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Compact Operators on some Generalized Mixed Norm Spaces

A. Alotaibi[a], E. Malkowsky[b,c], H. Nergiz[c]

[a]*Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia*
[b]*Department of Mathematics, University of Giessen, Arndtstrasse 2, D–35392 Giessen, Germany*
[c]*Department of Mathematics, Faculty of Arts and Sciences, Fatih University, 34500 Büyükçekmece, Istanbul, Turkey*

Abstract. We establish identities or estimates for the Hausdorff measure of noncompactness of operators from some generalized mixed norm spaces into any of the spaces c_0, c, ℓ_1, and $[\ell_1, \ell_\infty]^{<m(\mu)>}$. Furthermore we give necessary and sufficient conditions for the operators in these cases to be compact. Our results are complementary to those in [1, 3, 13].

1. Introduction and Notations

We use the following standard notations.

Let X and Y be normed spaces. Then $S_X = \{x \in X : \|x\| = 1\}$ and $\overline{B}_X = \{x \in X : \|x\| \leq 1\}$ denote the unit sphere and closed unit ball in X, and $\mathcal{B}(X, Y)$ is the set of all bounded linear operators $L : X \to Y$ which is a Banach space with the operator norm given by $\|L\| = \sup\{\|L(x)\| : x \in S_X\}$, whenever Y is a Banach space.

A sequence $(b_n)_{n=1}^\infty$ in a linear metric space X is called a Schauder basis if, for every $x \in X$, there exists a unique sequence $(\lambda_n)_{n=1}^\infty$ of scalars such that $x = \sum_{n=1}^\infty \lambda_n b_n$.

Let ω denote the set of all complex sequences $x = (x_k)_{k=1}^\infty$. We write ℓ_∞, c, c_0 and ϕ for the set of all bounded, convergent, null and finite sequences, respectively, and $\ell_p = \{x \in \omega : \sum_{k=1}^\infty |x_k|^p < \infty\}$ for $1 \leq p < \infty$. Let e and $e^{(n)}$ ($n \in \mathbb{N}$) be the sequences with $e_k = 1$ for all k, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \neq n$.

A BK space is a Banach sequence space with the property that convergence implies coordinatewise convergence; a BK space $X \supset \phi$ is said to have AK if $x^{[m]} = \sum_{k=1}^m x_k e^{(k)} \to x$ ($m \to \infty$) for every sequence $x = (x_k)_{k=1}^\infty \in X$. It is well known that the sets ℓ_∞, c and c_0 are BK spaces with $\|x\|_\infty = \sup_k |x_k|$, ℓ_p ($1 \leq p < \infty$) is a BK space with $\|x\|_p = (\sum_{k=1}^\infty |x_k|^p)^{1/p}$, c_0 and ℓ_p ($1 < p < \infty$) have AK, every sequence $x = (x_k)_{k=1}^\infty \in c$ has a unique representation $x = \xi \cdot e + \sum_{k=1}^\infty (x_k - \xi) e^{(k)}$, where $\xi = \lim_{k\to\infty} x_k$.

Let $A = (a_{nk})_{n,k=1}^\infty$ be an infinite matrix of complex numbers, X and Y be subsets of ω and $x \in \omega$. We write $A_n = (a_{nk})_{k=1}^n$ for the sequence in the nth row of A, $A_n x = \sum_{k=1}^\infty a_{nk} x_k$ and $Ax = (A_n x)_{n=1}^\infty$ (provided all the series $A_n x$ converge). The set $X^\beta = \{a \in \omega : \sum_{k=1}^\infty a_k x_k$ converges for all $x \in X\}$ is called the β–dual of X. Finally (X, Y) is the class of all matrices A for which $A_n \in X^\beta$ for all n and $Ax \in Y$ for all $x \in X$.

If X is a normed space, then we write $\|a\|_X^* = \sup\{|\sum_{k=1}^\infty a_k x_k| : x \in S_X\}$ provided the expression on the right–hand side exists and is finite which is the case whenever X is a BK space and $a \in X^\beta$ ([17, Theorem 7.2.9]).

