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Binding Numbers for all Fractional (a, b, k)-Critical Graphs

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Abstract. Let G be a graph of order n, and let a, b and k nonnegative integers with $2 \leq a \leq b$. A graph G is called all fractional (a, b, k)-critical if after deleting any k vertices of G the remaining graph of G has all fractional $[a, b]$-factors. In this paper, it is proved that G is all fractional (a, b, k)-critical if $n \geq \frac{(a+b-1)(a+b-3)+a}{a} + \frac{ak}{a-1}$ and $bind(G) > \frac{(a+b-1)(n-1)}{an-ak-(a+b)+2}$. Furthermore, it is shown that this result is best possible in some sense.

1. Introduction

We consider finite undirected graphs without loops or multiple edges. Let G be a graph with a vertex set $V(G)$ and an edge set $E(G)$. For $x \in V(G)$, the set of vertices adjacent to x in G is said to be the neighborhood of x, denoted by $N_G(x)$. For any $X \subseteq V(G)$, we write $N_G(X) = \bigcup_{x \in X} N_G(x)$. For two disjoint subsets S and T of $V(G)$, we denote by $e_G(S, T)$ the number of edges with one end in S and the other end in T. Thus $e_G(x, V(G) \setminus \{x\}) = d_G(x)$ is the degree of x and $\delta(G) = \min\{d_G(x) : x \in V(G)\}$ is the minimum degree of G. For $S \subseteq V(G)$, we use $G[S]$ to denote the subgraph of G induced by S, and $G - S$ to denote the subgraph obtained from G by deleting vertices in S together with the edges incident to vertices in S. A vertex set $S \subseteq V(G)$ is called independent if $G[S]$ has no edges. The binding number of G is defined as

$$bind(G) = min\{\frac{|N_G(X)|}{|X|} : \emptyset \neq X \subseteq V(G), N_G(X) \neq V(G)\}.$$

Let g and f be two integer-valued functions defined on $V(G)$ with $0 \leq g(x) \leq f(x)$ for each $x \in V(G)$. A (g, f)-factor of a graph G is defined as a spanning subgraph F of G such that $g(x) \leq d_F(x) \leq f(x)$ for each $x \in V(G)$. We say that G has all (g, f)-factors if G has an r-factor for every $r : V(G) \to Z^+$ such that $g(x) \leq r(x) \leq f(x)$ for each $x \in V(G)$ and $r(V(G))$ is even.

A fractional (g, f)-indicator function is a function h that assigns to each edge of a graph G a real number in the interval [0,1] so that for each vertex x we have $g(x) \leq h(E_x) \leq f(x)$, where $E_x = \{e : e = xy \in E(G)\}$ and $h(E_x) = \sum_{e \in E_x} h(e)$. Let h be a fractional (g, f)-indicator function of a graph G. Set $E_h = \{e : e \in E(G), h(e) > 0\}$. If G_h is a spanning subgraph of G such that $E(G_h) = E_h$, then G_h is called a fractional (g, f)-factor of G. h

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is also called the indicator function of G_h. If $h(e) \in \{0,1\}$ for every e, then G_h is just a (g,f)-factor of G. A fractional (g,f)-factor is a fractional f-factor if $g(x) = f(x)$ for each $x \in V(G)$. A fractional (g,f)-factor is a fractional $[a,b]$-factor if $g(x) = a$ and $f(x) = b$ for each $x \in V(G)$. We say that G has all fractional (g,f)-factors if G has a fractional r-factor for every $r : V(G) \to Z^+$ such that $g(x) \le r(x) \le f(x)$ for each $x \in V(G)$. All fractional (g,f)-factors are said to be all fractional $[a,b]$-factors if $g(x) = a$ and $f(x) = b$ for each $x \in V(G)$. A graph G is all fractional (a,b,k)-critical if after deleting any k vertices of G the remaining graph of G has all fractional $[a,b]$-factors.

Many authors have investigated factors [1,2,8] and fractional factors [3,4,7,10] of graphs. The following results on all (g,f)-factors, all fractional $[a,b]$-factors and all fractional (a,b,k)-critical graphs are known.

