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Stability of Solitary Waves for a Generalized Higher-Order Shallow Water Equation

Nurhan Dündar[a], Necat Polat[a]

[a] *Department of Mathematics, Dicle University, 21280, Diyarbakir, Turkey*

Abstract. In this work, we consider solitary wave solutions of a generalized higher-order shallow water equation. We investigate the existence and stability of solitary waves of the equation.

1. Introduction

Nonlinear evolution equations arise not only from many fields of mathematics, but also from other branches of science. Therefore, nonlinear evolution equations have attracted a lot of interest of many mathematicians and scientists in nonlinear sciences. Navier-Stokes equations, Cahn-Hilliard equations, Boussinesq-type equations and nonlinear Schrödinger equations are examples of nonlinear evolution equations. These equations have been studied by many authors (see [6, 13–15, 17] and the references therein).

In this paper, we consider the following Cauchy problem for a nonlinear evolution equation

$$\begin{cases} u_t - \alpha^2 u_{xxt} + (g(u))_x + \gamma \left(u - \alpha^2 u_{xx}\right)_{xxx} = \alpha^2 \left(\frac{h'(u)}{2} u_x^2 + h(u) u_{xx}\right)_x, & t > 0, x \in \mathbb{R}, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1}$$

where $g(u), h(u) : \mathbb{R} \to \mathbb{R}$ are given function, α and γ are constants.

Eq. (1) describes the generalized integrable shallow water equation with strong dispersive term. The strong dispersive term $\gamma \left(u - \alpha^2 u_{xx}\right)_{xxx}$ corresponds to the Lagrangian averaged Navier-Stokes alpha equations for turbulence and can provide analytical control over the solutions [18].

For $g(u) = 2\omega u + \frac{3}{2}u^2$ and $h(u) = u$, Eq. (1) becomes the the following equation

$$u_t - \alpha^2 u_{xxt} + 2\omega u_x + 3uu_x + \gamma \left(u - \alpha^2 u_{xx}\right)_{xxx} = \alpha^2 \left(2u_x u_{xx} + uu_{xxx}\right). \tag{2}$$

In [18], Tian et al. studied the well-posedness of Eq. (2) by applying Kato's semigroup approach. Moreover, they got the precise blow-up scenario and gave an explosion criterion of strong solutions of Eq. (2) with rather general initial data.

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Email addresses: nurhandundar@hotmail.com (Nurhan Dündar), npolat@dicle.edu.tr (Necat Polat)

If $\alpha = 1$, $\gamma = 1$ and $\omega = 0$ Eq. (2) becomes the following fifth-order shallow water equation

$$u_t - u_{xxt} + u_{xxx} + 3uu_x - u_{xxxxx} = 2u_x u_{xx} + uu_{xxx}, \tag{3}$$

which is a higher-order modification of the following Camassa-Holm equation

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}.$$

The well-posedness of the Cauchy problem of Eq. (3) in Sobolev spaces has been studied by several authours (see [7, 8, 19] and the references therein).

In Eq. (2) if the strong dispersive term $\gamma\left(u - \alpha^2 u_{xx}\right)_{xxx}$ is rewritten as the weak dispersive term γu_{xxx}, Eq. (2) becomes the following Dullin-Gottwald-Holm equation:

$$u_t - \alpha^2 u_{xxt} + 2\omega u_x + 3uu_x + \gamma u_{xxx} = \alpha^2 \left(2u_x u_{xx} + uu_{xxx}\right), \quad t > 0,\ x \in \mathbb{R}, \tag{4}$$

which was derived by Dullin, Gottwald and Holm using asymptotic expansions directly in the Hamiltonian for Euler's equations in the shallow water regime in [2].

Recently, the well-posedness problem for the following generalization of the DGH equation:

$$\begin{cases} u_t - \alpha^2 u_{xxt} + h(u)_x + \gamma u_{xxx} = \alpha^2 \left(\frac{g'(u)}{2} u_x^2 + g(u) u_{xx}\right)_x, & t > 0,\ x \in \mathbb{R}, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{5}$$

has been studied in [12]. In [3], authors studied the blow-up of solutions for the (5). Also, they proved the stability of solitary wave solutions with the help of the orbital stability theory [5].

It seems that the (1) is a better generalization of shallow water equation. In [4], Dündar and Polat studied the well-posedness by applying Kato's semigroup approach.

