cdots, x_k, y_k) = \theta \cdot \sum_{i=1}^{d+1} (\|x_i\|^p \cdot \|y_i\|^p).$$

When $p < 1$, we have

$$\lim_{n\to\infty} d^{-2n} \phi(d^n x_1, d^n y_1, \cdots, d^n x_k, d^n y_k) = \lim_{n\to\infty} \frac{d^{2pn}}{d^{2n}} \theta \cdot \sum_{i=1}^{d+1} (\|x_i\|^p \cdot \|y_i\|^p) = 0$$

for all $x_1, ..., x_k, y_1, ..., y_k \in \mathcal{A}$. By applying Theorem 4.4, there exists a unique proper CQ^*-algebra homomorphism $h : \mathcal{A} \to \mathcal{B}$ such that

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - h(x_1), \cdots, f(x_k) - h(x_k))\|_k \leq \sup_{k\in\mathbb{N}} \frac{d + d^p}{d - d^p} \theta \sum_{l=1}^{k} \|x_l\|^p$$

for all $x_1, ..., x_k \in \mathcal{A}$. □

5. Stability of derivations on proper CQ^*-algebras

We investigate the Hyers-Ulam stability of derivations in proper multi-CQ^*-algebras associated with the additive functional inequality. In this section, we assume that $(\mathcal{A}, \|.\|)$ is a Banach algebra such that $(\mathcal{A}^k, \|.\|_k)$ is a multi-Banach algebra.

Theorem 5.1. *Let $f : \mathcal{A} \to \mathcal{A}$ be a mapping. If there exists a function $\varphi : \mathcal{X}^{kd+k} \to [0, \infty)$ satisfying* (3) *and*

$$\begin{aligned}&\left\|\left(\sum_{i=1}^{d} f(x_{1i}) + \mu f(x_{1d+1}), \cdots, \sum_{i=1}^{d} f(x_{ki}) + \mu f(x_{k\,d+1})\right)\right\|_k \\ \leq &\left\|\left(mf\left(\frac{\sum_{i=1}^{d} x_{1i} + \mu x_{1d+1}}{m}\right), \cdots, mf\left(\frac{\sum_{i=1}^{d} x_{ki} + \mu x_{k\,d+1}}{m}\right)\right)\right\|_k \\ &+\varphi(x_{11}, \cdots, x_{k\,d+1}),\end{aligned} \tag{27}$$

for all $\mu \in \mathbb{T}^1$ and all $x_{11}, \cdots, x_{k\,d+1} \in \mathcal{A}$. If, in addition, there exists a function $\psi : \mathcal{A}^{2k} \to [0, \infty)$ satisfying

$$\begin{aligned}&\|(f(x_1y_1) - f(x_1)y_1 - x_1f(y_1), \cdots, f(x_ky_k) - f(x_k)y_k - x_kf(y_k))\|_k \\ \leq&\psi(x_1, y_1, \cdots, x_k, y_k),\end{aligned} \tag{28}$$

$$\lim_{n\to\infty} \sup_{k\in\mathbb{N}} d^{2n}\psi(d^{-n}x_1, d^{-n}y_1, \cdots, d^{-n}x_k, d^{-n}y_k) = 0 \tag{29}$$

for all $x_1, ..., x_k, y_1, ..., y_k \in \mathcal{A}$ whenever the multiplication is defined. Then there exists a unique derivation $\delta : \mathcal{A} \to \mathcal{A}$ satisfying

$$\begin{aligned}&\sup_{k\in\mathbb{N}} \|(f(x_1) - \delta(x_1), \cdots, f(x_k) - \delta(x_k))\|_k \\ \leq &\sup_{k\in\mathbb{N}} \widetilde{\varphi}(x_1, x_1, \cdots, -dx_1, \cdots, x_k, x_k, \cdots, -dx_k)\end{aligned} \tag{30}$$

for all $x_1, ..., x_k \in \mathcal{A}$.