Theorem 1.1. (*Niessen* [6]). *G has all (g,f)-factors if and only if*

$$g(S) + \sum_{x\in T} d_{G-S}(x) - f(T) - h_G(S,T,g,f) = \begin{cases} -1, & if\ f \neq g \\ 0, & if\ f = g \end{cases}$$

for all disjoint subsets $S,T \subseteq V(G)$, where $h_G(S,T,g,f)$ denotes the number of components C of $G-(S\cup T)$ such that there exists a vertex $v \in V(C)$ with $g(v) < f(v)$ or $e_G(V(C),T) + f(V(C)) \equiv 1$ (mod 2).

Theorem 1.2. (*Lu* [5]). *Let $a \le b$ be two positive integers. Let G be a graph with order $n \ge \frac{2(a+b)(a+b-1)}{a}$ and minimum degree $\delta(G) \ge \frac{(a+b-1)^2+4b}{4a}$. If $|N_G(x)| \cup |N_G(y)| \ge \frac{bn}{a+b}$ for any two nonadjacent vertices x and y in G, then G has all fractional $[a,b]$-factors.*

Theorem 1.3. (*Zhou* [9]). *Let a, b and k be nonnegative integers with $1 \le a \le b$, and let G be a graph of order n with $n \ge a+k+1$. Then G is all fractional (a,b,k)-critical if and only if for any $S \subseteq V(G)$ with $|S| \ge k$*

$$a|S| + \sum_{x\in T} d_{G-S}(x) - b|T| \ge ak,$$

where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) < b\}$.

Using Theorem 3, Zhou [9] obtained a neighborhood condition for graphs to be all fractional (a,b,k)-critical graphs.

Theorem 1.4. (*Zhou* [9]). *Let a,b,k,r be nonnegative integers with $1 \le a \le b$ and $r \ge 2$. Let G be a graph of order n with $n > \frac{(a+b)(r(a+b)-2)+ak}{a}$. If $\delta(G) \ge \frac{(r-1)b^2}{a} + k$, and $|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_r)| \ge \frac{bn+ak}{a+b}$ for any independent subset $\{x_1, x_2, \cdots, x_r\}$ in G, then G is all fractional (a,b,k)-critical.*

2. Main Result and Its Proof

In this paper, we proceed to study the existence of all fractional (a,b,k)-critical graphs and obtain a binding number condition for graphs to be all fractional (a,b,k)-critical. Our main result is the following theorem.

Theorem 2.1. *Let a, b and k be nonnegative integers with $2 \le a \le b$, and let G be a graph of order n with $n \ge \frac{(a+b-1)(a+b-3)+a}{a} + \frac{ak}{a-1}$. If $bind(G) > \frac{(a+b-1)(n-1)}{an-ak-(a+b)+2}$, then G is all fractional (a,b,k)-critical.*

Proof. Suppose that G satisfies the assumption of Theorem 2.1, but it is not all fractional (a,b,k)-critical. Then by Theorem 1.3, there exists some subset S of $V(G)$ with $|S| \ge k$ such that

$$a|S| + \sum_{x\in T} d_{G-S}(x) - b|T| \le ak - 1, \tag{1}$$

where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) < b\}$. Clearly, $T \neq \emptyset$ by (1). Define

$$h = \min\{d_{G-S}(x) : x \in T\}.$$

In terms of the definition of T, we obtain $0 \leq h \leq b-1$.

Now in order to prove the correctness of Theorem 2.1, we shall deduce some contradictions according to the following two cases.

Case 1. $h = 0$.