The aim of this paper is to investigate the existence and stability of solitary wave solutions of (1) when $g(u) = 2\omega u + \frac{p+2}{2} u^{p+1}$, $h(u) = u^p$, where $p > 0$ is an integer and $\omega > 0$. Without loss of generality we assume that $\alpha = \gamma = 1$. In this instance, (1) becomes the following problem:

$$\begin{cases} u_t - u_{xxt} + 2\omega u_x + \left(\frac{p+2}{2} u^{p+1}\right)_x + u_{xxx} - u_{xxxxx} = \left(\frac{p}{2} u^{p-1} u_x^2 + u^p u_{xx}\right)_x, & t > 0,\ x \in \mathbb{R}, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{6}$$

Our paper is organized as follows: In Section 2, we give two conservation laws for (1). They are important for to prove stability of solitary waves. In Section 3, we show the existence of solitary waves of (6). For this, we will use the method of concentrated compactness developed by Lions [11] to solve a constrained minimization problem . In Section 4, we prove the stability of solitary wave solutions of (6). We show that there is a function $d(c)$ of the wave speed such that the solitary waves are stable whenever $d(c)$ is convex. The proof uses a compactness argument similar to those in [1] and [16] (see also [9, 10]).

Notations: Throughout this paper, we use the following notations.

Let $L^p = L^p(\mathbb{R})$ be the Lebesgue measurable space with

$$\|f\|_{L^p} = \left(\int_{\mathbb{R}} |f(x)|^p \, dx\right)^{\frac{1}{p}}.$$

The space L^∞ is defined as the space of all measurable functions f on $\mathbb{R}$ such that

$$\|f\|_{L^\infty} = ess \sup_{x \in \mathbb{R}} |g(x)|.$$

Let $H^s = H^s(\mathbb{R})$ be Sobolev space with

$$\|f\|_{H^s} = \left(\int_{\mathbb{R}} \left(1 + |\xi|^2\right)^s \left|\widehat{f}(\xi)\right|^2 d\xi\right)^{\frac{1}{2}},$$

where $\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f(x)\, dx$.

2. Conservation Laws

First, we give the following local well-posedness theorem:

Theorem 2.1. *[4] Assume that* $g, h \in C^{[s]+1}(\mathbb{R})$ *, and* $h(0) = g(0) = 0$*. Given* $u_0 \in H^s$*,* $s > \frac{3}{2}$*, there exists a maximal* $T = T(u_0) > 0$*, and a unique strong solution u to (1) such that*

$$u = u(u_0, .) \in C([0, T); H^s) \cap C^1\left([0, T); H^{s-1}\right).$$

Moreover, the solution depends continuously on the initial data, i.e., the mapping

$$u_0 \to u(u_0, .) : H^s \to C([0, T); H^s) \cap C^1\left([0, T); H^{s-1}\right)$$

is continuous.

For a solution, we want all derivatives involved in the equation to exist and satisfy the equation with initial/boundary conditions at each point of the domain. Such a solution is called a classical solution. Certain specific partial differential equations such as the wave equation can be solved in the classical sense; but if we wish to study conservation laws and recover the underlying physics, we must allow for solutions which are not continously differentiable or even not continous. As in the case of conservation laws, some equations can be described in weaker forms and may be satisfied by functions that are not sufficiently smooth. Moreover, a solution that starts smooth may eventually become singular as in the case of shock waves. To overcome this difficulty, we allow for generalized or weak solutions. In the classical (smooth) category, there is no ambiguity as to what it means for a function u to solve an equation; but once one is in a low regularity class, there are several competing notions of solution, in particular the notions of a strong solution and a weak solution. To oversimplify a bit, both strong and weak solutions solve (1) in a distributional sense, but strong solutions are also continuous in time.

Applying the operator $\left(I - \partial_x^2\right)^{-1}$ to both sides of (1) we obtain

$$\begin{cases} u_t + bh(u)u_x + au_{xxx} = -\partial_x\left(I - \partial_x^2\right)^{-1}\left(g(u) + \frac{b}{2}h'(u)u_x^2\right) + \left(I - \partial_x^2\right)^{-1}(bh(u)u_x) \\ u(x, 0) = u_0(x), \end{cases}$$

where $t > 0, x \in \mathbb{R}$ and $a = \frac{\gamma}{\alpha^3}$ and $b = \frac{1}{\alpha}$. Hence u is a solution of (1) in the sense of distribution. In particular, if $s \geq 5$, u is also a classical solution.

Now, we give two useful conservation laws. We note that Eq. (1) can be written as the following Hamiltonian form:

$$u_t + JF'(u) = 0,$$

where $J = \left(I - \partial_x^2\right)^{-1}\partial_x$ is a skew-symmetric operator and

$$F(u) = \frac{1}{2}\int_{\mathbb{R}}\left(2G(u) + \alpha^2 h(u)u_x^2 - \gamma u_x^2 - \gamma\alpha^2 u_{xx}^2\right)dx$$

is the Hamiltonian, where $G'(s) = g(s)$. We note that the functional $F(u)$ is formally conserved. Moreover, the other conserved quantity is

$$E(u) = \frac{1}{2}\int_{\mathbb{R}}\left(u^2 + \alpha^2 u_x^2\right)dx.$$

Both $E(u)$ and $F(u)$ are very important to the stability analysis.

