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Normal Extensions of a First Order Differential Operator

Z. I. Ismailov[a], M. Sertbaş[a], B. Ö. Güler[a]

[a]*Department of Mathematics, Science Faculty, Karadeniz Technical University, Trabzon, Turkey*

Abstract. In the paper of W.N. Everitt and A. Zettl [26] in scalar case, all selfadjoint extensions of the minimal operator generated by Lagrange-symmetric any order quasi-differential expression with equal deficiency indexes in terms of boundary conditions are described by Glazman-Krein-Naimark method for regular and singular cases in the direct sum of corresponding Hilbert spaces of functions. In this work, by using the method of Calkin-Gorbachuk theory all normal extensions of the minimal operator generated by fixed order linear singular multipoint differential expression $l = (l_-, l_1, \ldots, l_n, l_+)$, $l_\mp = \frac{d}{dt} + A_\mp$, $l_k = \frac{d}{dt} + A_k$ where the coefficients $A_\mp$, A_k are selfadjoint operator in separable Hilbert spaces $H_\mp$, $H_k, k = 1, \ldots, n$, $n \in \mathbb{N}$ respectively, are researched in the direct sum of Hilbert spaces of vector-functions

$$L_2(H_-, (-\infty, a)) \oplus L_2(H_1, (a_1, b_1)) \oplus \ldots \oplus L_2(H_n, (a_n, b_n)) \oplus L_2(H_+, (b, +\infty))$$

$-\infty < a < a_1 < b_1 < \ldots < a_n < b_n < b < +\infty$. Moreover, the structure of the spectrum of normal extensions is investigated.
Note that in the works of A. Ashyralyev and O. Gercek [2, 3] the mixed order multipoint nonlocal boundary value problem for parabolic-elliptic equation is studied in weighed Hölder space in regular case.

1. Introduction

The modelings of many physical phenomenons are expressed as differential operators. Therefore operator theory plays an exceptionally important role in modern mathematics, especially in the modeling of processes of multi-particle quantum mechanics, quantum field theory, the multipoint boundary value problems for differential equations[1-3,16-18,28].

Although the first studies of the theory multipoint differential operators were performed at the beginning of twentieth century, most of them which are about the investigation of the theory and application to spectral problems, have been found since 1950 ([19,21,25-28]).

It is well known that basic results of normal extensions of formal normal operator had been established and developed in [4,6,7]. Unfortunately, applications of this theory to differential operators in Hilbert space have not received the attentions it deserves.

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Email addresses: zameddin.ismailov@gmail.com (Z. I. Ismailov), m.erolsertbas@gmail.com (M. Sertbaş), boguler@ktu.edu.tr (B. Ö. Güler)

In this study, general representation of all normal extensions of the formally normal minimal operator, which is generated by first order linear singular differential-operator expressions with selfadjoint coefficient are described in the direct sum of Hilbert spaces of vector functions in term of boundary values. Also, the spectrum structure of these extensions is investigated.

2. Description of Normal Extensions

Let $H_{\mp}$, $H_k, k = 1, 2, \ldots, n$, $n \in \mathbb{N}$ be separable Hilbert spaces, $-\infty < a < a_1 < b_1 < \ldots < a_n < b_n < b < +\infty$ and $dimH_- = dimH_+ \leq +\infty$. In the direct sum of Hilbert spaces

$$L_2(H_-, (-\infty, a)) \oplus L_2(H_1, (a_1, b_1)) \oplus \ldots \oplus L_2(H_n, (a_n, b_n)) \oplus L_2(H_+, (b, +\infty))$$

of vector-functions we consider the linear multipoint differential expression

$$l(u) = \begin{cases} u'_- + A_- u_-, & -\infty < t < a \\ u'_k + A_k u_k, & a_k < t < b_k,\ k = 1, \ldots, n \\ u'_+ + A_+ u_+, & b < t < +\infty, \end{cases} \tag{1}$$

where $u = (u_-, u_1, \ldots, u_n, u_+), A_{\mp} : H_{\mp} \to H_{\mp}, A_k : H_k \to H_k$, are linear selfadjoint operators and $A_- \leq 0$, $A_+ \geq 0$, $A_k \geq E_k$, (E_k is the identical operator in H_k) $k = 1, \ldots, n$.

