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# Existence of Positive Solutions for Nonlinear Third-Order *m*-Point Boundary Value Problems on Time Scales

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**Abstract.** In this paper, we investigate the existence of double positive solutions for nonlinear third-order $m$-point boundary value problems with $p$-Laplacian on time scales. By using double fixed point theorem, we establish results on the existence of two positive solutions with suitable growth conditions imposed on the nonlinear term. As an application, we give an example to demonstrate our main result.

## 1. Introduction

The theory of time scales was introduced by Stefan Hilger [9] in his PhD thesis in 1988. Theoretically, this new theory has not only unify continuous and discrete equations, but has also exhibited much more complicated dynamics on time scales. Moreover, the study of dynamic equations on time scales has led to several important applications, for example, insect population models, biology, neural networks, heat transfer, and epidemic models, see [1–3, 5, 6, 12].

The study of multi-point boundary value problem for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [10, 11]. Motivated by the study of Il'in and Moiseev [10, 11], Gupta [7] studied nonlinear three-point boundary value problems for nonlinear ordinary differential equations. Since then, more general nonlinear multi-point boundary value problems have been studied by several authors. We refer the reader to [8, 13–17] for some references along this line.

In [14], Ma considered the existence and multiplicity of positive solutions for the $m$-boundary value problems

$$\begin{cases} (p(t)u')' - q(t)u + f(t,u) = 0, \quad 0 < t < 1, \\ au(0) - bp(0)u'(0) = \displaystyle\sum_{i=1}^{m-2} \alpha_i u(\xi_i), \\ cu(1) + dp(1)u'(1) = \displaystyle\sum_{i=1}^{m-2} \beta_i u(\xi_i). \end{cases}$$

The main tool is Guo-Krasnoselskii fixed point theorem.

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In [16], Xu was concerned with the existence of positive solutions for the following third-order $p$-Laplacian $m$-point boundary value problems on time scales

$$\begin{cases} (\phi_p(u^{\Delta\nabla}(t)))^{\nabla} + a(t)f(t,u(t)) = 0, \quad t \in [0,T]_{\mathbb{T}}, \\ \beta u(0) - \gamma u^{\Delta}(0) = 0,\ u(T) = \sum_{i=1}^{m-2} a_i u(\xi),\ \phi_p(u^{\Delta\nabla}(0)) = \sum_{i=1}^{m-2} b_i \phi_p(u^{\Delta\nabla}(\xi_i)) \end{cases}$$

Xu obtained the existence of positive solutions by using fixed-point theorem in cones.

In [8], Han and Kang were concerned with the existence of multiple positive solutions of the third-order $p$-Laplacian dynamic equation on time scales

$$\begin{cases} \left(\phi_p\left(u^{\Delta\Delta}(t)\right)\right)^{\nabla} + f(t,u(t)) = 0,\ t \in [a,b], \\ \alpha u(\rho(a)) - \beta u^{\Delta}(\rho(a)) = 0,\ \gamma u(b) + \delta u^{\Delta}(b) = 0,\ u^{\Delta\Delta}(\rho(a)) = 0. \end{cases}$$

By using fixed-point theorems in cones, they obtained the existence of multiple positive solutions for singular nonlinear boundary value problem.

In [17], Yang and Yan studied the following third-order Sturm-Liouville boundary value problem with p-Laplacian

$$\begin{cases} (\phi_p(u''(t)))' + f(t,u(t)) = 0, \quad t \in (0,1), \\ \alpha u(0) - \beta u'(0) = 0, \\ \gamma u(1) + \delta u'(1) = 0, \\ u''(0) = 0, \end{cases}$$

By using the fixed point index method, they established the existence of at least one or at least two positive solutions for the third-order Sturm-Liouville boundary value problem with $p$-Laplacian.

Motivated by the above results, in this study, we consider the following third-order $p$-Laplacian boundary value problem (BVP) on time scales:

$$\begin{cases} \left(\phi_p\left(u^{\Delta\Delta}(t)\right)\right)^{\Delta} + q(t)f(t,u(t)) = 0,\ t \in [0,1]_{\mathbb{T}}, \\ au(0) - bu^{\Delta}(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \\ cu(1) + du^{\Delta}(1) = \sum_{i=1}^{m-2} \beta_i u(\xi_i), \\ u^{\Delta\Delta}(0) = 0, \end{cases} \tag{1.1}$$

where $\mathbb{T}$ is a time scale, $0, 1 \in \mathbb{T}$, $[0,1]_{\mathbb{T}} = [0,1] \cap \mathbb{T}$, $\phi_p(s)$ is a $p$-Laplacian operator, i.e., $\phi_p(s) = |s|^{p-2}s$ for $p > 1$, $(\phi_p)^{-1}(s) = \phi_q(s)$ where $\frac{1}{p} + \frac{1}{q} = 1$.