Theorem 2.2. *Let u be a solution of (1). Then the functionals*

$$E(u) = \frac{1}{2}\int_{\mathbb{R}} \left(u^2 + \alpha^2 u_x^2\right) dx, \quad F(u) = \frac{1}{2}\int_{\mathbb{R}} \left(2G(u) + \alpha^2 h(u) u_x^2 - \gamma u_x^2 - \gamma\alpha^2 u_{xx}^2\right) dx$$

are constant with respect to t, where $G'(s) = g(s)$.

Proof. Multiplying both sides of Eq. (1) by u and integrating by parts with respect to x, we obtain

$$\frac{dE(u)}{dt} = \int_{\mathbb{R}} u\left(u_t - \alpha^2 u_{xxt}\right) dx = 0.$$

Set $v(x,t) = \int_{-\infty}^{x} u_t(y,t)\, dy$. Consider the equalities

$$\frac{d}{dt}\int_{\mathbb{R}} G(u)\, dx = \int_{\mathbb{R}} g(u) u_t dx = \int_{\mathbb{R}} g(u) v_x dx = -\int_{\mathbb{R}} (g(u))_x v dx,$$

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} \alpha^2 \frac{1}{2} h(u) u_x^2 dx &= \alpha^2 \int_{\mathbb{R}} \frac{1}{2} h'(u) u_t u_x^2 dx + \alpha^2 \int_{\mathbb{R}} h(u) u_x u_{xt} dx \\ &= \alpha^2 \int_{\mathbb{R}} \frac{1}{2} h'(u) u_x^2 v_x dx + \alpha^2 \int_{\mathbb{R}} h(u) u_x v_{xx} dx \\ &= -\alpha^2 \int_{\mathbb{R}} \frac{1}{2} h'(u) u_x^2 v_x dx - \alpha^2 \int_{\mathbb{R}} h(u) u_{xx} v_x dx \\ &= \alpha^2 \int_{\mathbb{R}} \left(\frac{1}{2} h'(u) u_x^2 + h(u) u_{xx}\right)_x v dx,\end{aligned}$$

$$\frac{d}{dt}\int_{\mathbb{R}} \frac{1}{2}\gamma u_x^2 dx = \int_{\mathbb{R}} \gamma u_x u_{xt} dx = \int_{\mathbb{R}} \gamma u_x v_{xx} dx = \int_{\mathbb{R}} \gamma u_{xxx} v dx$$

and

$$\frac{d}{dt}\int_{\mathbb{R}} \frac{1}{2}\gamma\alpha^2 u_{xx}^2 dx = \int_{\mathbb{R}} \gamma\alpha^2 u_{xx} u_{xxt} dx = \int_{\mathbb{R}} \gamma\alpha^2 u_{xx} v_{xxx} dx = -\int_{\mathbb{R}} \gamma\alpha^2 u_{xxxxx} v dx.$$

Combining the above equalities, we have

$$\begin{aligned}\frac{dF(u)}{dt} &= \frac{d}{dt}\int_{\mathbb{R}} \frac{1}{2}\left(2G(u) + \alpha^2 h(u) u_x^2 - \gamma u_x^2 - \gamma\alpha^2 u_{xx}^2\right) dx \\ &= -\int_{\mathbb{R}} \left((g(u))_x - \alpha^2 \left(\frac{1}{2} h'(u) u_x^2 + h(u) u_{xx}\right)_x + \gamma u_{xxx} - \gamma\alpha^2 u_{xxxxx}\right) v dx.\end{aligned}$$

Multiplying both sides Eq. (1) by v and integrating we get

$$\int_{\mathbb{R}} v\left(u_t - \alpha^2 u_{xxt}\right) dx = -\int_{\mathbb{R}} \left((g(u))_x - \alpha^2 \left(\frac{1}{2} h'(u) u_x^2 + h(u) u_{xx}\right)_x + \gamma u_{xxx} - \gamma\alpha^2 u_{xxxxx}\right) v dx.$$

$$\int_{\mathbb{R}} \left(v v_x - \alpha^2 v v_{xxx}\right) dx = \frac{dF(u)}{dt} = 0.$$

□

3. Existence of Solitary Waves

In this section, we investigate the existence of solitary waves of the Eq. (6).