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Email addresses: mathker11@hotmail.com (A. Alotaibi), eberhard.malkowsky@math.uni-giessen.de (E. Malkowsky), h.sakirkaya@hotmail.com (H. Nergiz)

Throughout let $(k(\nu))_{\nu=0}^{\infty}$ and $(m(\mu))_{\mu=0}^{\infty}$ be sequences of integers with $1 = k(0) < k(1) < \dots$ and $1 = m(0) < m(1) < \dots$, $I_\nu = \{k \in \mathbb{N} : k(\nu) \le k \le k(\nu+1)-1\}$, $M_\mu = \{m \in \mathbb{N} : m(\mu) \le m \le m(\mu+1)-1\}$ and $x^{<\nu>} = \sum_{k\in I_\nu} x_k e^{(k)}$ be the ν–block of the sequence $x = (x_k)_{k=1}^{\infty}$ for $\nu = 0, 1, \dots$. We write $\sum_\nu$ and $\max_\nu$ for the sum and maximum taken over all $k \in I^{<\nu>}$, $\sum_\mu$ and $\max_\mu$ for the sum and maximum taken over all $m \in M_\mu$, and consider the sets ([8, Example 2.1 (a)]) for $1 \le p < \infty$ and $1 \le r < \infty$

$$[\ell_r, \ell_p]^{<k(\nu)>} = \left\{x \in \omega : \left(\|x^{<\nu>}\|_p\right)_{\nu=0}^{\infty} \in \ell_r\right\},$$

$$[\ell_r, \ell_\infty]^{<k(\nu)>} = \left\{x \in \omega : (\|x^{<\nu>}\|_\infty)_{\nu=0}^{\infty} \in \ell_r\right\},$$

$$[c_0, \ell_p]^{<k(\nu)>} = \left\{x \in \omega : \lim_{\nu\to\infty} \|x^{<\nu>}\|_p = 0\right\} \quad \text{and}$$

$$[\ell_\infty, \ell_p]^{<k(\nu)>} = \left\{x \in \omega : \sup_{\nu} \|x^{<\nu>}\|_p < \infty\right\}.$$

These sets generalize the mixed norm spaces $\ell(r, p)$ introduced and studied by Hedlund [6] and Kellogg [10]; they are also closely related to the sets w_∞^p and w_0^p of sequences that are strongly bounded and strongly convergent to 0 with index p by the Cesàro method of order 1, introduced and studied by Maddox [11], the Cesàro sequence spaces $ces(p)$ studied by Jagers [7], and the sets of strong weighted means studied by Grosse–Erdmann [4, 5].

In this paper, we establish identities or estimates for the Hausdorff measure of noncompactness of operators $L \in \mathcal{B}([\ell_r, \ell_p]^{<k(\nu)>}, Y)$ for $1 \le r < \infty$ and $1 < p \le \infty$, or $1 < r < \infty$ and $p = 1$, when Y any of the spaces c_0, c, ℓ_1 and $[\ell_1, \ell_\infty]^{<m(\mu)>}$. Furthermore, we give necessary and sufficient conditions for the operators in these cases to be compact. Our results are complementary to those in [1, 3, 13].

2. Basic Results

Here we collect and prove the important basic results.

If $1 \le p \le \infty$ then, as usual, its conjugate number is $q = \infty$ if $p = 1$, $q = p/(p-1)$ if $1 < p < \infty$, and $q = 1$ if $p = \infty$. Throughout, let $1 \le r < \infty$ and $1 < p \le \infty$, or $1 < r < \infty$ and $p = 1$, and s and q denote the conjugate numbers of r and p, respectively. We refer to Remark 3.4 for the other cases.

The following results are known ([8, Example 3.4 (a)]). If $1 \le p, r < \infty$ then the sets $[\ell_r, \ell_p]^{<k(\nu)>}$ and $[c_0, \ell_p]^{<k(\nu)>}$ are BK spaces with AK with

$$\|x\|_{[\ell_r,\ell_p]^{<k(\nu)>}} = \left\|\left(\|x^{<k(\nu)>}\|_p\right)_{\nu=0}^{\infty}\right\|_r \text{ and } \|x\|_{[\ell_\infty,\ell_p]^{<k(\nu)>}} = \left\|\left(\|x^{<k(\nu)>}\|_p\right)_{\nu=0}^{\infty}\right\|_\infty;$$

also $[\ell_\infty, \ell_p]^{<k(\nu)>}$ is a BK space with $\|\cdot\|_{[\ell_\infty,\ell_p]^{<k(\nu)>}}$; moreover $[c_0, \ell_p]^{<k(\nu)>}$ is a closed subspace of $[\ell_\infty, \ell_p]^{<k(\nu)>}$; finally, the sets $[\ell_r, \ell_\infty]^{<k(\nu)>}$ are BK spaces with AK with

$$\|x\|_{[\ell_r,\ell_\infty]^{<k(\nu)>}} = \left\|\left(\|x^{<k(\nu)>}\|_\infty\right)_{\nu=0}^{\infty}\right\|_r.$$

We need the following result on the equality of the norms $\|\cdot\|_X^*$ and $\|\cdot\|_{X^\beta}$.

