Filomat 28:5 (2014), 1027–1047
DOI 10.2298/FIL1405027A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Stability of Difference Schemes for Bitsadze-Samarskii Type Nonlocal Boundary Value Problem Involving Integral Condition

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Abstract. In this study, the stable difference schemes for the numerical solution of Bitsadze-Samarskii type nonlocal boundary-value problem involving integral condition for the elliptic equations are studied. The second and fourth orders of the accuracy difference schemes are presented. A procedure of modified Gauss elimination method is used for solving these difference schemes for the two-dimensional elliptic differential equation. The method is illustrated by numerical examples.

1. Introduction. Fourth-order of the accuracy difference scheme

Many problems in fluid mechanics, dynamics, elasticity and other areas of engineering, physics and biological systems lead to partial differential equations of elliptic type. The importance of nonlocal problems appears to have been first noted in the literature by Bitsadze-Samarskii. The problem studied in these papers constitutes a direct generalization of the classical boundary value problems [1]-[2]. Methods of solutions Bitsadze-Samarskii type nonlocal boundary-value problems for elliptic differential equations have been studied extensively by many researchers [6]-[22]. Furthermore investigations of the difference schemes for the approximate solution of boundary-value problems were carried out in [23]-[27]. In the present paper, we consider the Bitsadze-Samarskii type nonlocal boundary-value problem with integral condition,

$$\begin{cases} -\frac{d^2u(t)}{dt^2} + Au(t) = f(t), \ 0 < t < 1, \\ u(0) = \varphi, \ u(1) = \int\limits_0^1 \rho(\lambda)u(\lambda)d\lambda + \psi \end{cases} \tag{1.1}$$

for the differential equation of elliptic type in a Hilbert space H with the self-adjoint positive definite operator A with a closed domain $D(A) \subset H$. Here, let $f(t)$ be a given abstract continuous function defined on $[0, 1]$ with values in H, φ, and ψ are elements of $D(A)$ and $\rho(t)$ is a scalar function. A function $u(t)$ is called a solution of problem (1.1) if the following conditions are satisfied:

i; $u(t)$ is a twice continuously differentiable on the segment $[0, 1]$.

ii; The element $u(t)$ belongs to $D(A)$ for all $t \in [0, 1]$, and the function $Au(t)$ is continuous on the segment $[0, 1]$.

iii; $u(t)$ satisfies the equation and nonlocal boundary conditions (1.1).

2010 *Mathematics Subject Classification*. Primary 65N12; Secondary 65M12, 65J10
Keywords. Elliptic equation, difference scheme, stability
Received: 13 November 2013; Revised: 13 February 2014; Accepted: 07 March 2014
Communicated by Bahattin Yıldız (Guest Editor)
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The fourth order of the accuracy difference scheme

$$\begin{cases} -\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+Au_k+\frac{\tau^2}{12}A^2u_k=\varphi_k, \\ \varphi_k=f(t_k)+\frac{\tau^2}{12}(\frac{f(t_{k+1})-2f(t_k)+f(t_{k-1})}{\tau^2}+Af(t_k)), \quad t_k=k\tau,\ 1\leq k\leq N-1, \\ u_0=\varphi,\ u_N=\frac{\tau}{3}\left(\rho\left(t_0\right)u_0+\rho\left(t_N\right)u_N\right) \\ +\frac{\tau}{3}\left(4\sum\limits_{k=1}^{\frac{N}{2}}\rho\left(t_{2k-1}\right)u_{2k-1}+2\sum\limits_{k=1}^{\frac{N}{2}-1}\rho\left(t_{2k}\right)u_{2k}\right)+\psi \end{cases} \tag{1.2}$$

for the approximate solution of (1.1) is presented. The stability and almost coercive stability estimates for the solution of this difference scheme are established. For applications, the stability and the almost coercive stability estimates for solutions of difference schemes for approximate solutions of nonlocal boundary-value problems for elliptic equations are obtained. Since A is a self-adjoint positive definite operator, we have that $C=\frac{A}{2}+\frac{A^2}{12}$ is also self- adjoint positive definite operator. It follows that the operator $R=(I+\tau B)^{-1}$ exists for $B=\frac{\tau C}{2}+\sqrt{\frac{\tau^2C^2}{4}+C}$ and is defined on the whole space H is a bounded operator. Here, I is the identity operator. Namely, the stability estimates of solution of difference scheme (1.2) is established under the assumption:

$$\left(\frac{\tau}{3}\left|\rho\left(t_0\right)\right|+\frac{\tau}{3}\left|\rho\left(t_N\right)\right|+\frac{8\tau}{3}\sum_{k=1}^{\frac{N}{2}}\left|\rho\left(t_{2k-1}\right)\right|+\frac{4\tau}{3}\sum_{k=1}^{\frac{N}{2}-1}\left|\rho\left(t_{2k}\right)\right|\right)\leq 1. \tag{1.3}$$

2. The Bitsadze-Samarskii Type Nonlocal Boundary Value Problem with the Integral Condition for Elliptic Difference Equations

Firstly, let us give lemma that will be needed below.

Lemma 2.1. The following estimates hold [5]:

$$\begin{cases} \left\|\left(I-R^{2N}\right)^{-1}\right\|_{H\to H}\leq M(\delta), \\ \|R^k\|_{H\to H}\leq M(\delta)(1+\delta\tau)^{-k},\ k\tau\|BR^k\|_{H\to H}\leq M(\delta), k\geq 1, \delta>0, \\ \|B^\beta\left(R^{k+r}-R^k\right)\|_{H\to H}\leq M(\delta)\frac{(r\tau)^\alpha}{(k\tau)^{\alpha+\beta}}, 1\leq k<k+r\leq N, 0\leq\alpha,\beta\leq 1. \end{cases} \tag{2.1}$$

Lemma 2.2. Suppose A is the positive operator in Hilbert space H. Then the following estimate holds [5]:

$$\sum_{j=1}^{N-1}\left\|(I-R)R^{j-1}\right\|_{H\to H}\leq M\min\left(\ln(\frac{1}{\tau}), 1+\tau\left|\ln\|B\|_{H\to H}\right|\right), \tag{2.2}$$

where M does not depend on τ.

Lemma 2.3. The operator

$$I-\frac{\tau}{3}\rho\left(t_N\right)-\frac{4\tau}{3}\sum_{k=1}^{\frac{N}{2}}\rho\left(t_{2k-1}\right)\left(I-R^{2N}\right)^{-1}\left(R^{N-2k+1}-R^{N+2k-1}\right)-\frac{2\tau}{3}\sum_{k=1}^{\frac{N}{2}-1}\rho\left(t_{2k}\right)\left(I-R^{2N}\right)^{-1}\left(R^{N-2k}-R^{N+2k}\right)$$

has an inverse

$$G_\tau=\left(I-\frac{\tau}{3}\rho\left(t_N\right)-\frac{4\tau}{3}\sum_{k=1}^{\frac{N}{2}}\rho\left(t_{2k-1}\right)\left(I-R^{2N}\right)^{-1}\left(R^{N-2k+1}-R^{N+2k-1}\right)-\frac{2\tau}{3}\sum_{k=1}^{\frac{N}{2}-1}\rho\left(t_{2k}\right)\left(I-R^{2N}\right)^{-1}\left(R^{N-2k}-R^{N+2k}\right)\right)^{-1}$$

and the following estimate is satisfied under the assumption (1.3)

$$\|G_\tau\|_{H\to H}\le M\left(\delta\right)\tau, \tag{2.3}$$

where M does not depend on τ.