Let $X = \{x : x \in T, d_{G-S}(x) = 0\}$. Obviously, $X \neq \emptyset$ and $N_G(V(G)\setminus S) \cap X = \emptyset$, and so $|N_G(V(G)\setminus S)| \leq n-|X|$. According to the definition of $bind(G)$ and the condition of Theorem 2.1, we have

$$\frac{(a+b-1)(n-1)}{an-ak-(a+b)+2} < bind(G) \leq \frac{|N_G(V(G)\setminus S)|}{|V(G)\setminus S|} \leq \frac{n-|X|}{n-|S|},$$

which implies

$$\begin{aligned}
(a+b-1)(n-1)|S| &> (a+b-1)(n-1)n-(an-ak-(a+b)+2)n+(an-ak-(a+b)+2)|X| \\
&= (b-1)(n-1)n+(b-2)n+akn+(an-ak-(a+b)+2)|X| \\
&\geq (b-1)(n-1)n+akn+(an-ak-(a+b)+2)|X| \\
&= (b-1)(n-1)n+akn+[(n-1)+(a-1)n-ak-(a+b)+3]|X| \\
&\geq (b-1)(n-1)n+akn+[(n-1)+(a-1)\cdot\left(\frac{(a+b-1)(a+b-3)+a}{a}+\frac{ak}{a-1}\right) \\
&\quad -ak-(a+b)+3]|X| \\
&> (b-1)(n-1)n+akn+[(n-1)+(a-1)(a+b-3)-(a+b)+3]|X| \\
&\geq (b-1)(n-1)n+ak(n-1)+(n-1)|X|.
\end{aligned}$$

Thus, we obtain

$$|S| > \frac{(b-1)n+ak+|X|}{a+b-1}. \tag{2}$$

Using (1), (2) and $|S|+|T| \leq n$, we have

$$\begin{aligned}
ak-1 &\geq a|S|+\sum_{x\in T} d_{G-S}(x)-b|T| \\
&\geq a|S|+|T|-|X|-b|T| \\
&= a|S|-(b-1)|T|-|X| \\
&\geq a|S|-(b-1)(n-|S|)-|X| \\
&= (a+b-1)|S|-(b-1)n-|X| \\
&> (a+b-1)\cdot\frac{(b-1)n+ak+|X|}{a+b-1}-(b-1)n-|X| \\
&= ak,
\end{aligned}$$

which is a contradiction.

Case 2. $1 \leq h \leq b-1$.

Claim 1. $\delta(G) > \dfrac{(b-1)n+ak+a+b-2}{a+b-1}$.

Let v be a vertex of G with degree $\delta(G)$. Set $Y = V(G)\setminus N_G(v)$. Obviously, $Y \neq \emptyset$ and $v \notin N_G(Y)$. In terms of the definition of $bind(G)$, we have

$$\frac{(a+b-1)(n-1)}{an-ak-(a+b)+2} < bind(G) \leq \frac{|N_G(Y)|}{|Y|} \leq \frac{n-1}{n-\delta(G)},$$

which implies

$$\delta(G) > \frac{(b-1)n+ak+a+b-2}{a+b-1}.$$

This completes the proof of Claim 1.

Note that $\delta(G) \le |S| + h$. Then using Claim 1, we have

$$|S| \ge \delta(G) - h > \frac{(b-1)n + ak + a + b - 2}{a+b-1} - h. \tag{3}$$

Claim 2. $|T| \le \dfrac{an - ak - (a+b) + 1}{a+b-1} + h.$

Assume that $|T| \ge \dfrac{an - ak - (a+b) + 2}{a+b-1} + h$. We choose $u \in T$ such that $d_{G-S}(u) = h$ and let $Y = T \setminus N_{G-S}(u)$. Note that $|N_{G-S}(u)| = d_{G-S}(u) = h$. Thus, we obtain

$$\begin{aligned} |Y| &\ge |T| - d_{G-S}(u) \\ &= |T| - h \ge \frac{an - ak - (a+b) + 2}{a+b-1} > 0 \end{aligned}$$

and

$$N_G(Y) \ne V(G).$$

Combining these with the definition of $bind(G)$, we have

$$bind(G) \le \frac{|N_G(Y)|}{|Y|} \le \frac{n-1}{|T| - h} \le \frac{(a+b-1)(n-1)}{an - ak - (a+b) + 2},$$

which contradicts that the condition of Theorem 2.1. The proof of Claim 2 is completed.