By a solitary wave we mean a solution of (6) of the form $u(x,t) = \varphi(x-ct)$, where $c > 2\omega$ represents the speed of the wave. Inserting this into (6) and integrating once, taking integral constant zero, we see that φ must satisfiy

$$c\varphi - c\varphi'' - 2\omega\varphi - \frac{p+2}{2}\varphi^{p+1} - \varphi'' + \varphi'''' + \left(\frac{p}{2}\varphi^{p-1}\varphi'^2 + \varphi^p\varphi''\right) = 0. \tag{7}$$

We obtain solutions to the solitary wave equation (7) by solving a constrained minimization problem. Define the functionals

$$I(u) = I(u;\omega,c) = \int_{\mathbb{R}} \left((c-2\omega)u^2 + (c+1)u_x^2 + u_{xx}^2\right)dx$$

and

$$K(u) = \int_{\mathbb{R}} \left(u^{p+2} + u^p u_x^2\right)dx.$$

For $\lambda > 0$, we consider the following constrained minimization problem on H^2,

$$M_\lambda = \inf\left\{I(u) : u \in H^2,\ K(u) = \lambda\right\}. \tag{8}$$

If $\psi \in H^2$ achieves the minimum of problem (8), for some $\lambda > 0$, then by the Lagrange multiplier principle there exists $\vartheta \in \mathbb{R}$ such that ψ is a weak solution of the Euler-Lagrange equation $\delta I(\psi) = \vartheta\delta K(\psi)$, where $\delta I(\psi)$ and $\delta K(\psi)$ are the Fréchet derivatives of I and K at ψ. Namely, the function ψ is a weak solution of the Euler-Lagrange equation

$$(2c-4\omega)\psi - (2c+2)\psi'' + 2\psi'''' = \vartheta\left[(p+2)\psi^{p+1} - \left(p\psi^{p-1}(\psi')^2 + 2\psi^p\psi''\right)\right]$$

with a Lagrange multiplier ϑ. Hence $\varphi = \vartheta^{\frac{1}{p}}\psi$ is a solution of the solitary wave equation (7). Such solutions are called as ground states, and we denote the set of all ground states by N_c. By homogeneity of I and K, ground states also achieve minimum

$$m = m(\omega,c) = \inf\left\{\frac{I(u)}{K(u)^{\frac{2}{p+2}}} : u \in H^2,\ u \neq 0\right\},$$

and it follows that

$$M_\lambda = \lambda^{\frac{2}{p+2}} m. \tag{9}$$

Multiplying the solitary wave equation (7) by φ and integrating the resulting equation gives $I(\varphi) = \frac{p+2}{2}K(\varphi)$. Thus the set of ground states may be characterized by

$$N_c = \left\{\varphi \in H^2 : \frac{2}{p+2}I(\varphi) = K(\varphi) = \left(\frac{2}{p+2}m\right)^{\frac{p+2}{p}}\right\}. \tag{10}$$

We are now going to prove that N_c is nonempty.

We say that ψ_k is a minimizing sequence if for some $\lambda > 0$, $\lim_{k\to\infty} I(\psi_k) = M_\lambda$ and $\lim_{k\to\infty} K(\psi_k) = \lambda$.

Theorem 3.1. *Let $\{\psi_k\}$ be a minimizing sequence for some $\lambda > 0$. If $c > 2\omega$, then there exist a subsequence (renamed ψ_k) and scalars $y_k \in \mathbb{R}$ and $\psi \in H^2$ such that $\psi_k(.-y_k) \to \psi$ in H^2. The function ψ achieves the minimum $I(\psi) = M_\lambda$ subject to the constraint $K(\psi) = \lambda$.*

Proof. We prove the above theorem by applying the concentration compactness lemma of Lions [11]. From (9) we see that the subadditivity condition holds

$$M_\lambda < M_{\lambda_1} + M_{\lambda-\lambda_1}, \quad \text{for } \lambda_1 \in (0, \lambda).$$

Since $c > 2\omega$, the functional I satisfies the coercivity condition

$$I(u) \geq (c - 2\omega) \|u\|_{H^2}^2.$$

It is also clear that $(c+1)\|u\|_{H^2}^2 \geq I(u)$. That is to say, for $c > 2\omega$ the functional $I(u)$ is equivalent to $\|u\|_{H^2}^2$:

$$(c-2\omega)\|u\|_{H^2}^2 \leq I(u) \leq (c+1)\|u\|_{H^2}^2.$$

Let $\{\psi_k\}$ be a minimizing sequence. Then by coercivity of I, the squence $\{\psi_k\}$ is bounded in H^2, so we define

$$\rho_k = \left|\partial_x^2 \psi_k\right|^2 + \left|\partial_x \psi_k\right|^2 + \left|\psi_k\right|^2,$$

then after extracting a subsequence, we may assume $\lim\limits_{k\to\infty} \int_{\mathbb{R}} \rho_k dx = L > 0$. By normalizing we may assume further that $\left\|\rho_k\right\|_{L^1} = L$ for all k. By the concentration compactness lemma, a further subsequence ρ_k satisfies one of the following three conditions.
(i) Compactness: There exists $y_k \in \mathbb{R}$ such that for any $\varepsilon > 0$ there exists $R(\varepsilon)$ such that for all k

$$\int_{|x-y_k|\leq R} \rho_k dx \geq \int_{\mathbb{R}} \rho_k dx - \varepsilon. \tag{11}$$