Theorem 2.1. For any φ_k, $1\le k\le N-1$, the solution of the problem (1.2) exists and the following formula holds:

$$u_k=(I-R^{2N})^{-1}\left\{\left(R^k-R^{2N-k}\right)\varphi+\left(R^{N-k}-R^{N+k}\right)u_N\right.$$

$$\left.-\left(R^{N-k}-R^{N+k}\right)\left(I+\tau B\right)\left(2I+\tau B\right)^{-1}B^{-1}\sum_{i=1}^{N-1}\left(R^{N-1-i}-R^{N-1+i}\right)\varphi_i\tau\right\}$$

$$+\left(I+\tau B\right)\left(2I+\tau B\right)^{-1}B^{-1}\sum_{i=1}^{N-1}\left(R^{|k-i|-1}-R^{k+i-1}\right)\varphi_i\tau \text{ for k} = 1,\cdots,N-1,$$

$$u_N=G_\tau\left(\left(\frac{\tau}{3}\rho\left(t_0\right)+\frac{4\tau}{3}\sum_{k=1}^{\frac{N}{2}}\rho\left(t_{2k-1}\right)\left(I-R^{2N}\right)^{-1}\left(R^{2k-1}-R^{2N-2k+1}\right)\right.\right.$$

$$\left.+\frac{2\tau}{3}\sum_{k=1}^{\frac{N}{2}-1}\rho\left(t_{2k}\right)\left(I-R^{2N}\right)^{-1}\left(R^{2k}-R^{2N-2k}\right)\right)\varphi+\left(I-R^{2N}\right)^{-1}$$

$$\times\left(-\frac{4\tau}{3}\sum_{k=1}^{\frac{N}{2}}\rho\left(t_{2k-1}\right)\left(R^{N-2k-1}-R^{N+2k-1}\right)\right.$$

$$\left.\times\left(I+\tau B\right)\left(2I+\tau B\right)^{-1}B^{-1}\sum_{i=1}^{N-1}\left(R^{N-1-i}-R^{N-1+i}\right)\varphi_i\tau\right)$$

$$+\frac{4\tau}{3}\sum_{k=1}^{\frac{N}{2}}\rho\left(t_{2k-1}\right)\left(I+\tau B\right)\left(2I+\tau B\right)^{-1}B^{-1}\sum_{i=1}^{2k-1}\left(R^{2k-2-i}-R^{2k-2+i}\right)\varphi_i\tau$$

$$+\frac{4\tau}{3}\sum_{k=1}^{\frac{N}{2}}\rho\left(t_{2k-1}\right)\left(I+\tau B\right)\left(2I+\tau B\right)^{-1}B^{-1}\sum_{i=2k}^{N-1}\left(R^{i-2k}-R^{2k-2+i}\right)\varphi_i\tau$$

$$+\frac{2\tau}{3}\sum_{k=1}^{\frac{N}{2}-1}\rho\left(t_{2k}\right)\left(I+\tau B\right)\left(2I+\tau B\right)^{-1}B^{-1}\sum_{i=1}^{2k}\left(R^{2k-1-i}-R^{2k-1+i}\right)\varphi_i\tau$$

$$+\frac{2\tau}{3}\sum_{k=1}^{\frac{N}{2}-1}\rho\left(t_{2k}\right)(I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{i=2k+1}^{N-1}\left(R^{i-2k-1}-R^{2k-1+i}\right)\varphi_i\tau+\psi\Bigg).$$

Here,

$$G_\tau=\left(I-\frac{\tau}{3}\rho\left(t_N\right)-\frac{4\tau}{3}\sum_{k=1}^{\frac{N}{2}}\rho\left(t_{2k-1}\right)\left(I-R^{2N}\right)^{-1}\left(R^{N-2k+1}-R^{N+2k-1}\right)-\frac{2\tau}{3}\sum_{k=1}^{\frac{N}{2}-1}\rho\left(t_{2k}\right)\left(I-R^{2N}\right)^{-1}\left(R^{N-2k}-R^{N+2k}\right)\right)^{-1}.$$

Proof. The difference scheme

$$\begin{cases} -\frac{1}{\tau^2}[u_{k+1}-2u_k+u_{k-1}]+Au_k=\varphi_k, 1\le k\le N-1, N\tau=1,\\ u_0=\varphi,\ u_N \text{ is given}\end{cases} \tag{2.4}$$

has a solution and the following formula holds [5]:

$$u_k=(I-R^{2N})^{-1}\left\{\left(R^k-R^{2N-k}\right)\varphi+\left(R^{N-k}-R^{N+k}\right)u_N\right. \tag{2.5}$$

$$\left.-\left(R^{N-k}-R^{N+k}\right)(I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}\left(R^{N-1-i}-R^{N-1+i}\right)\varphi_i\tau\right\}$$

$$+(I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}\left(R^{|k-i|-1}-R^{k+i-1}\right)\varphi_i\tau.$$

The fourth order of the accuracy difference scheme can be written as,

$$\begin{cases} -\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+Cu_k=\varphi_k,\\ \varphi_k=f(t_k)+\frac{\tau^2}{12}\left(\frac{f(t_{k+1})-2f(t_k)+f(t_{k-1})}{\tau^2}+Af(t_k)\right),\quad t_k=k\tau,\ 1\le k\le N-1,\\ u_0=\varphi,\ u_N=\frac{\tau}{3}\left(\rho\left(t_0\right)u_0+\rho\left(t_N\right)u_N\right)\\ +\frac{\tau}{3}\left(4\sum\limits_{k=1}^{\frac{N}{2}}\rho\left(t_{2k-1}\right)u_{2k-1}+2\sum\limits_{k=1}^{\frac{N}{2}-1}\rho\left(t_{2k}\right)u_{2k}\right)+\psi.\end{cases}$$

Applying formula (2.5) and the nonlocal boundary condition, we obtain

$$u_N=\frac{\tau}{3}\left(\rho\left(t_0\right)u_0+\rho\left(t_N\right)u_N\right)+\frac{\tau}{3}\Bigg(4\sum_{k=1}^{\frac{N}{2}}\rho\left(t_{2k-1}\right)$$

$$\times\Big[(I-R^{2N})^{-1}\left\{\left(R^{2k-1}-R^{2N-2k+1}\right)\varphi+\left(R^{N-2k+1}-R^{N+2k-1}\right)u_N\right.$$

$$\left.-\left(R^{N-2k-1}-R^{N+2k-1}\right)(I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}\left(R^{N-1-i}-R^{N-1+i}\right)\varphi_i\tau\right\}$$

$$+ (I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}\left(R^{|2k-1-i|-1}-R^{2k+i-2}\right)\varphi_i\tau\Bigg]$$

$$+2\sum_{k=1}^{\frac{N}{2}-1}\rho(t_{2k})\Big[(I-R^{2N})^{-1}\Big\{\left(R^{2k}-R^{2N-2k}\right)\varphi+\left(R^{N-2k}-R^{N+2k}\right)u_N$$

$$-\left(R^{N-2k}-R^{N+2k}\right)(I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}\left(R^{N-1-i}-R^{N-1+i}\right)\varphi_i\tau\Big\}$$

$$+(I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}\left(R^{|2k-i|-1}-R^{2k+i-1}\right)\varphi_i\tau\Big]\Bigg)+\psi.$$

Since the operator

$$I-\frac{\tau}{3}\rho(t_N)-\frac{4\tau}{3}\sum_{k=1}^{\frac{N}{2}}\rho(t_{2k-1})\left(I-R^{2N}\right)^{-1}\left(R^{N-2k+1}-R^{N+2k-1}\right)-\frac{2\tau}{3}\sum_{k=1}^{\frac{N}{2}-1}\rho(t_{2k})\left(I-R^{2N}\right)^{-1}\left(R^{N-2k}-R^{N+2k}\right)$$

has an inverse G_τ, it follows that

$$u_N=G_\tau\Bigg(\Bigg(\frac{\tau}{3}\rho(t_0)+\frac{4\tau}{3}\sum_{k=1}^{\frac{N}{2}}\rho(t_{2k-1})\left(I-R^{2N}\right)^{-1}\left(R^{2k-1}-R^{2N-2k+1}\right) \tag{2.6}$$

$$+\frac{2\tau}{3}\sum_{k=1}^{\frac{N}{2}-1}\rho(t_{2k})\left(I-R^{2N}\right)^{-1}\left(R^{2k}-R^{2N-2k}\right)\Bigg)\varphi+\left(I-R^{2N}\right)^{-1}$$

$$\times\Bigg(-\frac{4\tau}{3}\sum_{k=1}^{\frac{N}{2}}\rho(t_{2k-1})\left(R^{N-2k-1}-R^{N+2k-1}\right)$$

$$\times(I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}\left(R^{N-1-i}-R^{N-1+i}\right)\varphi_i\tau$$