In the Hilbert space $L_2(H_-, (-\infty, a)) \oplus L_2(H_1, (a_1, b_1)) \oplus \ldots \oplus L_2(H_n, (a_n, b_n)) \oplus L_2(H_+, (b, +\infty))$ the operators $L_0 := L_0(1, 1, 1) = L_{-0} \oplus L_{10} \oplus \ldots \oplus L_{n0} \oplus L_{+0}$ and $L := L(1, 1, 1) = L_- \oplus L_1 \oplus \ldots \oplus L_n \oplus L_+$ are called minimal and maximal (multipoint) operators generated by the differential expression (1), respectively.

Now we give some notations for convenience as follows

$$L_2(1, 1, 1) := L_2(H_-, (-\infty, a)) \oplus L_2(H_1, (a_1, b_1)) \oplus \ldots \oplus L_2(H_n, (a_n, b_n)) \oplus L_2(H_+, (b, +\infty))$$
$$L_2(1, 0, 1) := L_2(H_-, (-\infty, a)) \oplus 0_1 \oplus \ldots \oplus 0_n \oplus L_2(H_+, (b, +\infty))$$
$$L_2(0, 1_k, 0) := 0_- \oplus 0_1 \oplus \ldots \oplus 0_{k-1} \oplus L_2(H_k, (a_k, b_k)) \oplus 0_{k+1} \oplus \ldots \oplus 0_+, \quad k = 1, \ldots, n.$$

It is known that any symmetric operator with equal deficiency indices has at least one space of boundary values [9]. Therefore, construct a space of boundary values for the minimal operators $M_0(1, 0, 1)$ and $M_0(0, 1_k, 0)$, $k = 1 \ldots, n$ generated by linear singular differential expressions of first order in the form

$$(m_-(u_-), 0_1, \ldots, 0_n, m_+(u_+)) = \left(-i\frac{du_-}{dt}, 0_1, \ldots, 0_n, -i\frac{du_+}{dt}\right),$$
$$(0_-, 0_1, \ldots, 0_{k-1}, m_k(u_k), 0_{k+1}, \ldots, 0_n, 0_+) = \left(0_-, 0_1, \ldots, 0_{k-1}\ -i\frac{du_k}{dt}, 0_{k+1}, \ldots, 0_n, 0_+\right),$$

in the direct sum $L_2(1, 0, 1)$ and $L_2(0, 1_k, 0)$, respectively. Moreover the minimal operators $M_0(1, 0, 1)$ and $M_0(0, 1_k, 0)$ are closed symmetric operators in $L_2(1, 0, 1)$ and $L_2(0, 1_k, 0)$ with deficiency indices $(\dim H_-, \dim H_+)$ and $(\dim H_k, \dim H_k)$.

Note that since Hilbert spaces H_- and H_+ are separable and $dimH_- = dimH_+$, then there exits a unitary operator $V : H_- \to H_+$ [15].

Lemma 2.1. *The triplet* $(H_+, \gamma_1, \gamma_2)$, *where*

$$\gamma_1 : D(M_0^*) \to H_+, \quad \gamma_1(u) = \frac{1}{i\sqrt{2}}(u_+(b) + Vu_-(a)),$$
$$\gamma_2 : D(M_0^*) \to H_+, \quad \gamma_2(u) = \frac{1}{\sqrt{2}}(u_+(b) - Vu_-(a)),$$
$$u = (u_-, 0_1, \ldots, 0_n, u_+) \in D(M_0^*)$$

is a space of boundary values of the minimal operator $M_0(1,0,1)$ *in* $L_2(1,0,1)$.