Lemma 2.1. *Let $a \in ([\ell_r, \ell_p]^{<k(\nu)>})^\beta$. Then we have*

$$\|a\|^*_{[\ell_r,\ell_p]^{<k(\nu)>}} = \|a\|_{[\ell_s,\ell_q]^{<k(\nu)>}}. \tag{1}$$

Proof. We write $Z = [\ell_r, \ell_p]^{<k(\nu)>}$, for short, and obtain $Z^\beta = [\ell_s, \ell_q]^{<k(\nu)>}$ by [8, Example 4.3 (a)], and so $\|a\|_{[\ell_s,\ell_q]^{<k(\nu)>}} < \infty$. Also, since the sets $[\ell_r, \ell_p]^{<k(\nu)>}$ are BK spaces, it follows that $\|a\|^*_{[\ell_r,\ell_p]^{<k(\nu)>}} < \infty$ by [17, Theorem 7.2.9].

First we show

$$\|a\|_Z^* \le \|a\|_{Z^\beta}. \tag{2}$$

Let $a \in Z^\beta$ and $x \in Z$ be given. If $1 < r < \infty$ and $1 < p \leq \infty$ then we obtain, applying Hölder's inequality twice,

$$\begin{aligned}\left|\sum_{k=1}^{\infty} a_k x_k\right| &\leq \sum_{\nu=0}^{\infty} \textstyle\sum_\nu |a_k x_k| \leq \displaystyle\sum_{\nu=0}^{\infty} \|a^{<\nu>}\|_q \cdot \|x^{<\nu>}\|_p \\ &\leq \left(\sum_{\nu=0}^{\infty} \|a^{<\nu>}\|_q^s\right)^{1/s} \cdot \left(\sum_{\nu=0}^{\infty} \|x^{<\nu>}\|_p^r\right)^{1/r} = \|a\|_{Z^\beta} \cdot \|x\|_Z.\end{aligned}$$

If $r = 1$ and $1 < p \leq \infty$ then we obtain by Hölder's inequality

$$\begin{aligned}\left|\sum_{k=1}^{\infty} a_k x_k\right| &\leq \sum_{\nu=0}^{\infty} \|a^{<\nu>}\|_q \cdot \|x^{<\nu>}\|_p \leq \sup_\nu \|a^{<\nu>}\|_q \cdot \left(\sum_{\nu=0}^{\infty} \|x^{<\nu>}\|_p\right) \\ &= \|a\|_{Z^\beta} \cdot \|x\|_Z,\end{aligned}$$

Finally, if $1 < r < \infty$ and $p = 1$ then we obtain by Hölder's inequality

$$\begin{aligned}\left|\sum_{k=1}^{\infty} a_k x_k\right| &\leq \sum_{\nu=0}^{\infty} \|a^{<\nu>}\|_\infty \cdot \|x^{<\nu>}\|_1 \leq \left(\sum_{\nu=0}^{\infty} \|a^{<\nu>}\|_\infty^s\right)^{1/s} \cdot \left(\sum_{\nu=0}^{\infty} \|x^{<\nu>}\|_1^r\right)^{1/r} \\ &= \|a\|_{Z^\beta} \cdot \|x\|_Z.\end{aligned}$$

Therefore (2) holds in each case.
Now we show

$$\|a\|_Z^* \geq \|a\|_{Z^\beta}. \tag{3}$$

Since the identity in (1) is trivial for $a = 0$, we assume $a \neq 0$.
If $1 < r < \infty$ and $1 < p \leq \infty$, then we write $A = (\sum_{\nu=0}^{\infty} \|a^{<\nu>}\|_q^s)^{1/r}$, and define the sequence x by

$$x_k = \begin{cases} |a_k|^{q/p} \cdot \|a^{<\nu>}\|_q^{-q/p+s-1} \cdot A^{-1} \cdot \operatorname{sgn}(a_k) & (k \in I_\nu) \\ \text{for those } \nu \text{ for which } \|a^{<\nu>}\|_q \neq 0 & \\ 0 & (\text{otherwise}) \end{cases} \quad (\nu = 0, 1, \dots).$$