$$-\frac{2\tau}{3}\sum_{k=1}^{\frac{N}{2}-1}\rho(t_{2k})\left(R^{N-2k}-R^{N+2k}\right)(I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}\left(R^{N-1-i}-R^{N-1+i}\right)\varphi_i\tau\Bigg)$$

$$+\frac{4\tau}{3}\sum_{k=1}^{\frac{N}{2}}\rho(t_{2k-1})(I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{i=1}^{2k-1}\left(R^{2k-2-i}-R^{2k-2+i}\right)\varphi_i\tau$$

$$+\frac{4\tau}{3}\sum_{k=1}^{\frac{N}{2}}\rho(t_{2k-1})(I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{i=2k}^{N-1}\left(R^{i-2k}-R^{2k-2+i}\right)\varphi_i\tau$$

$$+\frac{2\tau}{3}\sum_{k=1}^{\frac{N}{2}-1}\rho\left(t_{2k}\right)(I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{i=1}^{2k}\left(R^{2k-1-i}-R^{2k-1+i}\right)\varphi_i\tau$$

$$\left.+\frac{2\tau}{3}\sum_{k=1}^{\frac{N}{2}-1}\rho\left(t_{2k}\right)(I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{i=2k+1}^{N-1}\left(R^{i-2k-1}-R^{2k-1+i}\right)\varphi_i\tau+\psi\right).$$

Theorem 2.1. is proved.
Let $F([0,1]_\tau, H)$ be the linear space of the mesh functions $\varphi^\tau=\{\varphi_k\}_1^{N-1}$ with values in the Hilbert space H. We denote by $C([0,1]_\tau, H)$ Banach space with the norm

$$\left\|\varphi^\tau\right\|_{C([0,1]_\tau,H)}=\max_{1\le k\le N-1}\left\|\varphi_k\right\|_H.$$

The nonlocal boundary-value problem (1.2) is said to be stable in $F([0,1]_\tau, H)$ if we have the inequality

$$\|u^\tau\|_{F([0,1]_\tau,H)}\le M(\delta)\left[\left\|\varphi^\tau\right\|_{F([0,1]_\tau,H)}+\left\|\varphi\right\|_H+\left\|\psi\right\|_H\right].$$

Theorem 2.2. The solution of the difference scheme (1.2) satisfies the stability estimate with the assumption (1.3)

$$\|u^\tau\|_{C([0,1]\tau,H)}\le M(\delta)\left[\left\|\varphi^\tau\right\|_{C([0,1]\tau,H)}+\left\|\psi\right\|_H+\left\|\varphi\right\|_H\right], \quad (2.7)$$

where M does not depend on φ^τ, φ, ψ and τ.
Proof. We have that

$$\|u^\tau\|_{C([0,1]\tau,H)}\le M(\delta)\left[\left\|\varphi^\tau\right\|_{C([0,1]\tau,H)}+\left\|\psi\right\|_H+\|u_N\|_H\right] \quad (2.8)$$

for the solution of difference scheme (2.4).
The proof of (2.7) is based on (2.8) and on the estimate

$$\|u_N\|_H\le M(\delta)\left[\left\|\varphi^\tau\right\|_{C([0,1]\tau,H)}+\left\|\psi\right\|_H+\|u_N\|_H\right].$$

Using the formula (2.6), the estimates (2.1), (2.3) and the triangle inequality, we get

$$\|u_N\|_H\le\|G_\tau\|_{H\to H}\left\|\left(I-R^{2N}\right)^{-1}\right\|_{H\to H}\left[\left(\left\|I-R^{2N}\right\|_{H\to H}\frac{\tau}{3}\left|\rho\left(t_0\right)\right|\right.\right.$$

$$+\frac{4\tau}{3}\sum_{k=1}^{\frac{N}{2}}\left|\rho\left(t_{2k-1}\right)\right|\left(\left\|R^{2k-1}\right\|_{H\to H}+\left\|R^{2N-2k+1}\right\|_{H\to H}\right)$$

$$\left.+\frac{2\tau}{3}\sum_{k=1}^{\frac{N}{2}-1}\left|\rho\left(t_{2k}\right)\right|\left(\left\|R^{2k}\right\|_{H\to H}+\left\|R^{2N-2k}\right\|_{H\to H}\right)\right)\left\|\varphi\right\|_H$$

$$+\frac{4\tau}{3}\sum_{k=1}^{\frac{N}{2}}\left|\rho\left(t_{2k-1}\right)\right|\left(\left\|R^{N-2k-1}\right\|_{H\to H}+\left\|R^{N+2k-1}\right\|_{H\to H}\right)\left\|B^{-1}\right\|_{H\to H}$$

$$\times \left\|(I+\tau B)(2I+\tau B)^{-1}\right\|_{H\to H}\sum_{i=1}^{N-1}\left(\left\|R^{N-i-1}\right\|_{H\to H}+\left\|R^{N+i-1}\right\|_{H\to H}\right)\left\|\varphi_i\right\|_H\tau$$

$$+\frac{2\tau}{3}\sum_{k=1}^{\frac{N}{2}-1}\left|\rho(t_{2k})\right|\left(\left\|R^{N-2k}\right\|_{H\to H}+\left\|R^{N+2k}\right\|_{H\to H}\right)\left\|B^{-1}\right\|_{H\to H}$$

$$\times \left\|(I+\tau B)(2I+\tau B)^{-1}\right\|_{H\to H}\sum_{i=1}^{N-1}\left(\left\|R^{N-i-1}\right\|_{H\to H}+\left\|R^{N+i-1}\right\|_{H\to H}\right)\left\|\varphi_i\right\|_H\tau$$

$$+\left\|I-R^{2N}\right\|_{H\to H}\left(\frac{4\tau}{3}\sum_{k=1}^{\frac{N}{2}}\left|\rho(t_{2k-1})\right|\left\|(I+\tau B)(2I+\tau B)^{-1}\right\|_{H\to H}\left\|B^{-1}\right\|_{H\to H}\right.$$

$$\times\sum_{i=1}^{2k-1}\left(\left\|R^{2k-2-i}\right\|_{H\to H}+\left\|R^{2k-2+i}\right\|_{H\to H}\right)\left\|\varphi_i\right\|_H\tau$$

$$+\frac{4\tau}{3}\sum_{k=1}^{\frac{N}{2}}\left|\rho(t_{2k-1})\right|\left\|(I+\tau B)(2I+\tau B)^{-1}\right\|_{H\to H}\left\|B^{-1}\right\|_{H\to H}$$

$$\times\sum_{i=2k}^{N-1}\left(\left\|R^{i-2k}\right\|_{H\to H}+\left\|R^{2k-2+i}\right\|_{H\to H}\right)\left\|\varphi_i\right\|_H\tau$$

$$+\frac{2\tau}{3}\sum_{k=1}^{\frac{N}{2}-1}\left|\rho(t_{2k})\right|\left\|(I+\tau B)(2I+\tau B)^{-1}\right\|_{H\to H}\left\|B^{-1}\right\|_{H\to H}$$

$$\times\sum_{i=1}^{2k}\left(\left\|R^{2k-i-1}\right\|_{H\to H}+\left\|R^{2k+i-1}\right\|_{H\to H}\right)\left\|\varphi_i\right\|_H\tau$$

$$+\frac{2\tau}{3}\sum_{k=1}^{\frac{N}{2}-1}\left|\rho(t_{2k})\right|\left\|(I+\tau B)(2I+\tau B)^{-1}\right\|_{H\to H}\left\|B^{-1}\right\|_{H\to H}$$

$$\left.\left.\times\sum_{i=2k+1}^{N-1}\left(\left\|R^{i-2k-1}\right\|_{H\to H}+\left\|R^{2k+i-1}\right\|_{H\to H}\right)\left\|\varphi_i\right\|_H\tau+\left\|\psi\right\|_H\right)\right].$$

From (1.3) it follows that

$$\|u_N\|_H \le M_2(\delta)\left[\left\|\varphi^\tau\right\|_{C([0,1]_\tau,H)}+\left\|\varphi\right\|_H+\left\|\psi\right\|_H\right].$$

So, Theorem 2.2. is proved.