Proof. For arbitrary $u = (u_-, u_1, \dots u_n, u_+)$ and $v = (v_-, v_1, \dots, v_n, v_+)$ from $D((L_{-0} \oplus 0_1 \oplus \dots \oplus 0_n \oplus L_{+0})^*)$ the validity of the equality

$$(Mu, v)_{L_2(1,0,1)} - (u, Mv)_{L_2(1,0,1)} = (\gamma_1(u), \gamma_2(v))_{H_+} - (\gamma_2(u), \gamma_1(v))_{H_+}$$

can be easily verified. Now for any given elements $f, g \in H_+$, we will find the function $u = (u_-, u_1, \dots, u_n, u_+) \in D((L_{-0} \oplus 0_1 \oplus \dots \oplus 0_n \oplus L_{+0})^*)$ such that

$$\gamma_1(u) = \frac{1}{i\sqrt{2}}(Vu_-(a) + u_+(b)) = f \quad \text{and} \quad \gamma_2(u) = \frac{1}{\sqrt{2}}(Vu_-(a) - u_+(b)) = g$$

that is,

$$Vu_-(a) = (if + g)/\sqrt{2} \quad \text{and} \quad u_+(b) = (if - g)/\sqrt{2}.$$

Since $V : H_- \to H_+$ is an isomorphism, so that we can choose the functions $u_-(t)$, $u_+(t)$ in the following form

$$\begin{aligned} &u_-(t) = e^{\frac{t-a}{2}} V^*(if + g)/\sqrt{2}, && \text{with} \quad t < a; \\ &u_k(t) = 0_k, && \text{with} \quad a_k < t < b_k,\ k = 1, \dots, n; \\ &u_+(t) = e^{\frac{b-t}{2}}(if - g)/\sqrt{2}, && \text{with} \quad t > b \end{aligned}$$

then it is clear that $(u_-, u_1, \dots, u_n, u_+) \in D((M_{-0} \oplus 0_1 \oplus \dots \oplus 0_n \oplus M_{+0})^*)$ and $\gamma_1(u) = f$, $\gamma_2(u) = g$. □

Lemma 2.2. *The triplet* $\left(H_k, \gamma_1^{(k)}, \gamma_2^{(k)}\right)$,

$$\gamma_1^{(k)} : D((M_{k0})^*) \to H_k, \quad \gamma_1^{(k)}(u_k) = \frac{1}{i\sqrt{2}}(u_k(a_k) + u_k(b_k)),$$
$$\gamma_2^{(k)} : D((M_{k0})^*) \to H_k, \quad ,\gamma_2^{(k)}(u_k) = \frac{1}{\sqrt{2}}(u_k(a_k) - u_k(b_k)),\ u_k \in D((M_{k0})^*)$$

is a space of boundary values of the minimal operator M_{k0} *in the Hilbert space* $L_2(0, 1_k, 0)$, $k = 1, \dots, n$.

Theorem 2.3. *If the minimal operators* L_{-0}, L_{+0} *and* $L_{k0}, k = 1, \dots, n$ *are formally normal, then the following correlations*

$$\begin{aligned} &D(L_{-0}) \subset W_2^1(H_-, (-\infty, a)), \quad A_- D(L_{-0}) \subset L_2(H_-, (-\infty, a)), \\ &D(L_{k0}) \subset W_2^1(H_k, (a_k, b_k)), \quad A_k D(L_{k0}) \subset L_2(H_k, (a_k, b_k)), \\ &D(L_{+0}) \subset W_2^1(H_+, (b, \infty)), \quad A_+ D(L_{+0}) \subset L_2(H_+, (b, \infty)). \end{aligned}$$

are true.

Proof. In acceptation of theorem, for any $u_- \in D(L_{-0}) \subset D(L_{-0}^*)$

$$u_-' + A_- u_- \in L_2(H_-, (-\infty, a)) \text{ and } -u_-' + A_- u_- \in L_2(H_-, (-\infty, a)),$$

are true, so

$$u_-' \in L_2(H_-, (-\infty, a)) \quad \text{and} \quad A_- u_- \in L_2(H_-, (-\infty, a)).$$

These mean that

$$D(L_{-0}) \subset W_2^1(H_-, (-\infty, a)) \text{ and } A_- D(L_{-0}) \subset L_2(H_-, (-\infty, a)).$$

The other parts of theorem can be proved similarly. □

The following result can be easily established.