Proof. By Theorem 3.5, we have a unique $\mathbb{C}$-linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ defined by

$$\delta(x) := \lim_{n\to\infty} d^n f\left(\frac{x}{d^n}\right)$$

for all $x \in \mathcal{A}$ which satisfies (30).

Now we show that $\delta(xy) = \delta(x)\delta(y)$ for all $x, y \in \mathcal{A}$ whenever the multiplication is defined.

Replacing x_i, y_i by $d^{-n}x_i, d^{-n}y_i$, $1 \leq i \leq k$, respectively, and multiplying by d^{2n} in (28), we get

$$\begin{aligned}&\Big\|(d^{2n}[f(d^{-n}x_1d^{-n}y_1) - d^{-n}f(d^{-n}x_1)y_1 - d^{-n}x_1f(d^{-n}y_1)], \\ &\cdots, d^{2n}[f(d^{-n}x_kd^{-n}y_k) - d^{-n}f(d^{-n}x_k)y_k - d^{-n}x_kf(d^{-n}y_k)])\Big\|_k \\ \leq&d^{2n}\psi(d^{-n}x_1, d^{-n}y_1, \cdots, d^{-n}x_k, d^{-n}y_k)\end{aligned} \tag{31}$$

for all $x_1, ..., x_k, y_1, ..., y_k \in \mathcal{A}$ whenever the multiplication is defined. We have

$$\begin{aligned}&\lim_{n\to\infty} d^{2n}f(d^{-n}xd^{-n}y) = \lim_{n\to\infty} d^{2n}f(d^{-2n}xy) = \delta(xy) \\ &\lim_{n\to\infty} d^{2n}f(d^{-n}x)d^{-n}y = \lim_{n\to\infty} d^n f(d^{-n}x) \cdot y = \delta(x)y \\ &\lim_{n\to\infty} d^{2n}d^{-n}xf(d^{-n}y) = \lim_{n\to\infty} x \cdot d^n f(d^{-n}y) = x\delta(y)\end{aligned}$$

for all $x, y \in \mathcal{A}$ whenever the multiplication is defined. If we let $n \to \infty$ in the above inequality then (31) gives $\delta(xy) = \delta(x)y - x\delta(y)$ for all $x, y \in \mathcal{A}$ whenever the multiplication is defined. □

Corollary 5.2. *Let θ, p be nonnegative real numbers with $p > 1$ and $f : \mathcal{A} \to \mathcal{A}$ be a mapping satisfying*

$$\left\|\left(\sum_{i=1}^{d} f(x_{1i}) + \mu f(x_{1d+1}), \cdots, \sum_{i=1}^{d} f(x_{ki}) + \mu f(x_{kd+1})\right)\right\|_k \tag{32}$$
$$\leq \left\|\left(mf\left(\frac{\sum_{i=1}^{d} x_{1i} + \mu x_{1d+1}}{m}\right), \cdots, mf\left(\frac{\sum_{i=1}^{d} x_{ki} + \mu x_{kd+1}}{m}\right)\right)\right\|_k$$
$$+\theta \cdot \sum_{l=1}^{k}\sum_{i=1}^{d+1} \|x_{li}\|^p,$$

for all $\mu \in \mathbb{T}^1$ and all $x_{11}, \cdots, x_{kd+1} \in \mathcal{A}$. If, in addition,

$$\|(f(x_1y_1) - f(x_1)y_1 - x_1 f(y_1), \cdots, f(x_ky_k) - f(x_k)y_k - x_k f(y_k))\|_k \tag{33}$$
$$\leq \theta \cdot \sum_{i=1}^{d+1} (\|x_i\|^{2p} + \|y_i\|^{2p})$$

for all $x_1, ..., x_k, y_1, ..., y_k \in \mathcal{A}$ whenever the multiplication is defined. Then there exists a unique derivation $\delta : \mathcal{A} \to \mathcal{A}$ satisfying

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - \delta(x_1), \cdots, f(x_k) - \delta(x_k))\|_k \leq \sup_{k\in\mathbb{N}} \frac{d^p + d}{d^p - d}\theta \sum_{l=1}^{k} \|x_l\|^p$$

for all $x_1, ..., x_k \in \mathcal{A}$.