Theorem 2.3. The solution of the difference problem (1.2) in $C([0,1]_\tau, H)$ under the assumption (1.3) obeys the almost coercive inequality

$$\|\{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\}_1^{N-1}\|_{C([0,1]_\tau,H)}+\|\left\{A\left(I+\frac{A\tau^2}{12}\right)u_k\right\}_1^{N-1}\|_{C([0,1]_\tau,H)}$$

$$\leq M(\delta)\left[\min\left\{\ln\frac{1}{\tau},1+|\ln\|B\|_{H\to H}|\right\}\|\varphi^{\tau}\|_{C([0,1]_\tau,H)}\right.$$
$$\left.+\left\|A\left(I+\frac{A\tau^2}{12}\right)\varphi\right\|_H+\left\|A\left(I+\frac{A\tau^2}{12}\right)\psi\right\|_H\right].$$

Here M does not depend on τ,ψ,φ and φ_k, $1\leq k\leq N-1$.

Proof. The proof of this theorem is based on the estimate

$$\|\{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\}_1^{N-1}\|_{C([0,1]_\tau,H)}+\|\{Au_k\}_1^N\|_{C([0,1]_\tau,H)}$$
$$\leq M(\delta)\left[\min\left\{\ln\frac{1}{\tau},1+|\ln\|B\|_{H\to H}|\right\}\|\varphi^{\tau}\|_{C([0,1]_\tau,H)}+\left\|A\varphi\right\|_H+\left\|A\psi\right\|_H\right]$$

and on the estimate

$$\|A\left(I+\frac{A\tau^2}{12}\right)u_N\|_H \tag{2.9}$$
$$\leq M(\delta)\left(\min\left\{\ln\frac{1}{\tau},1+|\ln\|B\|_{H\to H}|\right\}\|\varphi^{\tau}\|_{C([0,1]\tau,H)}\right.$$
$$\left.+\left\|A\left(I+\frac{A\tau^2}{12}\right)\varphi\right\|_H+\|A\left(I+\frac{A\tau^2}{12}\right)\psi\|_H\right)$$

for the solutions of difference scheme (1.2). Using the formula (2.6), and $A=B^2R$ we obtain

$$A\left(I+\frac{A\tau^2}{12}\right)u_N=J_1+J_2,$$

where

$$J_1=G_\tau\left(\left(\frac{\tau}{3}\rho(t_0)+\frac{4\tau}{3}\sum_{k=1}^{\frac{N}{2}}\rho(t_{2k-1})\left(I-R^{2N}\right)^{-1}\left(R^{2k-1}-R^{2N-2k+1}\right)\right.\right.$$
$$\left.\left.+\frac{2\tau}{3}\sum_{k=1}^{\frac{N}{2}-1}\rho(t_{2k})\left(I-R^{2N}\right)^{-1}\left(R^{2k}-R^{2N-2k}\right)\right)A\left(I+\frac{A\tau^2}{12}\right)\varphi+A\left(I+\frac{A\tau^2}{12}\right)\psi\right),$$

$$J_2=G_\tau\left(I-R^{2N}\right)^{-1}\left(-\frac{4\tau}{3}\sum_{k=1}^{\frac{N}{2}}\rho(t_{2k-1})\left(R^{N-2k-1}-R^{N+2k-1}\right)\right.$$

$$\times(I+\tau B)(2I+\tau B)^{-1}\sum_{i=1}^{N-1}B\left(R^{N+1-i}-R^{N+1+i}\right)\varphi_i\tau$$

$$\left.-\frac{2\tau}{3}\sum_{k=1}^{\frac{N}{2}-1}\rho(t_{2k})\left(R^{N-2k}-R^{N+2k}\right)(I+\tau B)(2I+\tau B)^{-1}\sum_{i=1}^{N-1}B\left(R^{N+1-i}-R^{N+1+i}\right)\varphi_i\tau\right)$$

$$+\frac{4\tau}{3}\sum_{k=1}^{\frac{N}{2}}\rho(t_{2k-1})(I+\tau B)(2I+\tau B)^{-1}\sum_{i=1}^{2k-1}B\left(R^{2k-i}-R^{2k+i}\right)\varphi_i\tau$$

$$+\frac{4\tau}{3}\sum_{k=1}^{\frac{N}{2}}\rho\left(t_{2k-1}\right)(I+\tau B)(2I+\tau B)^{-1}\sum_{i=2k}^{N-1}B\left(R^{i-2k+2}-R^{2k+i}\right)\varphi_i\tau$$

$$+\frac{2\tau}{3}\sum_{k=1}^{\frac{N}{2}-1}\rho\left(t_{2k}\right)(I+\tau B)(2I+\tau B)^{-1}\sum_{i=1}^{2k}B\left(R^{2k+1-i}-R^{2k+1+i}\right)\varphi_i\tau$$

$$\left.+\frac{2\tau}{3}\sum_{k=1}^{\frac{N}{2}-1}\rho\left(t_{2k}\right)(I+\tau B)(2I+\tau B)^{-1}\sum_{i=2k+1}^{N-1}B\left(R^{i-2k+1}-R^{2k+1+i}\right)\varphi_i\tau\right).$$

To this end (1.2) suffices to show that

$$\|J_1\|_H\leq M(\delta)\left(\left\|A\left(I+\frac{A\tau^2}{12}\right)\varphi\right\|_H+\|A\left(I+\frac{A\tau^2}{12}\right)\psi\|_H\right) \tag{2.10}$$

and

$$\|J_2\|_H\leq M(\delta)\min\left\{\ln\frac{1}{\tau},1+|\ln\|B\|_{H\to H}|\right\}\|\varphi^\tau\|. \tag{2.11}$$

The estimate (2.10) follows from the estimates (2.1), (2.3). Using the estimates (2.1), (2.3), (2.2) and under the assumption (1.3) we obtain

$$\|J_2\|_H\leq\|G_\tau\|_{H\to H}\left\|\left(I-R^{2N}\right)^{-1}\right\|_{H\to H}$$

$$\times\left[\frac{4\tau}{3}\sum_{k=1}^{\frac{N}{2}}\left|\rho\left(t_{2k-1}\right)\right|\left(\left\|R^{N-2k-1}\right\|_{H\to H}+\left\|R^{N+2k-1}\right\|_{H\to H}\right)\right.$$

$$\times\left\|(I+\tau B)(2I+\tau B)^{-1}\right\|_{H\to H}\sum_{i=1}^{N-1}\left(\left\|(I-R)R^{N-i}\right\|_{H\to H}+\left\|(I-R)R^{N+i}\right\|_{H\to H}\right)\left\|\varphi_i\right\|_H$$

$$+\frac{2\tau}{3}\sum_{k=1}^{\frac{N}{2}-1}\left|\rho\left(t_{2k}\right)\right|\left(\left\|R^{N-2k}\right\|_{H\to H}+\left\|R^{N+2k}\right\|_{H\to H}\right)$$

$$\times\left\|(I+\tau B)(2I+\tau B)^{-1}\right\|_{H\to H}\sum_{i=1}^{N-1}\left(\left\|(I-R)R^{N-i}\right\|_{H\to H}+\left\|(I-R)R^{N+i}\right\|_{H\to H}\right)\left\|\varphi_i\right\|_H$$

$$+\left\|I-R^{2N}\right\|_{H\to H}\left(\frac{4\tau}{3}\sum_{k=1}^{\frac{N}{2}}\left|\rho\left(t_{2k-1}\right)\right|\left\|(I+\tau B)(2I+\tau B)^{-1}\right\|_{H\to H}\right.$$

$$\times\sum_{i=1}^{2k-1}\left(\left\|(I-R)R^{2k-1-i}\right\|_{H\to H}+\left\|(I-R)R^{2k-1+i}\right\|_{H\to H}\right)\left\|\varphi_i\right\|_H$$

$$+\frac{4\tau}{3}\sum_{k=1}^{\frac{N}{2}}\left|\rho\left(t_{2k-1}\right)\right|\left\|(I+\tau B)\left(2I+\tau B\right)^{-1}\right\|_{H\to H}$$

$$\times\sum_{i=2k}^{N-1}\left(\left\|(I-R)\,R^{i-2k+1}\right\|_{H\to H}+\left\|(I-R)\,R^{2k-1+i}\right\|_{H\to H}\right)\left\|\varphi_i\right\|_H$$

$$+\frac{2\tau}{3}\sum_{k=1}^{\frac{N}{2}-1}\left|\rho\left(t_{2k}\right)\right|\left\|(I+\tau B)\left(2I+\tau B\right)^{-1}\right\|_{H\to H}$$

$$\times\sum_{i=1}^{2k}\left(\left\|(I-R)\,R^{2k-i}\right\|_{H\to H}+\left\|(I-R)\,R^{2k+i}\right\|_{H\to H}\right)\left\|\varphi_i\right\|_H$$

$$+\frac{2\tau}{3}\sum_{k=1}^{\frac{N}{2}-1}\left|\rho\left(t_{2k}\right)\right|\left\|(I+\tau B)\left(2I+\tau B\right)^{-1}\right\|_{H\to H}$$

$$\left.\left.\times\sum_{i=2k+1}^{N-1}\left(\left\|(I-R)\,R^{i-2k}\right\|_{H\to H}+\left\|(I-R)\,R^{2k+i}\right\|_{H\to H}\right)\left\|\varphi_i\right\|_H\right)\right]$$

$$\leq M_1\min\left\{\ln\frac{1}{\tau},1+|\ln\|\,B\,\|_{H\to H}|\right\}\|\,\varphi^\tau\,\|.$$