(ii) Vanishing: For every $R > 0$,

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}} \int_{|x-y|\leq R} \rho_k dx = 0. \tag{12}$$

(iii) Dichotomy: There exists some $l \in (0, L)$ such that for any $\varepsilon > 0$ there exist $R > 0$ and $R_k \to \infty$, $y_k \in \mathbb{R}$ and k_0 such that

$$\left|\int_{|x-y_k|\leq R} \rho_k dx - l\right| < \varepsilon \quad \text{and} \quad \left|\int_{R<|x-y_k|<R_k} \rho_k dx\right| < \varepsilon \tag{13}$$

for $k \geq k_0$.
Our purpose is to show that both vanishing and dichotomy ruled out, and therefore ρ_k is compact. First suppose that (ii) holds. By the Sobolev inequality we have

$$\int_{|x-y|\leq 1} \left|\psi_k\right|^{p+2} dx \leq C\left(\int_{|x-y|\leq 1} \rho_k dx\right)^{\frac{p+2}{2}},$$

$$\int_{|x-y|\leq 1} \left|\psi_k^p (\partial_x \psi_k)^2\right| dx \leq C\left(\int_{|x-y|\leq 1} \left|\psi_k\right|^{p+2} dx\right)^{\frac{p}{p+2}} . \left(\int_{|x-y|\leq 1} \left|\partial_x \psi_k\right|^{p+2} dx\right)^{\frac{2}{p+2}}$$

for all $y \in \mathbb{R}$ and $\{\psi_k\}$ is bounded in H^2. We obtain that for any $\psi_k \in H^2$

$$\int_{|x-y|\leq 1} \left|\psi_k\right|^{p+2} dx \leq C\left(\sup_{y\in\mathbb{R}} \int_{|x-y|\leq 1} \rho_k dx\right)^{\frac{p}{2}} . \left\|\psi_k\right\|_{H^2}^2,$$

$$\int_{|x-y|\le 1}\left|\psi_k^p\left(\partial_x\psi_k\right)^2\right|dx \le C\left\|\psi_k\right\|_{H^2}^2\left[C\left(\sup_{y\in\mathbb{R}}\int_{|x-y|\le 1}\rho_k dx\right)^{\frac{p}{2}}.\left\|\psi_k\right\|_{H^2}^2\right]^{\frac{p}{p+2}}.$$

Hence from (12) (with $R = 1$), we arrive at the contradiction that $\lim_{k\to\infty} K(\psi_k) = 0$. Hence vanishing cannot occur.

Next suppose (iii) holds. Then we may define cutoff functions δ_1 and δ_2 with support on $|x| \le 2$ and $|x| \ge \frac{1}{2}$, respectively and with $\delta_1(x) = 1$ for $|x| \le 1$ and $\delta_2(x) = 1$ for $|x| \ge 1$, in such a way that the functions

$$\psi_{k,1}(x) = \delta_1\left(\frac{|x-y_k|}{R}\right)\psi_k(x)$$

and

$$\psi_{k,2}(x) = \delta_2\left(\frac{|x-y_k|}{R_k}\right)\psi_k(x)$$

satisfy

$$\begin{aligned} I(\psi_k) &= I(\psi_{k,1}) + I(\psi_{k,2}) + O(\varepsilon),\\ K(\psi_k) &= K(\psi_{k,1}) + K(\psi_{k,2}) + O(\varepsilon) \end{aligned} \tag{14}$$

for $k \ge k_0$. Since $\{\psi_k\}$ is bounded in H^2 it follows that $\psi_{k,1}$ and $\psi_{k,2}$ are also bounded in H^2 independently of ε. Consequently $K(\psi_{k,1})$ and $K(\psi_{k,2})$ are bounded and we can pass to subsequences to define

$$\lambda_1(\varepsilon) = \lim_{k\to\infty} K(\psi_{k,1}) \quad \text{and} \quad \lambda_2(\varepsilon) = \lim_{k\to\infty} K(\psi_{k,2}).$$

As $\lambda_1(\varepsilon)$ and $\lambda_2(\varepsilon)$ are bounded independently of ε, we can choose a sequence $\varepsilon_j \to 0$ such that both limits

$$\lambda_1 = \lim_{j\to\infty}\lambda_1\left(\varepsilon_j\right) \quad \text{and} \quad \lambda_2 = \lim_{j\to\infty}\lambda_2\left(\varepsilon_j\right)$$

exist. Certainly, $\lambda_1 + \lambda_2 = \lambda$, and there are three cases to consider now.