Proof. Apply Theorem 5.1, the proof is the same of that of Corollary 4.2. □

Corollary 5.3. *Let θ, p be nonnegative real numbers with $p > 1$ and $f : \mathcal{A} \to \mathcal{A}$ be a mapping satisfying*

$$\left\|\left(\sum_{i=1}^{d} f(x_{1i}) + \mu f(x_{1d+1}), \cdots, \sum_{i=1}^{d} f(x_{ki}) + \mu f(x_{kd+1})\right)\right\|_k \tag{34}$$
$$\leq \left\|\left(mf\left(\frac{\sum_{i=1}^{d} x_{1i} + \mu x_{1d+1}}{m}\right), \cdots, mf\left(\frac{\sum_{i=1}^{d} x_{ki} + \mu x_{kd+1}}{m}\right)\right)\right\|_k$$
$$+\theta \cdot \sum_{l=1}^{k}\sum_{i=1}^{d+1} \|x_{li}\|^p,$$

for all $\mu \in \mathbb{T}^1$ and all $x_{11}, \cdots, x_{kd+1} \in \mathcal{A}$. If, in addition,

$$\|(f(x_1y_1) - f(x_1)y_1 - x_1 f(y_1), \cdots, f(x_ky_k) - f(x_k)y_k - x_k f(y_k))\|_k \tag{35}$$
$$\leq \theta \cdot \sum_{i=1}^{d+1} (\|x_i\|^{p} \cdot \|y_i\|^{p})$$

for all $x_1, ..., x_k, y_1, ..., y_k \in \mathcal{A}$ whenever the multiplication is defined. Then there exists a unique derivation $\delta : \mathcal{A} \to \mathcal{A}$ satisfying

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - \delta(x_1), \cdots, f(x_k) - \delta(x_k))\|_k \leq \sup_{k\in\mathbb{N}} \frac{d^p + d}{d^p - d}\theta \sum_{l=1}^{k} \|x_l\|^p$$

for all $x_1, ..., x_k \in \mathcal{A}$.

Theorem 5.4. *Let $f : \mathcal{A} \to \mathcal{A}$ be a mapping with $f(0) = 0$. If there exists a function $\varphi : \mathcal{X}^{kd+k} \to [0, \infty)$ satisfying* (4) *and* (27). *If, in addition, there exists a function $\psi : \mathcal{A}^{2k} \to [0, \infty)$ satisfying*

$$\begin{aligned}&\|(f(x_1y_1) - f(x_1)f(y_1), \cdots, f(x_ky_k) - f(x_k)f(y_k))\|_k \\ \leq&\psi(x_1, y_1, \cdots, x_k, y_k),\end{aligned} \tag{36}$$

$$\lim_{n\to\infty} \sup_{k\in\mathbb{N}} d^{-2n}\psi(d^n x_1, d^n y_1, \cdots, d^n x_k, d^n y_k) = 0 \tag{37}$$

for all $x_1, ..., x_k, y_1, ..., y_k \in \mathcal{A}$ whenever the multiplication is defined. Then there exists a unique derivation $\delta : \mathcal{A} \to \mathcal{A}$ satisfying

$$\begin{aligned}&\sup_{k\in\mathbb{N}} \|(f(x_1) - \delta(x_1), \cdots, f(x_k) - \delta(x_k))\|_k \\ \leq&\sup_{k\in\mathbb{N}} \widetilde{\varphi}(x_1, x_1, \cdots, -dx_1, \cdots, x_k, x_k, \cdots, -dx_k)\end{aligned} \tag{38}$$

for all $x_1, ..., x_k \in \mathcal{A}$.