Hence, from the estimates (2.11) and (2.10), it follows (2.9). Theorem 2.3. is proved.

Secondly, we will give the application of Theorems 2.2. and 2.3. We consider the mixed boundary-value problem for elliptic equation

$$\begin{cases} -u_{tt}-(a(x)u_x)_x+\delta u=f(t,x), 0<t<1, 0<x<1, \\ u(t,0)=u(t,1),\ u_x(t,1)=u_x(t,0), 0\leq t\leq 1, \\ u(0,x)=\varphi(x), u(1,x)=\int\limits_0^1\rho(\lambda)u(\lambda,x)d\lambda+\psi\left(x\right), 0\leq x\leq 1, \end{cases} \tag{2.12}$$

where $a(x)$, $\varphi(x)$, $\psi(x)$ and $f(t,x)$ are given sufficiently smooth functions and $a(x)\geq a>0$, $a(1)=a(0)$, $\delta=const>0$. The discretization of problem (2.12) is carried out in two steps. In the first step, let us define the grid space

$$[0,1]_h=\{x:x_n=nh, 0\leq n\leq M,\ Mh=1\}.$$

We introduce the Hilbert space $L_{2h}=L_2([0,1]_h)$ of the grid functions $\varphi^h(x)=\{\varphi_n\}_{n=1}^{M-1}$ defined on $[0,1]_h$, equipped with the norms

$$\left\|\varphi^h\right\|_{L_{2h}}=(\sum_{x\in[0,1]_h}\left|\varphi^h(x)\right|^2h)^{\frac{1}{2}},$$

$$\left\|\varphi^h\right\|_{W_{2h}^2}=\left\|\varphi^h\right\|_{L_{2h}}+\left(\sum_{x\in[0,1]_h}\left|(\varphi^h\left(x\right))_x\right|^2h\right)^{1/2}+\left(\sum_{x\in[0,1]_h}\left|(\varphi^h(x))_{\bar{x},x}\right|^2h\right)^{1/2}.$$

To the differential operator A generated by the problem (2.12) we assign the difference operator A_h^x by the formula

$$A_h^x\varphi^h(x) = \{-(a(x)\varphi_{\bar{x}})_{x,n} + \delta\varphi_n\}_1^{M-1}, \tag{2.13}$$

acting in the space of the grid functions $\varphi^h(x) = \{\varphi_n\}_1^{M-1}$ satisfying the conditions $\varphi_0 = \varphi_M$, $\varphi_1 - \varphi_0 = \varphi_M - \varphi_{M-1}$. It is known that A_h^x is a self-adjoint positive definite operator in L_{2h}. With the help of A_h^x, we arrive at the nonlocal boundary-value problem

$$\begin{cases} -\frac{d^2u^h(t,x)}{dt^2} + A_h^x u^h(t,x) = f^h(t,x),\ 0 < t < 1,\ x \in [0,1]_h, \\ u^h(0,x) = \varphi^h(x);\ u^h(1,x) = \int\limits_0^1 \rho(t)\, u^h(t,x)dt + \psi^h(x),\ x \in [0,1]_h \end{cases} \tag{2.14}$$

for an infinite system of ordinary differential equations. In the second step, equation (2.14) is replaced, by the difference scheme found next:

$$\begin{cases} -\frac{u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)}{\tau^2} + A_h^x u_k^h(x) + \frac{\tau^2}{12}\left(A_h^x\right)^2 u_k^h(x) = \varphi_k^h, \\ \varphi_k^h = f^h(t_k,x) + \frac{\tau^2}{12}\left(\frac{f^h(t_{k+1},x)-2f^h(t_k,x)+f^h(t_{k-1},x)}{\tau^2} + A_h^x f^h(t_k,x)\right), \\ t_k = k\tau,\ 1 \le k \le N-1, N\tau = 1,\ x \in [0,1]_h, \\ u_0^h(x) = \varphi^h(x);\ u_N^h(x) = \frac{\tau}{3}\left(\rho(t_0)\, u_0^h(x) + \rho(t_N)\, u_N^h(x)\right) \\ +\frac{\tau}{3}\left(4\sum\limits_{k=1}^{\frac{N}{2}} \rho(t_{2k-1})\, u_{2k-1}^h(x) + 2\sum\limits_{k=1}^{\frac{N}{2}-1} \rho(t_{2k})\, u_{2k}^h(x)\right) + \psi^h(x),\ x \in [0,1]_h. \end{cases} \tag{2.15}$$

Theorem 2.4. Let τ and h be sufficiently small positive numbers. Then under the assumption (1.3), the solution of the difference scheme (2.15) satisfies the following stability and almost coercivity estimates:

$$\max_{1\le k\le N-1}\left\|u_k^h\right\|_{L_{2h}} \le M_1(\delta)\left[\max_{1\le k\le N-1}\left\|\varphi_k^h\right\|_{L_{2h}} + \left\|\psi^h\right\|_{L_{2h}} + \left\|\varphi^h\right\|_{L_{2h}}\right],$$

$$\max_{1\le k\le N-1} \|\tau^{-2}\left(u_{k+1}^h - 2u_k^h + u_{k-1}^h\right)\|_{L_{2h}} + \max_{1\le k\le N-1}\|\left(u_k^h\right)\|_{W_{2h}^2}$$

$$\le M_2(\delta)\left[\ln\frac{1}{\tau+|h|}\max_{1\le k\le N-1}\|\varphi_k^h\|_{L_{2h}} + \left\|\varphi^h\right\|_{W_{2h}^2} + \left\|\psi^h\right\|_{W_{2h}^2}\right].$$

Here, M_1 and M_2 independent on $\tau, h, \psi^h(x), \varphi^h(x)$ and $\varphi_k^h(x), 1 \le k \le N-1$.
The proof of Theorem 2.4. is based on abstract Theorems 2.2. and 2.3., on the estimate

$$\min\left\{\ln\frac{1}{\tau}, 1 + \left|\ln\|B_h^x\|_{L_{2h}\to L_{2h}}\right|\right\} \le M\ln\frac{1}{\tau+|h|}, \tag{2.16}$$

and on the symmetry properties of the difference operator A_h^x defined by the formula (2.13) in L_{2h}.