Lemma 2.4. *Every normal extension of $L_0(1,1,1)$ in $L_2(1,1,1)$ is a direct sum of normal extensions of the minimal operator $L_0(1,0,1)$ in $L_2(1,0,1)$ and minimal operator L_{k0} in $L_2(H_k,(a_k,b_k)), k=1,\dots,n$.*

Furthermore, the next theorem can be proved by using the method in [9-14,22-24], Lemma 2.1 and Lemma 2.2.

Theorem 2.5. *Assume that*

$$(-A_-)^{1/2}W_2^1(H_-,(-\infty,a)) \subset W_2^1(H_-,(-\infty,a))$$
$$A_k^{1/2}W_2^1(H_k,(a_k,b_k)) \subset W_2^1(H_k,(a_k,b_k)),$$
$$A_+^{1/2}W_2^1(H_+,(b,\infty)) \subset W_2^1(H_+,(b,\infty))$$

$\tilde{L}$ is a normal extension of the minimal operator L_0 in the Hilbert space $L_2(1,1,1)$ generated by differential expression (1) iff there exits $W_0 : H_- \to H_+$, $W_k, A_k^{-1/2}W_kA_k^{-1/2} : H_k \to H_k$ are unitary operators and the boundary conditions

$$u_+(b) = W_0u_-(a), \quad u_-(a) \in \ker(-A_-)^{1/2}, \quad u_+(b) \in \ker A_+^{1/2}, \tag{2}$$
$$u_k(b_k) = W_ku_k(a_k), \tag{3}$$

are held. Also, these unitary operators are uniquely determined by the extension $\tilde{L}$ i.e., $\widetilde{L} = L_W$, $W = (W_0,W_1,\dots,W_n)$.

Corollary 2.6. *If A_- or A_+ is an injective operator, then the minimal operator $L_0(1,1,1)$ have not any normal extension, i.e. it is maximally formal normal in $L_2(1,1,1)$.*

Corollary 2.7. *If there is at least one normal extension of the minimal operator $L_0(1,1,1)$, then the relations*

$$\dim\ker(-A_-)^{1/2} = \dim\ker A_+^{1/2} > 0$$

are true.

3. The Spectrum of the Normal Extensions

The structure of the spectrum of the normal extension L_W in $L_2(1,1,1)$ will be researched in this section. In this case by the Lemma 2.4 it is easy to see that

$$L_W = L_{W_0} \oplus L_{W_1} \oplus \dots \oplus L_{W_n},$$

where L_{W_0} and L_{W_k} are normal extensions of the minimal operators $L_0(1,0,1)$ and $L_0(0,1_k,0)$ in the Hilbert spaces L_2 and $L_2(0,1_k,0)$, $k=1,\dots,n$ respectively. Also, it will be supposed that $A_- = A_-^* \leq 0$, $A_+ = A_+^* \geq 0, A_k = A_k^* \geq 0$ and $0 \in \sigma_p((-A_-)^{1/2}) \cap \sigma_p(A_+^{1/2})$.

Theorem 3.1. *The point spectrum of any normal extension L_{W_0} of the minimal operator $L_0(1,0,1)$ in the Hilbert space $L_2(1,0,1)$ is empty, i.e. $\sigma_p(L_{W_0}) = \emptyset$.*

Proof. Assume that λ is an eigenvalue for the normal operator L_{W_0}. In this case, the following problem is obtained

$$L_{W_0}u = \lambda u, \quad \lambda = \lambda_r + i\lambda_i \in \mathbb{C}, \quad u = (u_-,0_1,\dots,0_n,u_+) \in D(L_{W_0}).$$