Proof. We omit the proof because it is similar to the proof of Theorem 5.1. □

Corollary 5.5. *Let θ, p be nonnegative real numbers with $p < 1$ and $f : \mathcal{A} \to \mathcal{A}$ be a mapping satisfying*

$$\begin{aligned}&\left\|\left(\sum_{i=1}^{d} f(x_{1i}) + \mu f(x_{1d+1}), \cdots, \sum_{i=1}^{d} f(x_{ki}) + \mu f(x_{kd+1})\right)\right\|_k \\ \leq&\left\|\left(mf\left(\frac{\sum_{i=1}^{d} x_{1i} + \mu x_{1d+1}}{m}\right), \cdots, mf\left(\frac{\sum_{i=1}^{d} x_{ki} + \mu x_{kd+1}}{m}\right)\right)\right\|_k \\ &+\theta \cdot \sum_{l=1}^{k}\sum_{i=1}^{d+1} \|x_{li}\|^p,\end{aligned} \tag{39}$$

for all $\mu \in \mathbb{T}^1$ and all $x_{11}, \cdots, x_{kd+1} \in \mathcal{A}$. If, in addition,

$$\begin{aligned}&\|(f(x_1y_1) - f(x_1)y_1 - x_1f(y_1), \cdots, f(x_ky_k) - f(x_k)y_k - x_kf(y_k))\|_k \\ \leq&\theta \cdot \sum_{i=1}^{d+1}(\|x_i\|^{2p} + \|y_i\|^{2p})\end{aligned} \tag{40}$$

for all $x_1, ..., x_k, y_1, ..., y_k \in \mathcal{A}$ whenever the multiplication is defined. Then there exists a unique derivation $\delta : \mathcal{A} \to \mathcal{A}$ satisfying

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - \delta(x_1), \cdots, f(x_k) - \delta(x_k))\|_k \leq \sup_{k\in\mathbb{N}} \frac{d + d^p}{d - d^p}\theta \sum_{l=1}^{k} \|x_l\|^p$$

for all $x_1, ..., x_k \in \mathcal{A}$.

Corollary 5.6. *Let θ, p be nonnegative real numbers with $p < 1$ and $f : \mathcal{A} \to \mathcal{A}$ be a mapping satisfying*

$$\begin{aligned}&\left\|\left(\sum_{i=1}^{d} f(x_{1i}) + \mu f(x_{1d+1}), \cdots, \sum_{i=1}^{d} f(x_{ki}) + \mu f(x_{kd+1})\right)\right\|_k \\ \leq&\left\|\left(mf\left(\frac{\sum_{i=1}^{d} x_{1i} + \mu x_{1d+1}}{m}\right), \cdots, mf\left(\frac{\sum_{i=1}^{d} x_{ki} + \mu x_{kd+1}}{m}\right)\right)\right\|_k \\ &+\theta \cdot \sum_{l=1}^{k}\sum_{i=1}^{d+1} \|x_{li}\|^p,\end{aligned} \tag{41}$$

for all $\mu \in \mathbb{T}^1$ *and all* $x_{11}, \cdots, x_{kd+1} \in \mathcal{A}$. *If, in addition,*

$$\begin{aligned} &\|(f(x_1y_1) - f(x_1)y_1 - x_1f(y_1), \cdots, f(x_ky_k) - f(x_k)y_k - x_kf(y_k))\|_k \\ \leq&\theta \cdot \sum_{i=1}^{d+1}(\|x_i\|^p \cdot \|y_i\|^p) \end{aligned} \tag{42}$$

for all $x_1, ..., x_k, y_1, ..., y_k \in \mathcal{A}$ *whenever the multiplication is defined. Then there exists a unique derivation* $\delta : \mathcal{A} \to \mathcal{A}$ *satisfying*

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - \delta(x_1), \cdots, f(x_k) - \delta(x_k))\|_k \leq \sup_{k\in\mathbb{N}} \frac{d + d^p}{d - d^p}\theta \sum_{l=1}^{k} \|x_l\|^p$$

for all $x_1, ..., x_k \in \mathcal{A}$.

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