Let Ω be the unit open cube in $\mathbb{R}^n(x = (x_1,\cdots,x_n) : 0 < x_k < 1, 1 \le k \le n)$ with boundary S, $\overline{\Omega} = \Omega \cup S$. In $[0,1]\times\Omega$, the Dirichlet-Bitsadze-Samarskii type mixed boundary-value problem for the multidimensional elliptic equation

$$\begin{cases} -u_{tt} - \sum\limits_{r=1}^{n} (a_r(x)u_{x_r})_{x_r} = f(t,x), 0 < t < 1,\ x = (x_1, \ldots, x_n) \in \Omega, \\ u(0,x) = \varphi(x),\ u(1,x) = \int\limits_0^1 \rho(\lambda) u(\lambda, x) d\lambda + \psi(x),\ x \in \overline{\Omega}, \\ u(t,x) \mid_{x \in S} = 0,\ x \in \overline{\Omega} \end{cases} \tag{2.17}$$

is considered. We will study the problem (2.17) under the assumption (1.3). Here, $a_r(x), (x \in \Omega), \psi(x), \varphi(x)$ $(x \in \overline{\Omega})$ and $f(t,x)$ $(t \in (0,1),\ x \in \Omega)$ are smooth functions and $a_r(x) \geq a > 0$. The discretization of problem (2.17) is carried out in two steps. In the first step let us define the grid sets

$$\overline{\Omega}_h = \{x = x_m = (h_1 m_1, \cdots, h_m m_m), m = (m_1, \cdots, m_m), 0 \leq m_r \leq N_r,$$

$$h_r N_r = 1,\ r = 1, \cdots, m\}, \Omega_h = \widetilde{\Omega}_h \cap \Omega, S_h = \widetilde{\Omega}_h \cap S.$$

We introduce the Hilbert space $L_{2h} = L_2(\widetilde{\Omega}_h)$ of the grid functions $\varphi^h(x) = \{\varphi(h_1 m_1, \cdots, h_m m_m)\}$ defined on $\widetilde{\Omega}_h$, equipped with the norms

$$\left\|\varphi^h\right\|_{L_{2h}} = (\sum_{x \in \overline{\Omega}_h} \left|\varphi^h(x)\right|^2 h_1 \cdots h_m)^{1/2},$$

$$\begin{aligned} \left\|\varphi^h\right\|_{W_{2h}^2} &= \left\|\varphi^h\right\|_{L_{2h}} + \left(\sum_{x \in \widetilde{\Omega}_h} \sum_{r=1}^{m} \left|(\varphi^h(x))_{x_r}\right|^2 h_1 \cdots h_m\right)^{1/2} \\ &+ \left(\sum_{x \in \widetilde{\Omega}_h} \sum_{r=1}^{m} \left|(\varphi^h(x))_{x_r \overline{x}_r,\, m_r}\right|^2 h_1 \cdots h_m\right)^{1/2}. \end{aligned}$$

To the differential operator A generated by the problem (2.17), we assign the difference operator A_h^x by the formula

$$A_h^x u^h(x) = -\sum_{r=1}^{m} (a_r(x) u^h_{\bar{x}_r})_{x_r, j_r}, \tag{2.18}$$

acting in the space of the grid functions $u^h(x)$, satisfying the conditions $u^h(x) = 0$ for all $x \in S_h$. It is known that A_h^x is a self-adjoint positive definite operator in L_{2h}. With the help of A_h^x, we arrive at the nonlocal boundary-value problem for an infinite system of ordinary differential equations

$$\begin{cases} -\frac{d^2 u^h(t,x)}{dt^2} + A_h^x u^h(t,x) = f^h(t,x),\ 0 < t < 1,\ x \in \widetilde{\Omega}_h, \\ u^h(0,x) = \varphi^h(x);\ u^h(1,x) = \int\limits_0^1 \rho(t) u^h(t,x) dt + \psi^h(x),\ x \in \widetilde{\Omega}_h. \end{cases} \tag{2.19}$$

In the second step, (2.19) is replaced, by the difference scheme (1.2), and we obtain the fourth order of accuracy difference scheme

$$\begin{cases} -\frac{u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)}{\tau^2}+A_h^x u_k^h(x)+\frac{\tau^2}{12}\left(A_h^x\right)^2 u_k^h(x)=\varphi_k^h(x), \\ \varphi_k^h(x)=f^h(t_k,x)+\frac{\tau^2}{12}\left(\frac{f^h(t_{k+1},x)-2f^h(t_k,x)+f^h(t_{k-1},x)}{\tau^2}+A_h^x f^h(t_k,x)\right), \\ t_k=k\tau,\ 1\le k\le N-1, N\tau=1, x\in\Omega_h, \\ u_0^h(x)=\varphi^h(x);\ u_N^h(x)=\frac{\tau}{3}\left(\rho(t_0)u_0^h(x)+\rho(t_N)u_N^h(x)\right) \\ +\frac{\tau}{3}\left(4\sum\limits_{k=1}^{\frac{N}{2}}\rho(t_{2k-1})u_{2k-1}^h(x)+2\sum\limits_{k=1}^{\frac{N}{2}-1}\rho(t_{2k})u_{2k}^h(x)\right)+\psi^h(x),\ x\in\overline{\Omega_h}. \end{cases} \tag{2.20}$$

Theorem 2.5. Let τ and $|h|=\sqrt{h_1^2+...+h_n^2}$ be sufficiently small positive numbers. Then under the assumption (1.3) , the solution of the difference scheme (2.20) satisfies the following stability and almost coercive stability estimates:

$$\max_{1\le k\le N-1}\left\|u_k^h\right\|_{L_{2h}}\le M_4(\delta)\left[\max_{1\le k\le N-1}\left\|\varphi_k^h\right\|_{L_{2h}}+\left\|\psi^h\right\|_{L_{2h}}+\left\|\varphi^h\right\|_{L_{2h}}\right],$$

$$\max_{1\le k\le N-1}\left\|\tau^{-2}\left(u_{k+1}^h-2u_k^h+u_{k-1}^h\right)\right\|_{L_{2h}}+\max_{1\le k\le N-1}\left\|u_k^h\right\|_{W_{2h}^2}$$

$$\le M_5(\delta)\left[\ln\frac{1}{\tau+|h|}\max_{1\le k\le N-1}\left\|\varphi_k^h\right\|_{L_{2h}}+\left\|\varphi^h\right\|_{W_{2h}^2}+\left\|\psi^h\right\|_{W_{2h}^2}\right].$$

Here, M_4 and M_5 independent on $\tau, h, \psi^h(x), \varphi^h(x)$ and $\varphi_k^h(x), 1\le k\le N-1$.
The proof of Theorem 2.5. is based on Theorems 2.2. and 2.3., on the estimate (2.16), on the symmetry properties of the difference operator A_h^x defined by the formula (2.18) in L_{2h}, along with the theorem on the coercivity inequality for the solution of the elliptic difference problem in L_{2h} [4].

3. Numerical Results

We consider the Bitsadze-Samarskii type nonlocal boundary-value problem for the elliptic equation

$$\begin{cases} -\frac{\partial^2 u(t,x)}{\partial t^2}-\frac{\partial^2 u(t,x)}{\partial x^2}+u=\pi^2\exp(-t)\sin(\pi x), \\ 0<t<1,\ 0<x<1, u(0,x)=\sin(\pi x), \\ u(1,x)=\int_0^1 e^{-\lambda}u(\lambda,x)d\lambda+(\exp(-1)+\frac{1}{2}\exp(-2)-\frac{1}{2})\sin(\pi x), 0<x<1, \\ u(t,0)=u(t,1)=0,\ 0<t<1. \end{cases} \tag{3.1}$$

The exact solution of this problem is

$$u(t,x)=\exp(-t)\sin(\pi x).$$

In the present part for the approximate solutions of the Bitsadze-Samarskii type nonlocal boundary-value problem (3.1), we will use the second and fourth orders of the accuracy difference schemes with grid intervals $\tau=\frac{1}{N}, h=\frac{1}{M}$ for t and x, respectively. For the approximate solution of the Bitsadze-Samarskii type nonlocal boundary problem (3.1), we consider the set $[0,1]_\tau \times [0,1]_h$ of a family of grid points depending on the small parameters τ and h,

$$\begin{aligned} [0,1]_\tau \times [0,1]_h &= \{(t_k, x_n) : t_k = k\tau,\ \ 1 \le k \le N-1,\ \ N\tau = 1, \\ x_n &= nh,\ \ 1 \le n \le M-1, Mh = 1\}. \end{aligned}$$