Because L_{W_0} is a normal operator, the equation

$$L_{W_0}^*u = \lambda u, \quad \lambda = \lambda_r - i\lambda_i \in \mathbb{C}, \quad u = (u_-,0_1,\dots,0_n,u_+) \in D(L_{W_0})$$

is true. From these equations, the solution is like that

$$u_-(\lambda;t) = e^{i\lambda_i(t-a)}f_-^*, \quad t<a,\ f_1^*\in H_{-1/2}((-A_-)),$$
$$u_+(\lambda;t) = e^{i\lambda_i(t-b)}f_+^*, \quad t>b,\ f_3^*\in H_{-1/2}(A_+)$$
$$f_3^* = W_1 f_1^*.$$

Because $\sigma_p((A_-))\cap\sigma_p(A_2^{1/2}) = 0$, it is clear that $\lambda_r = 0$. If $u_-(\lambda;.)\in L_2(H_-,(-\infty,a))$ and $u_+(\lambda;.)\in L_2(H_+,(b,\infty))$, then $f_-^*=0$, $f_+^*=0$. This implies that $u_-=0$ and $u_+=0$ in $L_2(1,1,1)$, so $\sigma_p(L_{W_1})=\emptyset$. □

Since residual spectrum of any normal operators in any Hilbert space is empty, it is sufficient to investigate the continuous spectrum of the normal extensions L_{W_0} of the minimal operator $L_0(1,0,1)$ in the Hilbert space $L_2(1,0,1)$.

Theorem 3.2. *For any normal extension L_{W_0} of the minimal operator $L_0(1,0,1)$ in the Hilbert space $L_2(1,0,1)$ the relations $i\mathbb{R}\subset\sigma_c(L_{W_0})\subset\sigma(A_-)\cup\sigma(A_+)+i\mathbb{R}$ are held.*

Proof. For the spectrum of normal operators spectrum [8], the following relation is true,

$$\sigma(L_{W_0})\subset\sigma(\operatorname{Re}L_{W_0})+i\sigma(\operatorname{Im}L_{W_0}),$$

where $\operatorname{Re}(L_{W_0}) = \overline{\frac{L_{W_0}+L_{W_0}^*}{2}}$ and $\operatorname{Im}(L_{W_0}) = \overline{\frac{L_{W_0}-L_{W_0}^*}{2i}}$ are selfadjoint operators.

Now consider selfadjoint operator $T: L_2(1,0,1)\to L_2(1,0,1)$,

$$Tu(t) := \begin{cases} A_-u_-(t), & -\infty<t<a \\ 0_k, & a_k<t<b_k, k=1,\dots,n \\ A_+u_+(t), & b<t<+\infty, \end{cases}.$$

Hence, $\frac{L_{W_0}+L_{W_0}^*}{2}$ is densely defined symmetric bounded and $\frac{L_{W_0}+L_{W_0}^*}{2}\subset T$ in the Hilbert space $L_2(1,0,1)$. Therefore, $\operatorname{Re}(L_{W_0})=T$ is obtained. Also, the operator $T=A_-\otimes E\oplus A_+\otimes E$ can be written, so from [5] and [29] the spectrum of T is equal to $\sigma(A_-)\cup\sigma(A_+)$. Furthermore $\sigma(\operatorname{Im}(L_{W_0}))=\mathbb{R}$ can be easily seen.

Finally, for any $\lambda = i\lambda_i\in\mathbb{C}$ the general solution of the boundary value problem

$$u_-' + A_-u_- = i\lambda_i u_- + f_-, \quad u_-, f_-\in L_2(H_-,(-\infty,a)),$$
$$u_+' + A_+u_+ = i\lambda_i u_+ + f_+, \quad u_+, f_+\in L_2(H_+,(b,\infty)),\ \lambda_i\in\mathbb{R},$$
$$u_+(b) = W_0u_-(a),\ u_-(a)\in\ker(-A_-)^{1/2}, u_+(b)\in\ker A_+^{1/2}$$

will be of the form

$$u_-(i\lambda_i;t) = e^{-(A_--i\lambda_i)(t-a)}f_{i\lambda_i} - \int_t^a e^{-(A_--i\lambda_i)(t-s)}f_-(s)ds, \quad t<a,$$
$$u_+(i\lambda_i;t) = \mathrm{e}^{-(A_+-i\lambda_i)(t-b)}g_{i\lambda_i} + \int_b^t \mathrm{e}^{-(A_+-i\lambda_i)(t-s)}f_+(s)\mathrm{ds}, \quad t>b,$$
$$g_{i\lambda_i} = W_0 f_{i\lambda_i}.$$