Now let $r = 1$ and $1 < p \leq \infty$ and $\mu \in \mathbb{N}$ be given and so large that $a^{<k(\mu)>} \neq 0$. Furthermore, let $\mu(\nu)$ denote the smallest integer for which $\max_{0 \leq \nu \leq \mu} \|a^{<\nu>}\|_q = \|a^{<\nu(\mu)>}\|_q$. We define the sequence $x^{(\mu)}$ by

$$x_k^{(\mu)} = \begin{cases} |a_k|^{q/p} \cdot \|a^{<\nu>}\|_q^{-q/p} \cdot \operatorname{sgn}(a_k) & (k \in I_{\nu(\mu)}) \\ 0 & (\text{otherwise}) \end{cases}.$$

Finally let $1 < r < \infty$ and $p = 1$. We write $A = (\sum_{\nu=0}^{\infty} \|a^{<\nu>}\|_\infty^s)^{1/r}$, and define the sequence x by

$$x_k = \begin{cases} \|a^{<\nu>}\|_\infty^{s-1} \cdot A^{-1} \cdot \operatorname{sgn}(a_{k(\nu)}) & (k = k(\nu)) \\ \text{where } k(\nu) \text{ is the smallest integer } k \in I_\nu & \\ \text{for which } |a_{k(\nu)}| = \max_\nu |a_k| & \\ 0 & (\text{otherwise}) \end{cases} \quad (\nu = 0, 1, \dots).$$

Then it follows that $\|x\|_Z, \|x^{(\mu)}\|_Z \leq 1$ for all μ, and

$$\left|\sum_{k=1}^{\infty} a_k x_k\right| = \left(\sum_{\nu=0}^{\infty} \|a^{<\nu>}\|_q^s\right)^{1/s} \text{ and } \left|\sum_{k=1}^{\infty} a_k x_k^{(\mu)}\right| = \|a^{<\nu(\mu)>}\|_q \text{ for all } \mu.$$

Thus (3) holds in each case. □

We need also need the following known result.

Proposition 2.2. ([17, Theorem 4.2.8], [9, Theorem 1.9]) *Let X and Y be BK spaces and X have AK. Then we have $\mathcal{B}(X,Y) = (X,Y)$, that is, every $A \in (X,Y)$ defines an operator $L \in \mathcal{B}(X,Y)$, where*

$$L(x) = Ax \text{ for all } x \in X, \tag{4}$$

and conversely every operator $L \in \mathcal{B}(X,Y)$ is given by and infinite matrix $A \in (X,Y)$ such that (4) holds.

Now we establish an identity and some inequalities for the norms of bounded linear operators. We write $\sup_N$ for the supremum taken over all finite subsets of $\mathbb{N}$ or $\mathbb{N}_0$, and $\mathcal{T}$ for the set of all sequences $(t_\mu)_{\mu=0}^\infty$ of integers such that for each μ there is one and only one $t_\mu \in M_\mu$.

Theorem 2.1. *(a) If $L \in \mathcal{B}([\ell_r,\ell_p]^{<k(\nu)>}, Y)$ where $Y = \ell_\infty, c, c_0$, then*

$$\|L\| = \sup_n \|A_n\|_{[\ell_s,\ell_q]^{<k(\nu)>}}. \tag{5}$$

(b) If $L \in \mathcal{B}([\ell_r,\ell_p]^{<k(\nu)>}, \ell_1)$, then

$$\sup_N \left\|\sum_{n\in N} A_n\right\|_{[\ell_s,\ell_q]^{<k(\nu)>}} \le \|L\| \le 4\cdot \sup_N \left\|\sum_{n\in N} A_n\right\|_{[\ell_s,\ell_q]^{<k(\nu)>}}. \tag{6}$$

(c) If $L \in \mathcal{B}([\ell_r,\ell_p]^{<k(\nu)>}, [\ell_1,\ell_\infty]^{<m(\mu)>})$, then

$$\sup_N \left[\sup_{t\in\mathcal{T}} \left\|\sum_{\mu\in N} A_{t_\mu}\right\|_{[\ell_s,\ell_q]^{<k(\nu)>}}\right] \le \|L\| \le 4\cdot \sup_N \left[\sup_{t\in\mathcal{T}} \left\|\sum_{\mu\in N} A_{t_\mu}\right\|_{[\ell_s,\ell_q]^{<k(\nu)>}}\right]. \tag{7}$$