Applying the second order of the accuracy difference scheme from [16] for the approximate solutions of the problem (3.1), we get

$$\begin{cases} -\frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2} - \frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2} + u_n^k = f(t_k, x_n), \\ 1 \le k \le N-1, 1 \le n \le M-1, \\ u_n^0 = \varphi(x_n),\ 0 \le n \le M, \\ u_n^N - \sum_{k=1}^N \frac{\tau}{2}\left(\exp(-(k-1)\tau)u_n^{k-1} + \exp(-k\tau)\, u_n^k\right) = \psi(x_n), \\ \psi(x_n) = (\exp(-1) + \frac{1}{2}\exp(-2) - \frac{1}{2})\sin(\pi x_n),\ 0 \le n \le M, \\ u_0^k = u_M^k = 0,\ 0 \le k \le N, \\ f(t_k, x_n) = \pi^2 \exp(-t_k)\sin(\pi x_n), \\ \varphi(x_n) = \sin(\pi x_n). \end{cases} \tag{3.2}$$

We have $(N+1)\times(M+1)$ system of linear equations in (3.2) and we will write them in the matrix form. We can rewrite this system as the following form

$$\begin{cases} \left(\frac{1}{h^2}\right)u_{n+1}^k + \left(-\frac{2}{\tau^2} - \frac{2}{h^2} - 1\right)u_n^k + \left(\frac{1}{h^2}\right)u_{n-1}^k \\ + \left(\frac{1}{\tau^2}\right)u_n^{k-1} + \left(\frac{1}{\tau^2}\right)u_n^{k+1} = f(t_k, x_n), \\ 1 \le k \le N-1,\ 1 \le n \le M-1, \\ u_n^0 = \varphi(x_n),\ 0 \le n \le M, \\ u_n^N - \sum_{k=1}^N \frac{\tau}{2}\left(\exp(-(k-1)\tau)u_n^{k-1} + \exp(-k\tau)\, u_n^k\right) = \psi(x_n), \\ \psi(x_n) = (\exp(-1) + \frac{1}{2}\exp(-2) - \frac{1}{2})\sin(\pi x_n),\ 0 \le n \le M, \\ u_0^k = u_M^k = 0,\ 0 \le k \le N, \\ \varphi(x) = \sin(\pi x_n). \end{cases} \tag{3.3}$$

We denote

$$a = \frac{1}{h^2},\ \ b = -\frac{2}{\tau^2} - \frac{2}{h^2} - 1,\ \ c = \frac{1}{\tau^2},$$

$$\varphi_n^k = \begin{cases} \sin(\pi x_n),\ k = 0, \\ f(t_k, x_n),\ 1 \le k \le N-1, \\ (\exp(-1) + \frac{1}{2}\exp(-2) - \frac{1}{2})\sin(\pi x_n),\ k = N, \end{cases}$$

$$\varphi_n = \begin{bmatrix} \varphi_n^0 \\ \varphi_n^1 \\ \cdot \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1},$$

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & a & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & a & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & a & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & a & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)}, C = A,$$

$$B = \begin{bmatrix} 1 & 0 & 0 & \cdot & 0 & 0 & 0 \\ c & b & c & \cdot & 0 & 0 & 0 \\ 0 & c & b & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & b & c & 0 \\ 0 & 0 & 0 & \cdot & c & b & c \\ -\frac{\tau}{2} & -\frac{\tau}{2}e^{-\tau} & -\frac{\tau}{2}2e^{-2\tau} & \cdot & -\frac{\tau}{2}2e^{-(N-2)\tau} & -\frac{\tau}{2}e^{-(N-1)\tau} & 1-\frac{\tau}{2}e^{-N\tau} \end{bmatrix}_{(N+1)\times(N+1)},$$

and D is an $(N+1)\times(N+1)$ identity matrix, and

$$U_s = \begin{bmatrix} u_s^0 \\ u_s^1 \\ \cdot \\ u_s^N \end{bmatrix}_{(N+1)\times 1},$$

where $s = n-1, n, n+1$. Then, (3.3) can be written as

$$\begin{cases} A\,U_{n+1} + B\,U_n + C\,U_{n-1} = D\varphi_n, \quad 1 \le n \le M-1, \\ U_0 = U_M = \tilde{0}. \end{cases} \tag{3.4}$$

So, we have a second order difference equation with respect to n with matrix coefficients. To solve this difference equation we have applied a procedure of modified Gauss elimination method for difference equation with respect to n matrix coefficients [3]. Hence, we seek a solution of the matrix equation in the following form

$$U_n = \alpha_{n+1}U_{n+1} + \beta_{n+1}, \quad n = M-1,\cdots,2,1 \tag{3.5}$$

where α_n $(n = 1,\cdots,M)$ are $(N+1)\times(N+1)$ square matrix and β_n $(n = 1,\cdots,M)$ are $(N+1)\times 1$ column matrix.

Here

$$\alpha_{n+1} = -(B + C\alpha_n)^{-1} A, \tag{3.6}$$

$$\beta_{n+1} = (B + C\alpha_n)^{-1}(D\varphi_n - C\beta_n), \ n = 1,2,\cdots,M-1. \tag{3.7}$$

For the solution of difference equations, we need to find α_1 and β_1. We can find them from $U_0 = \tilde{0} = \alpha_1 U_1 + \beta_1$. Thus, we have

$$\alpha_1 = \begin{bmatrix} 0 & 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 0 \end{bmatrix}_{(N+1)\times(N+1)}, \beta_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ 0 \end{bmatrix}_{(N+1)\times 1}.$$

For the first step, using formulas (3.6), (3.7) we can compute α_{n+1} and β_{n+1}, $1 \leq n \leq M-1$. Thus, using formula (3.5) and $U_M = \tilde{0}$, we can compute U_n, $1 \leq n \leq M-1$. Applying the fourth order of the accuracy difference scheme (1.2) for the approximate solutions of the problem (3.1), we obtain

$$\begin{cases} -\frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2} + \left(-\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2} + u_n^k\right) + \frac{\tau^2}{12}\left[\frac{-1}{h^2}\left(-\frac{u_{n+2}^k-2u_{n+1}^k+u_n^k}{h^2} + u_{n+1}^k\right)\right. \\ +\frac{2}{h^2}\left(-\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2} + u_n^k\right) - \frac{1}{h^2}\left(-\frac{u_n^k-2u_{n-1}^k+u_{n-2}^k}{h^2} + u_{n-1}^k\right) \\ \left.-\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2} + u_n^k\right] = f(t_k, x_n). \\ 1 \leq k \leq N-1, 2 \leq n \leq M-2, \\ u_n^0 = \varphi(x_n),\ 0 \leq n \leq M, \\ u_n^N - \sum_{k=1}^{\frac{N}{2}} \frac{\tau}{3}\left(\exp(-(2k-2)\tau)u_n^{2k-2} + 4\exp(-(2k-1)\tau)u_n^{2k-1} + \exp(-(2k)\tau)u_n^{2k}\right) \\ = \psi(x_n) = (\exp(-1) + \frac{1}{2}\exp(-2) - \frac{1}{2})\sin(\pi x_n),\ 0 \leq n \leq M, \\ u_0^k = u_M^k = 0,\ 0 \leq k \leq N, \\ u_1^k = \frac{4}{5}u_2^k - \frac{1}{5}u_3^k,\ 0 \leq k \leq N, \\ u_{M-1}^k = \frac{4}{5}u_{M-2}^k - \frac{1}{5}u_{M-3}^k,\ 0 \leq k \leq N, \\ f(t_k, x_n) = \pi^2\exp(-t_k)\sin(\pi x_n), \\ \varphi(x_n) = \sin(\pi x_n). \end{cases} \tag{3.8}$$