In this case,

$$e^{-(A_--i\lambda_i)(t-a)}f_{i\lambda_i}\in L_2(H,(-\infty,a)), \quad \mathrm{e}^{-(A_+-i\lambda_i)(t-b)}g_{i\lambda_i}\in L_2(H_+,(b,\infty))$$

for any $g_{i\lambda_i} \in H_-, f_{i\lambda_i} \in H_+$. If $f_-(t) = e^{i\lambda_i t}e^{-(t-a)}f^*$, $f^* \in \ker(-A_-)^{1/2}$, $t < a$, then

$$\int_t^a e^{-(A_- - i\lambda_i)(t-s)}f_-(s)ds = e^{-i\lambda_i t}\int_t^a e^{-(s-a)}f^* ds$$
$$= e^{-i\lambda_i t}(e^{-(t-a)} - 1)f^*, \quad t < a_1.$$

Therefore,

$$\int_{-\infty}^a \|e^{-i\lambda_i t}(e^{-(t-a)} - 1)f^*\|^2 dt = \int_{-\infty}^a \|e^{-i\lambda_i t}(e^{-(t-a)} - 1)f^*\|^2 dt$$
$$= \int_{-\infty}^a (e^{-2(t-a)} - 2e^{-(t-a)} + 1)dt\|f^*\|^2 = \infty.$$

Consequently, we have $f_-(t) \in L_2(H_-, (-\infty, a))$, $u_-(i\lambda_i; t) \notin L_2(H_-, (-\infty, a))$. This implies that for any $\lambda \in \mathbb{C}$, an operator $L_{W_0} - \lambda$ is one-to-one in $L_2(1, 0, 1)$, but it is not an onto transformation. Hence the proof is completed. □

Theorem 3.3. *The spectrum of the normal extension* L_{W_k} *of the minimal operator* $L_0(0, 1_k, 0)$ *in the Hilbert space* $L_2(0, 1_k, 0)$, $k = 1, \dots, n$ *is of the form*

$$\sigma(L_{W_k}) = \Big\{\lambda \in \mathbb{C}\colon \lambda = \frac{1}{a_k - b_k}(\ln|\mu| + i\ \arg\mu + 2n\pi i), n \in \mathbb{Z},$$
$$\mu \in \sigma(W_k^* e^{-A_k(b_k - a_k)}),\ 0 \le \arg\mu < 2\pi\Big\}$$

Theorem 3.4. *For the spectrum* $\sigma(L_W)$ *of any normal extension* L_W*, it is true that*

$$\sigma_p(L_W) = \left(\bigcup_{k=1}^n \sigma(L_{W_k})\right), \quad \sigma_c(L_W) = \left(\bigcup_{k=1}^n \sigma_p(L_{W_k})\right)^c \cap \left(\bigcup_{k=0}^n \sigma_c(L_{W_k})\right).$$

Proof. If S_k, $k = 1, \dots, m$, $m \in \mathbb{N}$ are linear closed operators in any Hilbert spaces $\mathfrak{H}_k$, by using [29] we have

$$\sigma_p\left(\bigoplus_{k=1}^m S_k\right) = \bigcup_{k=1}^m \sigma_p(S_k),$$
$$\sigma_c\left(\bigoplus_{k=1}^m S_k\right) = \left(\bigcup_{k=1}^m \sigma_p(S_k)\right)^c \cap \left(\bigcup_{k=1}^m \sigma_r(S_k)\right)^c \cap \left(\bigcup_{k=1}^m \sigma_c(S_k)\right).$$

Therefore, the relations of theorem is obtained by using last equalities, Theorem 3.1 and Theorem 3.3. □

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