Proof. Since all the spaces are *BK* spaces and each space $[\ell_r,\ell_p]^{<k(\nu)>}$ has *AK*, in each case, the operator L is given by an infinite matrix as in (4).
(a) If $L \in \mathcal{B}([\ell_r,\ell_p]^{<k(\nu)>}, \ell_\infty)$, then we have [14, Theorem 1.23] and (1)

$$\|L\| = \sup_n \|A_n\|^*_{[\ell_r,\ell_p]^{<k(\nu)>}} = \sup_n \|A_n\|_{[\ell_s,\ell_q]^{<k(\nu)>}}.$$

Since trivially $\mathcal{B}([\ell_r,\ell_p]^{<k(\nu)>}, Y) \subset \mathcal{B}([\ell_r,\ell_p]^{<k(\nu)>}, \ell_\infty)$ for $Y = c, c_0$ and the *BK* norms on ℓ_∞, c and c_0 are the same, (5) also holds for $Y = c, c_0$.
(b) If $L \in \mathcal{B}([\ell_r,\ell_p]^{<k(\nu)>}, \ell_1)$, then we have by [12, Theorem 1] and (1)

$$\sup_N \left\|\sum_{n\in N} A_n\right\|_{[\ell_s,\ell_q]^{<k(\nu)>}} = \sup_N \left\|\sum_{n\in N} A_n\right\|^*_{[\ell_r,\ell_p]^{<k(\nu)>}} \le \|L\| \le$$
$$4\cdot \sup_N \left\|\sum_{n\in N} A_n\right\|^*_{[\ell_r,\ell_p]^{<k(\nu)>}} = 4\cdot \sup_N \left\|\sum_{n\in N} A_n\right\|_{[\ell_s,\ell_q]^{<k(\nu)>}}.$$

(c) Let $L \in \mathcal{B}([\ell_r,\ell_p]^{<k(\nu)>}, [\ell_1,\ell_\infty]^{<m(\mu)>})$, and $x \in S_{[\ell_r,\ell_p]^{<k(\nu)>}}$ be given. Also, for each $\mu = 0, 1, \dots$, let $m_\mu \in M_\mu$ be such that $|A_{m_\mu}x| = \max_{m\in M_\mu} |A_m x|$. Then we have by a well–known inequality ([15])

$$\|L(x)\|_{[\ell_1,\ell_\infty]^{<m(\mu)>}} = \sum_{\mu=0}^\infty \left\|(Ax)^{<m(\mu)>}\right\|_\infty = \sum_{\mu=0}^\infty \left|A_{m_\mu}x\right| \le 4\cdot \sup_N \left|\sum_{\mu\in N} A_{m_\mu}x\right| \le$$

$$4 \cdot \sup_N \left\| \sum_{\mu \in N} A_{m_\mu} \right\|^*_{[\ell_r,\ell_p]^{<k(\nu)>}} \leq 4 \cdot \sup_N \left(\sup_{t \in \mathcal{T}} \left\| \sum_{\mu \in N} A_{t_\mu} \right\|^*_{[\ell_r,\ell_p]^{<k(\nu)>}} \right),$$

hence

$$\|L\| \leq 4 \cdot \sup_N \left(\sup_{t \in \mathcal{T}} \left\| \sum_{\mu \in N} A_{t_\mu} \right\|^*_{[\ell_r,\ell_p]^{<k(\nu)>}} \right). \tag{8}$$

We also have $\sum_{\mu=0}^{\infty} |A_{m_\mu} x| \geq |\sum_{\mu \in N} A_{m_\mu} x|$ for all finite subsets N of $\mathbb{N}_0$, for all $\mu \in M_\mu$ and for all $x \in S_{[\ell_r,\ell_p]^{<k(\nu)>}}$, hence $\|L\| \geq \| \sum_{\mu \in N} A_{t_\mu} \|^*_{[\ell_r,\ell_p]^{<k(\nu)>}}$ for all $t \in \mathcal{T}$, and so

$$\sup_N \left(\sup_{t \in \mathcal{T}} \left\| \sum_{\mu \in N} A_{t_\mu} \right\|^*_{[\ell_r,\ell_p]^{<k(\nu)>}} \right) \leq \|L\|. \tag{9}$$

Now (7) follows from (9), (8) and (1). □

3. Compact Operators

We recall that a linear operator L between Banach spaces X and Y is said to be compact, if its domain is all of X and, for every bounded sequence (x_n) in X, the sequence $(L(x_n))$ has a convergent subsequence in Y.