If $\lambda_1 \in (0,\lambda)$ then by (14) and $M_\lambda = \lambda^{\frac{2}{p+2}} m$

$$\begin{aligned} I(\psi_k) &= I(\psi_{k,1}) + I(\psi_{k,2}) + O(\varepsilon) \ge M_{K(\psi_{k,1})} + M_{K(\psi_{k,2})} + O\left(\varepsilon_j\right)\\ &= \left(K(\psi_{k,1})^{2/(p+2)} + K(\psi_{k,2})^{2/(p+2)}\right)m + O\left(\varepsilon_j\right). \end{aligned}$$

We first let $k \to \infty$ to obtain

$$M_\lambda \ge \left[\lambda_1^{2/(p+2)}\left(\varepsilon_j\right) + \lambda_2^{2/(p+2)}\left(\varepsilon_j\right)\right]m + O\left(\varepsilon_j\right).$$

Then letting $j \to \infty$, we arrive at

$$M_\lambda \ge M_{\lambda_1} + M_{\lambda-\lambda_1},$$

a contradiction.

If $\lambda_1 = 0$ (or equivalently, when $\lambda_1 = \lambda$), we have

$$\begin{aligned} I(\psi_{k,1}) &\ge (c-2\omega)\int_{\mathbb{R}}\left(\left|\psi_{k,1}\right|^2 + \left|\partial_x\psi_{k,1}\right|^2 + \left|\partial_x^2\psi_{k,1}\right|^2\right)dx\\ &= (c-2\omega)\left(\int_{|x-y_k|\le 2R}\left(\left|\psi_k\right|^2 + \left|\partial_x\psi_k\right|^2 + \left|\partial_x^2\psi_k\right|^2\right)dx + O\left(\varepsilon_j\right)\right)\\ &= (c-2\omega)\left(l + O\left(\varepsilon_j\right)\right). \end{aligned}$$

Thus

$$I(\psi_k) = I(\psi_{k,1}) + I(\psi_{k,2}) + O\left(\varepsilon_j\right)$$
$$\geq (c-2\omega)\left(l + O\left(\varepsilon_j\right)\right) + K(\psi_{k,2})^{2/(p+2} m + O\left(\varepsilon_j\right).$$

Letting k and $j \to \infty$ respectively, we obtain

$$M_\lambda \geq (c-2\omega) l + K(\psi_{k,2})^{2/(p+2)} m = (c-2\omega) l + M_\lambda > M_\lambda,$$

which is a contradiction.
Finally, $\lambda_1 > \lambda$ (or equivalently, when $\lambda_1 < 0$), we use the positivity of I to estimate

$$I(\psi_k) \geq I(\psi_{k,1}) + O\left(\varepsilon_j\right) \geq K(\psi_{k,1})^{2/(p+2} m + O\left(\varepsilon_j\right).$$

Letting k and $j \to \infty$ respectively, we obtain

$$M_\lambda \geq M_{\lambda_1} > M_\lambda,$$

which is a contradiction.
So there exist $y_k \in \mathbb{R}$ such that $\rho_k(. - y_k)$ is compact. Now set $\varphi_k = \psi_k(. - y_k)$. Since φ_k is bounded in H^2, a subsequence φ_k converges to some $\psi \in H^2$, and by the weak lower semicontinuity of I over H^2, we have

$$I(\psi) \leq \lim_{k\to\infty} I(\varphi_k) = M_\lambda.$$

Also, weak convergence in H^2, compactness of ρ_k, and Sobolev inequality imply strong convergence of φ_k to ψ in L^{p+2}. Therefore

$$K(\psi) = \lim_{k\to\infty} K(\varphi_k) = \lambda,$$

so $I(\psi) \geq M_\lambda$. Together with the inequality above, this implies $I(\psi) = M_\lambda$, so ψ is minimizer of I subject to the constraint $K(\varphi) = \lambda$. Finally, since I is equivalent to the norm on H^2, $\varphi_k \to \psi$, and $I(\varphi_k) \to I(\psi)$, it follows that φ_k converges to ψ in H^2. □

We now show that this weak solution is in fact a classical solution of (7).