According to (1), (3) and Claim 2, we obtain

$$\begin{aligned} ak - 1 &\ge a|S| + \sum_{x \in T} d_{G-S}(x) - b|T| \ge a|S| - (b-h)|T| \\ &> a \cdot \left(\frac{(b-1)n + ak + a + b - 2}{a+b-1} - h \right) - (b-h) \cdot \left(\frac{an - ak - (a+b) + 1}{a+b-1} + h \right) \\ &= \frac{(h-1)an + (a+b-h)ak - a}{a+b-1} - (h-1)(a+b-h), \end{aligned}$$

that is,

$$ak - 1 > \frac{(h-1)an + (a+b-h)ak - a}{a+b-1} - (h-1)(a+b-h). \tag{4}$$

Let $f(h) = \dfrac{(h-1)an + (a+b-h)ak - a}{a+b-1} - (h-1)(a+b-h)$. If $h = 1$, then by (4) we have $ak - 1 > f(h) = f(1) = ak - \dfrac{a}{a+b-1} > ak - 1$, which is a contradiction. In the following, we assume that $2 \le h \le b - 1$.

In view of $2 \le h \le b - 1$ and $n \ge \dfrac{(a+b-1)(a+b-3) + a}{a} + \dfrac{ak}{a-1}$, we have

$$\begin{aligned} f'(h) &= \frac{an - ak}{a+b-1} - (a+b-h) + (h-1) \\ &= 2h + \frac{an - ak}{a+b-1} - (a+b+1) \\ &\ge 4 + \frac{(a+b-1)(a+b-3) + a}{a+b-1} - (a+b+1) \\ &= \frac{a}{a+b-1} > 0. \end{aligned}$$

Thus, we obtain

$$f(h) \ge f(2). \tag{5}$$

From (4), (5) and $n \geq \frac{(a+b-1)(a+b-3)+a}{a} + \frac{ak}{a-1}$, we obtain

$$\begin{aligned} ak-1 &> f(h) \geq f(2) = \frac{an+(a+b-2)ak-a}{a+b-1} - (a+b-2) \\ &\geq \frac{(a+b-1)(a+b-3)+a+ak+(a+b-2)ak-a}{a+b-1} - (a+b-2) \\ &= ak-1, \end{aligned}$$

which is a contradiction. This completes the proof of Theorem 2.1.

Remark. In Theorem 2.1, the lower bound on the condition $bind(G)$ is best possible in the sense since we cannot replace $bind(G) > \frac{(a+b-1)(n-1)}{an-ak-(a+b)+2}$ with $bind(G) \geq \frac{(a+b-1)(n-1)}{an-ak-(a+b)+2}$, which is shown in the following example.

Let $b \geq a \geq 2, k \geq 0$ be three integers such that $a+b+k-1$ is even and $\frac{a(b-1)+b(b-2)+(a+b-1)k}{a}$ is a positive integer. Set $l = \frac{a+b+k-1}{2}$ and $m = \frac{a(b-1)+b(b-2)+(a+b-1)k}{a}$. We construct a graph $G = K_m \vee K_{2l}$. Then $n = m+2l = \frac{a(b-1)+b(b-2)+(a+b-1)k}{a} + a+b+k-1$. Let $X = V(lK_2)$, for any $x \in X$, then $|N_G(X \setminus x)| = n-1$. According to the definition of $bind(G)$, we obtain

$$bind(G) = \frac{|N_G(X \setminus x)|}{|X \setminus x|} = \frac{n-1}{2l-1} = \frac{n-1}{a+b+k-2} = \frac{(a+b-1)(n-1)}{an-ak-(a+b)+2}.$$

Let $S = V(K_m)$, $T = V(lK_2)$. Then $|S| = m \geq k$, $|T| = 2l$ and $\sum_{x \in T} d_{G-S}(x) = 2l$. Thus, we have

$$\begin{aligned} & a|S| + \sum_{x \in T} d_{G-S}(x) - b|T| = am - 2l(b-1) \\ = \ & a(b-1)+b(b-2)+(a+b-1)k - (b-1)(a+b+k-1) \\ = \ & ak-1 < ak. \end{aligned}$$

In terms of Theorem 1.3, G is not all fractional (a,b,k)-critical.

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