Let (X, d) be a metric space. Then we write $\mathcal{M}_X$ for the class of all bounded subsets of X, and $B_r(x_0) = \{x \in X : d(x, x_0) < r\}$ for the open ball of radius $r > 0$ with its centre in $x_0 \in X$. We recall that the Hausdorff measure of noncompactness is the map $\chi : \mathcal{M}_X \to [0, \infty)$ with

$$\chi(Q) = \inf \left\{ \varepsilon > 0 : Q \subset \bigcup_{k=1}^{n} B_{r_k}(x_k),\ x_k \in X,\ r_k < \varepsilon\ (k = 1, 2, \ldots n; n \in \mathbb{N}) \right\}.$$

If X is a linear metric space with a Schauder basis (b_k) then we define the operator $\mathcal{R}_n$ for each $n \in \mathbb{N}$ by $\mathcal{R}_n(x) = \sum_{k=n+1}^{\infty} \lambda_k b_k$ for all $x = \sum_{k=1}^{\infty} \lambda_k b_k \in X$.

We say that a norm $\| \cdot \|$ on a sequence space X is monotonous if, for all $x, \tilde{x} \in X$, $|x_k| \leq |\tilde{x}_k|$ for all k implies $\|x\| \leq \|\tilde{x}\|$.

We need the following result for the Hausdorff measure of noncompactness in certain BK spaces.

Proposition 3.1. *(a) If X is a BK space with monotonous norm and AK then we have*

$$\chi(Q) = \lim_{n \to \infty} \left(\sup_{x \in Q} \|\mathcal{R}_n(x)\| \right) \text{ for all } x \in \mathcal{M}_X. \tag{10}$$

(b) If $X = c$ then we have

$$\frac{1}{2} \cdot \lim_{n \to \infty} \left(\sup_{x \in Q} \|\mathcal{R}_n(x)\| \right) \leq \chi(Q) \leq \lim_{n \to \infty} \left(\sup_{x \in Q} \|\mathcal{R}_n(x)\| \right) \text{ for all } x \in \mathcal{M}_c. \tag{11}$$

Proof. If X is any BK space with a Schauder basis then (11) holds with lim sup instead of lim and 1/2 replaced by $1/a$, where $a = \limsup_{n \to \infty} \|\mathcal{R}_n\|$, the basis constant, by the Goldenštein–Gohberg–Markus theorem (e.g [16, Theorem 4.2]). The limits in (10) and (11) exist by [3, Lemma 1.10].
(a) If X has AK then $a = 1$ by [3, Lemma 1.10 (a)].
(b) If $X = c$, then $a = 2$ by [3, Lemma 1.10 (b)]. □

Now we recall the definition of the Hausdorff measure of noncompactness of operators between Banach spaces ([14, Definition 2.25]).
Let χ_1 and χ_2 be Hausdorff measures of noncompactness on the Banach spaces X and Y. An operator $L : X \to Y$ is said to be (χ_1, χ_2)–bounded if $L(Q) \in \mathcal{M}_Y$ for all $Q \in \mathcal{M}_X$ and there exists a non–negative real number c such that

$$\chi_2(L(Q)) \leq c \cdot \chi_1(Q) \text{ for all } Q \in \mathcal{M}_X. \tag{12}$$

If an operator L is (χ_1, χ_2)–bounded, then the number

$$\|L\|_\chi = \inf\{c \geq 0 : (12) \text{ holds }\}$$

is called the Hausdorff measure of noncompactness of L.
It is well known ([14, Theorem 2.25]) that if X and Y are Banach spaces and $L \in \mathcal{B}(X, Y)$ then

$$\|L\|_\chi = \chi(L(S_X)) \tag{13}$$

and

$$L \text{ is compact if and only if } \|L\|_\chi = 0. \tag{i}$$

Now we establish an identity and some inequalities for the Hausdorff measures of noncompactness of certain operators.