Lemma 3.2. *Suppose $\varphi \in H^2$ is a weak solution of (7). Then φ is a classical solution and $\varphi \in C^5$.*

Proof. Eq. (7) can be written as

$$c\varphi - c\varphi'' - 2\omega\varphi - \varphi'' + \varphi'''' = f(\varphi, \varphi', \varphi''),$$

where $f(\varphi, \varphi', \varphi'') = \frac{p+2}{2}\varphi^{p+1} - \left(\frac{p}{2}\varphi^{p-1}\varphi'^2 + \varphi^p\varphi''\right)$. Since $\varphi \in H^2$, both φ and φ' are in $L^\infty \cap L^2$ and thus $f(\varphi, \varphi', \varphi'') \in L^2$. Since φ is a weak solution of (7) this implies $\varphi \in H^4$ and therefore $f(\varphi, \varphi', \varphi'') \in C^1$ by Sobolev's lemma. Thus $\varphi \in C^5$. □

4. Stability

We define the function d of wavespeed $c > 2\omega$ as

$$d(c) = cE(\varphi) - F(\varphi),$$

where φ is any ground state solution of (7), i.e $\varphi \in N_c$. Then φ satisfies

$$cE'(\varphi) - F'(\varphi) = 0,$$

where E' and F' are the Fréchet derivatives of E and F, respectively. The functionals E and F are related to the functionals I and K used to obtain the solitary waves by the simple formula

$$cE(u) - F(u) = \frac{1}{2}(I(u) - K(u)). \tag{15}$$

By using $I(\varphi) = \frac{p+2}{2}K(\varphi)$ and (10), we obtain

$$d(c) = \frac{p}{2(p+2)}I(\varphi) = \frac{p}{4}K(\varphi) = \frac{p}{4}\left(\frac{2}{p+2}m\right)^{\frac{p+2}{p}}. \tag{16}$$

In this section we show that stability of the set of ground states is determined by the convexity of the function $d(c)$. We will use the following definition of stability throughout.

Definition 4.1. *A set $S \subset H^2$ is stable with respect to (6) if given $\varepsilon > 0$ there exists some $\delta > 0$ such that if u_0 satisfies*

$$\inf_{\varphi \in S} \left\| u_0(.) - \varphi(.) \right\|_{H^2} < \delta$$

then the solution $u(.,t)$ of (6) with initial data $u(.,0) = u_0(.)$ exists for all $t > 0$ and satisfies

$$\sup_{0 \le t < \infty} \inf_{\varphi \in S} \left\| u(.,t) - \varphi(. - y) \right\|_{H^2} < \varepsilon.$$

Otherwise, we say that S is unstable with respect to (6).

We state the basic properties of the function d.

Therefore, d is well defined, and we may deduce its properties by examining the function m.

Lemma 4.2. *Let $c > 2\omega$, then $m = m(\omega, c)$ is monotonically increasing in c.*

Proof. We assume that φ_{c_1}, $\varphi_{c_2}$are solutions of equation (7) corresponding to $c = c_1, c = c_2$, respectively. Without loss of generality, let $c_1 < c_2$, then we have

$$\begin{aligned}
m(\omega, c_1) &\le \frac{I(\varphi_{c_2}; \omega, c_1)}{K(\varphi_{c_2})^{\frac{2}{p+2}}} = \frac{\int_{\mathbb{R}} \left[(c_1 - 2\omega)\varphi_{c_2}^2 + (c_1 + 1)\left(\varphi_{c_2}'\right)^2 + \left(\varphi_{c_2}''\right)^2 \right] dx}{K(\varphi_{c_2})^{\frac{2}{p+2}}} \\
&= \frac{\int_{\mathbb{R}} \left[(c_2 - 2\omega)\varphi_{c_2}^2 + (c_2 + 1)\left(\varphi_{c_2}'\right)^2 + \left(\varphi_{c_2}''\right)^2 \right] dx}{K(\varphi_{c_2})^{\frac{2}{p+2}}} \\
&+ \frac{-c_2 \int_{\mathbb{R}} \left[\varphi_{c_2}^2 + \left(\varphi_{c_2}'\right)^2 \right] dx + c_1 \int_{\mathbb{R}} \left[\varphi_{c_2}^2 + \left(\varphi_{c_2}'\right)^2 \right] dx}{K(\varphi_{c_2})^{\frac{2}{p+2}}} \\
&= m(\omega, c_2) + (c_1 - c_2) \frac{\int_{\mathbb{R}} \left[\left(\varphi_{c_2}'\right)^2 + \varphi_{c_2}^2 \right] dx}{K(\varphi_{c_2})^{\frac{2}{p+2}}} \\
&\le m(\omega, c_2).
\end{aligned}$$

This shows that m is monotonically increasing in c, so that by (16) d must be monotonically increasing as well. □

Let

$$U_\varepsilon = \left\{ u \in H^2 : \inf_{\varphi \in N_c} \left\| u - \varphi \right\|_{H^2} < \varepsilon \right\}$$

denote the $\varepsilon-$neighborhood of the set of ground states N_c. It follows from (16) and $d(c)$ is monotonically increasing in c that

$$c(u) = d^{-1}\left(\frac{p}{4} K(u)\right). \tag{17}$$

The following lemma is helpful in order to prove the stability of solitary waves.