We have $(N+1)\times(M+1)$ system of linear equations in (3.8) and we will write them in the matrix form. We can rewrite this system as the following form

$$\begin{cases} \left(-\frac{\tau^2}{12h^4}\right) u_{n+2}^k + \left(\frac{-12h^2-4\tau^2-2h^2\tau^2}{12h^4}\right) u_{n+1}^k + \left(-\frac{1}{12h^2}\right) u_{n+1}^{k-1} \\ + \left(\frac{12h^4+4h^2\tau^2+6\tau^2 24h^2+\tau^2h^4-2\tau^2h^2}{12h^4} + \frac{2}{\tau^2}\right) u_n^k + \left(-\frac{1}{\tau^2}\right) u_n^{k+1} + \left(-\frac{1}{\tau^2}\right) u_n^{k-1} \\ + \left(\frac{-12h^2-4\tau^2-2\tau^2h^2}{12h^4}\right) u_{n-1}^k + \left(-\frac{\tau^2}{12h^4}\right) u_{n-2}^k = f(t_k, x_n), \\ 1 \le k \le N-1,\ 2 \le n \le M-2, \\ u_n^N - \sum_{k=1}^{\frac{N}{2}} \frac{\tau}{3} \left(\exp(-(2k-2)\,\tau) u_n^{2k-2} + 4\exp(-(2k-1)\,\tau) u_n^{2k-1} \right. \\ \left. + \exp(-(2k)\,\tau) u_n^{2k} = \psi(x_n). \right. \\ \psi(x_n) = (\exp(-1) + \frac{1}{2}\exp(-2) - \frac{1}{2}) \sin(\pi x_n),\ 0 \le n \le M, \\ u_n^0 = \varphi(x_n),\ 0 \le n \le M, \\ u_0^k = u_M^k = 0,\ 0 \le k \le N, \\ u_1^k = \frac{4}{5} u_2^k - \frac{1}{5} u_3^k,\ 0 \le k \le N, \\ u_{M-1}^k = \frac{4}{5} u_{M-2}^k - \frac{1}{5} u_{M-3}^k,\ 0 \le k \le N, \\ f(t_k, x_n) = \pi^2 \exp(-t_k) \sin(\pi x_n), \\ \varphi(x_n) = \sin(\pi x_n). \end{cases} \tag{3.9}$$

We denote

$$a = \frac{\tau^2}{12h^4},\ \ b = \frac{-12h^2-4\tau^2-2h^2\tau^2}{12h^4},$$

$$c = \frac{24h^4 + h^4\tau^4 + \tau^2h^4 + 6\tau^4 + 12\tau^2h^4 + 24\tau^2h^2}{12h^4\tau^2},$$

$$d = -\frac{1}{\tau^2},$$

$$e = \frac{-12h^2-4\tau^2-2\tau^2h^2}{12h^4},$$

$$\varphi_n^k = \begin{cases} \sin(\pi x_n),\ k = 0, \\ f(t_k, x_n),\ 1 \le k \le N-1, \\ (\exp(-1) + \frac{1}{2}\exp(-2) - \frac{1}{2}) \sin(\pi x_n),\ k = N, \end{cases}$$

$$\varphi_n = \begin{bmatrix} \varphi_n^0 \\ \varphi_n^1 \\ \cdot \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1},$$

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & a & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & a & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & a & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & a & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & b & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & b & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & b & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & b & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

and

$$C = \begin{bmatrix} 1 & 0 & 0 & \cdot & 0 & 0 & 0 \\ d & c & d & \cdot & 0 & 0 & 0 \\ 0 & d & c & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & c & d & 0 \\ 0 & 0 & 0 & \cdot & d & c & d \\ -\frac{\tau}{3} & -\frac{\tau}{3}e^{-\tau} & -\frac{2\tau}{3}e^{-2\tau} & \cdot & -\frac{2\tau}{3}e^{-(N-2)\tau} & -\frac{\tau}{3}e^{-(N-1)\tau} & 1-\frac{\tau}{3}e^{-N\tau} \end{bmatrix}_{(N+1)\times(N+1)},$$

$$D = \begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & e & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & e & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & e & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & e & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)}, E = A,$$

and $R = D$,

$$U_s = \begin{bmatrix} U_s^0 \\ U_s^1 \\ \cdot \\ U_s^N \end{bmatrix}_{(N+1)\times(1)},$$

where $s = n-2, n-1, n, n+1, n+2$.
Then, (3.9) can be written as

$$\begin{cases} A\,U_{n+2} + B\,U_{n+1} + C\,U_n + DU_{n-1} + EU_{n-2} = R\varphi_n, \quad 2 \le n \le M-2, \\ U_0 = U_M = \tilde{0}, \\ U_1 = \frac{4}{5}U_2 - \frac{1}{5}U_3, 0 \le k \le N, \\ U_{M-1} = \frac{4}{5}U_{M-2} - \frac{1}{5}U_{M-3},\ 0 \le k \le N. \end{cases}$$

For the solution of the last matrix equation, we use the modified Gauss elimination method. We seek a solution of the matrix equation by the following form:

$$U_n = \alpha_{n+1}U_{n+1} + \beta_{n+1}U_{n+2} + \gamma_{n+1}, \quad n = M-2,\cdots,2,1, \tag{3.10}$$

$$U_M = \tilde{0},$$

$$U_{M-1} = \left[(\beta_{M-2} + 5I) - (4I - \alpha_{M-2})\alpha_{M-1}\right]^{-1}\left[(4I - \alpha_{M-2})\gamma_{M-1} - \gamma_{M-2}\right],$$

$$U_{M-2} = \left[(\beta_{M-2} + 5I)\,U_{M-1} + \gamma_{M-2}\right]\left[(4I - \alpha_{M-2})\right]^{-1}.$$

Here, $\alpha_n, \beta_n \ \ (n = 1,\cdots,M)$ are $(N+1)\times(N+1)$ square matrix and γ_n-s are $(N+1)\times 1$ column matrix.

$$\alpha_{n+1} = -(C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n)^{-1}(B + D\beta_n + E\alpha_{n-1}\beta_n), \tag{3.11}$$

$$\beta_{n+1} = -(C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n)^{-1}(A), \tag{3.12}$$

$$\gamma_{n+1} = (C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n)^{-1} \times(R\varphi_n - D\gamma_n - E\alpha_{n-1}\gamma_n - E\alpha_{n-1}\gamma_n - E\gamma_{n-1}), \tag{3.13}$$

where $n = 2 : M-2$.
Here,

$$\alpha_1 = \begin{bmatrix} 0 & 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 0 \end{bmatrix}_{(N+1)\times(N+1)}, \beta_1 = \begin{bmatrix} 0 & 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 0 \end{bmatrix}_{(N+1)\times(N+1)}, \gamma_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ 0 \end{bmatrix}_{(N+1)\times 1},$$

$$\alpha_2 = \begin{bmatrix} \frac{4}{5} & 0 & 0 & \cdot & 0 \\ 0 & \frac{4}{5} & 0 & \cdot & 0 \\ 0 & 0 & \frac{4}{5} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \frac{4}{5} \end{bmatrix}_{(N+1)\times(N+1)}, \beta_2 = \begin{bmatrix} -\frac{1}{5} & 0 & 0 & \cdot & 0 \\ 0 & -\frac{1}{5} & 0 & \cdot & 0 \\ 0 & 0 & -\frac{1}{5} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & -\frac{1}{5} \end{bmatrix}_{(N+1)\times(N+1)},$$

$$\gamma_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ 0 \end{bmatrix}_{(N+1)\times 1}.$$

For the second step, using formulas (3.11), (3.12) and (3.13) we can compute α_{n+1}, β_{n+1}, and γ_{n+1}, $1 \le n \le M-1$. Thus, using formulas (3.10) and $U_M = \tilde{0}$, we can compute U_n, $1 \le n \le M-1$.

Now, we will give the results of the numerical analysis. In order to get the solution of (3.2) and (3.8) we use MATLAB program. The errors are computed by

$$E_M^N = \max_{1\le k\le N-1,} \left(\sum_{n=1}^{M-1} \left| u(t_k, x_n) - u_n^k \right|^2 h \right)^{\frac{1}{2}}$$

of numerical solutions for different values of M and N, where $u(t_k, x_n)$ represents the exact solution and u_n^k represents the numerical solution at (t_k, x_n). The results are shown in Table 1, respectively.

Table 1. Comparison of the errors of different difference schemes for (3.1)

Difference Schemes	$N = M = 20$	$N = M = 40$	$N = M = 80$
Second order DS (3.2)	6.245e-004	1.562e-004	3,906e-005
Fourth order DS (3.8)	1.326e-004	2.906e-005	7.158e-006

4. Conclusion

In this paper, the fourth order of the accuracy difference scheme for the approximate solution of the Bitsadze-Samarskii type nonlocal boundary-value problem with integral condition for elliptic equations is presented. Theorems on the stability estimates, almost coercive stability estimates for the solution of difference scheme for elliptic equations are proved. The theoretical statements for the solution of this difference scheme are supported by the results of a numerical example. As can be seen from Table 1, the fourth order of the accuracy difference scheme is more accurate than the second order of the accuracy difference scheme.

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