Theorem 3.1. *Let X and Y be BK spaces, X have AK, $L \in \mathcal{B}(X, Y)$, and $A \in (X, Y)$ be the matrix that represents L as in (4). Then we have*

$$\|L\|_\chi = \lim_{r\to\infty}\left(\sup_{n\geq r}\|A_n\|_X^*\right) \text{ if } Y = c_0; \tag{14}$$

$$\frac{1}{2}\cdot\lim_{r\to\infty}\left(\sup_{n\geq r}\left\|A_n - (\alpha_k)_{k=1}^\infty\right\|_X^*\right) \leq \|L\|_\chi \leq \lim_{r\to\infty}\left(\sup_{n\geq r}\left\|A_n - (\alpha_k)_{k=1}^\infty\right\|_X^*\right)$$
$$\text{if } Y = c, \text{ where } \alpha_k = \lim_{k\to\infty} a_{nk} \text{ for each } k; \tag{15}$$

$$\lim_{r\to\infty}\left(\sup_{N_r}\left\|\sum_{n\in N_r}A_n\right\|_X^*\right) \leq \|L\|_X \leq 4\cdot\lim_{r\to\infty}\left(\sup_{N_r}\left\|\sum_{n\in N_r}A_n\right\|_X^*\right) \text{for } Y = \ell_1,$$
$$\text{where the supremum is taken over all finite sets of integers} \geq r; \tag{16}$$

$$\lim_{r\to\infty}\left[\sup_{N_r^*}\left(\sup_{t\in\mathcal{T}}\left\|\sum_{\mu\in N_r^*}A_{t_\mu}\right\|_X^*\right)\right] \leq \|L\|_\chi \leq 4\cdot\lim_{r\to\infty}\left[\sup_{N_r^*}\left(\sup_{t\in\mathcal{T}}\left\|\sum_{\mu\in N_r^*}A_{t_\mu}\right\|_X^*\right)\right]$$
$$\text{for } Y = [\ell_1, \ell_\infty]^{<m(\mu)>}, \text{ where } N_r^* = \{\mu \in \mathbb{N}_0 : r \geq m(\mu)\}. \tag{17}$$

Proof. For each $r \in \mathbb{N}$, let $A^{(r)}$ denote the matrix obtained from the matrix A by replacing the first r rows by the zero sequence.
The identity in (14) holds by [1, Corollary 3.4].
The inequalities in (15) hold with lim sup by [2, Theorem 3.4] and the limit exists by [3, Lemma 1.10 (b)].
To prove the inequalities in (16), we first observe that we have by [12, Theorem 1] and the fact that $\mathcal{R}_{r-1} \circ L$ is given by $A^{(r)}$

$$\sup_{N_r}\left\|\sum_{n\in N_r}A_n\right\|_X^* = \sup_N\left\|\sum_{n\in N}A_n^{(r)}\right\|_X^* \leq \|\mathcal{R}_{r-1}\circ L\|_X^* = \left\|A^{(r)}\right\|_X^* \leq$$

$$4\cdot\sup_N\left\|\sum_{n\in N}A_n^{(r)}\right\|_X^* = \sup_{N_r}\left\|\sum_{n\in N_r}A_n\right\|_X^*.$$

Now the inequalities in (16) follow by (10) and (13).
Similarly, to prove the inequalities in (17), we obtain, using the same argument as in the proof of Theorem 2.1 with the norm $\|\cdot\|^*_{[\ell_r,\ell_p]^{<k(\nu)>}}$ replaced by the norm $\|\cdot\|^*_X$,

$$\sup_N\left[\sup_{t\in\mathcal{T}}\left\|\sum_{\mu\in N}A_{t_\mu}^{(r)}\right\|_X^*\right] \leq \|\mathcal{R}_{r-1}\circ L\| \leq 4\cdot\sup_N\left[\sup_{t\in\mathcal{T}}\left\|\sum_{\mu\in N}A_{t_\mu}^{(r)}\right\|_X^*\right].$$

As before, the inequalities in (17) follow by (10) and (13). □

We obtain as an immediate consequence of Lemma 2.1 and Theorem 3.1

Corollary 3.2. *Let $L\in\mathcal{B}([\ell_r,\ell_p]^{<k(\nu)>},Y)$ where Y is any of the spaces c_0, c, ℓ_1 and $[\ell_1,\ell_\infty]^{<m(\mu)>}$, and $A\in([\ell_r,\ell_p]^{<k(\nu)>},Y)$ be the matrix that represents L as in (4). Then the identity in (14) and the inequalities in (15)–(17) hold with the norm $\|\cdot\|^*_X$ replaced by the norm $\|\cdot\|_{[\ell_s,\ell_q]^{<k(\nu)>}}$.*

Furthermore we obtain the following characterizations of compact operators from Corollary 3.2 and the condition in (i).