Lemma 4.3. *If $d''(c) > 0$, then there exists $\varepsilon > 0$ such that for any $u \in U_\varepsilon$ and $\varphi \in N_c$ we have*

$$c(u)\left[E(u) - E(\varphi)\right] - \left[F(u) - F(\varphi)\right] \geq \frac{1}{4} d''(c) \left| c(u) - c \right|^2 .$$

Proof. By using $d'(c) = E(\varphi)$ and Taylor's formula, we have the expansion

$$d(\widetilde{c}) = d(c) + E(\varphi)(\widetilde{c} - c) + \frac{1}{2} d''(c)(\widetilde{c} - c)^2 + o\left(\left|\widetilde{c} - c\right|^2\right)$$

for $\widetilde{c}$ near c. By choosing ε sufficiently small the continuity of $c(u)$ implies that

$$\begin{aligned} d(c(u)) &\geq d(c) + E(\varphi)(c(u) - c) + \frac{1}{4} d''(c)(c(u) - c)^2 \\ &= c(u) E(\varphi) - F(\varphi) + \frac{1}{4} d''(c)(c(u) - c)^2 . \end{aligned}$$

It follows from (16) and (17) that $K\left(\varphi_{c(u)}\right) = \frac{4}{p} d(c(u)) = K(u)$ and $\varphi_{c(u)}$ minimizes $I(.;\omega, c(u))$ subject to this constraint, we then have

$$\begin{aligned} c(u) E(u) - F(u) &= \frac{1}{2}\left(I(u;\omega, c(u)) - K(u)\right) \\ &\geq \frac{1}{2}\left(I\left(\varphi_{c(u)};\omega, c(u)\right) - K\left(\varphi_{c(u)}\right)\right) = d(c(u)) \end{aligned}$$

and

$$c(u) E(u) - F(u) \geq c(u) E(\varphi) - F(\varphi) + \frac{1}{4} d''(c)(c(u) - c)^2 .$$

□

Theorem 4.4. *Let $c > 2\omega$. If $d''(c) > 0$ then the set of ground states N_c is stable.*

Proof. Suppose N_c is unstable and choose initial data v_k such that

$$\inf_{\varphi \in N_c} \left\| v_k - \varphi \right\|_{H^2} < \frac{1}{k},$$

and let $u_k(.,t)$ be the solution of (6) with $u_k(.,0) = v_k$. Then, by Theorem 2.1 u_k is continuous in t, and there exist some $\delta > 0$ and times t_k such that

$$\inf_{\varphi \in N_c} \left\| u_k(., t_k) - \varphi \right\|_{H^2} = \delta. \tag{18}$$

Since E and F are invariants of (6) and since N_c is bounded, we can find $\varphi_k \in N_c$ such that

$$\left|E\left(u_k\left(.,t_k\right)\right)-E\left(\varphi_k\right)\right|=\left|E\left(u_k\left(.,0\right)\right)-E\left(\varphi_k\right)\right|\rightarrow 0, \tag{19}$$

$$\left|F\left(u_k\left(.,t_k\right)\right)-F\left(\varphi_k\right)\right|=\left|F\left(u_k\left(.,0\right)\right)-F\left(\varphi_k\right)\right|\rightarrow 0 \tag{20}$$

as $k\rightarrow\infty$. By Lemma 4.3, if δ is sufficiently small, we have

$$\begin{aligned}&c\left(u_k\left(.,t_k\right)\right)\left[E\left(u_k\left(.,t_k\right)\right)-E\left(\varphi_k\right)\right]-\left[F\left(u_k\left(.,t_k\right)\right)-F\left(\varphi_k\right)\right]\\&\geq\frac{1}{4}d''\left(c\right)\left|c\left(u_k\left(.,t_k\right)\right)-c\right|^2,\end{aligned}$$

and therefore, by (19) and (20), $c\left(u_k\left(.,t_k\right)\right)\rightarrow c$ as $k\rightarrow\infty$.
The continuity of d implies that

$$\lim_{k\rightarrow\infty}K\left(u_k\left(.,t_k\right)\right)=\lim_{k\rightarrow\infty}\frac{4}{p}d\left(c\left(u_k\left(.,t_k\right)\right)\right)=\frac{4}{p}d\left(c\right). \tag{21}$$

Using (15), (19), (20) and the fact that $d\left(c\right)=cE\left(\varphi_k\right)-F\left(\varphi_k\right)$, we have

$$\lim_{k\rightarrow\infty}I\left(u_k\left(.,t_k\right)\right)=\frac{2\left(p+2\right)}{p}d\left(c\right). \tag{22}$$

Hence $u_k\left(.,t_k\right)$ is a minimizing sequence and therefore has a subsequence which converges in H^2 to some $\varphi\in N_c$. This contradicts with (18). The proof is completed. □

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