Corollary 3.3. *(a) If $L\in\mathcal{B}([\ell_r,\ell_p]^{<k(\nu)>},c_0)$ then L is compact if and only if*

$$\lim_{r\to\infty}\left(\sup_{n\geq r}\|A_n\|_{[\ell_s,\ell_q]^{<k(\nu)>}}\right)=0.$$

(b) If $L\in\mathcal{B}([\ell_r,\ell_p]^{<k(\nu)>},c_0)$ then L is compact if and only if

$$\lim_{r\to\infty}\left(\sup_{n\geq r}\left\|A_n-(\alpha_k)_{k=1}^\infty\right\|^*_{[\ell_s,\ell_q]^{<k(\nu)>}}\right)=0,\ \textit{where}\ \alpha_k=\lim_{n\to\infty}a_{nk}\ \textit{for each}\ k.$$

(c) If $L\in\mathcal{B}([\ell_r,\ell_p]^{<k(\nu)>},\ell_1)$ then L is compact if and only if

$$\lim_{r\to\infty}\left(\sup_{N_r}\left\|\sum_{n\in N_r}A_n\right\|_{[\ell_s,\ell_q]^{<k(\nu)>}}\right)=0.$$

(d) If $L\in\mathcal{B}([\ell_r,\ell_p]^{<k(\nu)>},[\ell_1,\ell_\infty]^{<m(\mu)>})$ then L is compact if and only if

$$\lim_{r\to\infty}\left[\sup_{N_r^*}\left(\sup_{t\in\mathcal{T}}\left\|\sum_{\mu\in N_r^*}A_{t_\mu}\right\|_{[\ell_s,\ell_q]^{<k(\nu)>}}\right)\right]=0.$$

We close with the following remark.

Remark 3.4. *The analogous results of those above in the case of $Y=[c_0,\ell_1]^{<m(\mu)>}$ can easily be obtained from Lemma 2.1 and [3, Theorem 3.6] with $\alpha_k=0$ for all k by observing that 2^ν can be replaced by $k(\nu)$ for $\nu=0,1,\dots,$ and $x\in[c_0,\ell_1]^{<2^\nu>}$ if and only if $(z_kx_k)_{k=1}^\infty\in w_0$ where $z_k=2^\nu$ for $2^\nu\leq k\leq 2^{\nu+1}-1$ and $\nu=0,1,\dots$.*
The results of Corollaries 3.2 and 3.3 also hold for $r=p=1$, since obviously $[\ell_r,\ell_p]^{<k(\nu)>}=[\ell_1,\ell_1]^{<k(\nu)>}=\ell_1$ and

$\|\cdot\|^*_{[\ell_r,\ell_p]^{<k(\nu)>}} = \|\cdot\|_\infty$. *Similarly, we have* $[c_0,\ell_\infty]^{<k(\nu)>} = c_0$ *and* $\|\cdot\|^*_{[c_0,\ell_1]^{<k(\nu)>}} = \|\cdot\|_{\ell_1}$*, and the results of Corollaries 3.2 and 3.3 also hold when* $X = [c_0,\ell_\infty]^{<k(\nu)>}$.
When $X = [c_0,\ell_p]^{<k(\nu)>}$ *for* $1 \leq p < \infty$ *then it easily follows ([11]) that* $\|\cdot\|^*_{[c_0,\ell_p]^{<k(\nu)>}} = \|\cdot\|_{[\ell_1,\ell_q]^{<k(\nu)>}}$*, and the results of Corollaries 3.2 and 3.3 follow from Theorem 3.1. Since* $\|\cdot\|^*_{[\ell_\infty,\ell_p]^{<k(\nu)>}} = \|\cdot\|_{[\ell_1,\ell_q]^{<k(\nu)>}}$*, the results of Corollaries 3.2 and 3.3 hold for those operators on* $[\ell_\infty,\ell_p]^{<k(\nu)>}$ *that are given by matrices.*
Since $Y = [\ell_\infty,\ell_1]^{<m(\mu)>}$ *does not have AK, we cannot use the Goldenštein–Gohberg–Markus theorem. Similarly as in the proof of [2, Theorem 4.1 (c)] we would obtain estimates for* $\|L\|_\chi$ *with the lower bound equal to* 0 *in each case, and only sufficient conditions for L to be compact, which are not necessary, in general.*

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