We assume that following conditions hold:

(C1) $a, b, c, d \in [0,\infty)$ with $ac + ad + bc > 0$; $\alpha_i, \beta_i \in [0,\infty)$, $\xi_i \in (0,1)_{\mathbb{T}}$ for $i \in \{1, 2, ..., m-2\}$,

(C2) $f \in C([0,1]_{\mathbb{T}} \times \mathbb{R}^+, \mathbb{R}^+)$,

(C3) $q \in C([0,1]_{\mathbb{T}}, \mathbb{R}^+)$.

By using the double fixed point theorem [4], we get the existence of at least two positive solution for the BVP (1.1). In fact, our result is also new when $\mathbb{T} = \mathbb{R}$ (the differential case) and $\mathbb{T} = \mathbb{Z}$ (the discrete case). Therefore, the result can be considered as a contribution to this field.

This paper is organized as follows. In Section 2, we provide some definitions and preliminary lemmas which are key tools for our main result. We give and prove our main result in Section 3. Finally, in Section 4, we give an example to demonstrate our main result.

## 2. Preliminaries

In this section, to state the main results of this paper, we need the following lemmas.
We define $\mathbb{B} = C[0,1]$, which is a Banach space with the norm

$$\|u\| = \max_{t\in[0,1]_{\mathbb{T}}} |u(t)|.$$

Define the cone $\mathcal{P} \subset \mathbb{B}$ by

$$\mathcal{P} = \{u \in \mathbb{B} : u(t) \text{ is nonnegative, nondecreasing and concave on } [0,1]_{\mathbb{T}}\}.$$

Denote by $\theta$ and $\varphi$, the solutions of the corresponding homogeneous equation

$$\left(\phi_p\left(u^{\Delta\Delta}(t)\right)\right)^{\Delta} = 0,\ t \in [0,1]_{\mathbb{T}}, \tag{2.1}$$

under the initial conditions

$$\begin{cases} \theta(0) = b, & \theta^{\Delta}(0) = a, \\ \varphi(1) = d, & \varphi^{\Delta}(1) = -c. \end{cases} \tag{2.2}$$

Using the initial conditions (2.2), we can deduce from equation (2.1) for $\theta$ and $\varphi$ the following equations:

$$\theta(t) = b + at,\ \varphi(t) = d + c(1-t). \tag{2.3}$$

Set

$$\Delta := \begin{vmatrix} -\sum_{i=1}^{m-2}\alpha_i(b + a\xi_i) & \rho - \sum_{i=1}^{m-2}\alpha_i(d + c(1-\xi_i)) \\ \rho - \sum_{i=1}^{m-2}\beta_i(b + a\xi_i) & -\sum_{i=1}^{m-2}\beta_i(d + c(1-\xi_i)) \end{vmatrix}, \tag{2.4}$$

and

$$\rho := ad + ac + bc. \tag{2.5}$$

**Lemma 2.1.** *Let* $(C1)-(C3)$ *hold. Assume that*

(C4) $\Delta \neq 0$.

*If* $u \in C[0,1]$ *is a solution of the equation*

$$u(t) = \int_0^1 G(t,s)\,\phi_q\left(\int_0^s q(\tau)f(\tau,u(\tau))\,\Delta\tau\right)\Delta s + A(f)(b+at) + B(f)(d + c(1-t)), \tag{2.6}$$

*where*

$$G(t,s) = \frac{1}{\rho}\begin{cases} (b + a\sigma(s))(d + c(1-t)), & \sigma(s) \le t, \\ (b + at)(d + c(1-\sigma(s))), & t \le s, \end{cases} \tag{2.7}$$

$$A(f) := \frac{1}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2}\alpha_i\left(\int_0^1 G(\xi_i,s)\,\phi_q\left(\int_0^s q(\tau)f(\tau,u(\tau))\,\Delta\tau\right)\Delta s\right) & \rho - \sum_{i=1}^{m-2}\alpha_i(d + c(1-\xi_i)) \\ \sum_{i=1}^{m-2}\beta_i\left(\int_0^1 G(\xi_i,s)\,\phi_q\left(\int_0^s q(\tau)f(\tau,u(\tau))\,\Delta\tau\right)\Delta s\right) & -\sum_{i=1}^{m-2}\beta_i(d + c(1-\xi_i)) \end{vmatrix}, \tag{2.8}$$

*and*

$$B(f) := \frac{1}{\Delta}\begin{vmatrix} -\sum_{i=1}^{m-2}\alpha_i\,(b+a\xi_i) & \sum_{i=1}^{m-2}\alpha_i\left(\int_0^1 G\,(\xi_i,s)\,\phi_q\left(\int_0^s q(\tau)f\,(\tau,u(\tau))\,\triangle\tau\right)\triangle s\right) \\ \rho-\sum_{i=1}^{m-2}\beta_i\,(b+a\xi_i) & \sum_{i=1}^{m-2}\beta_i\left(\int_0^1 G\,(\xi_i,s)\,\phi_q\left(\int_0^s q(\tau)f\,(\tau,u(\tau))\,\triangle\tau\right)\triangle s\right)\end{vmatrix}, \tag{2.9}$$

*then u is a solution of the boundary value problem* (1.1).

*Proof.* Let $u$ satisfies the integral equation (2.6), then $u$ is a solution of the boundary value problem (1.1). Then we have

$$u(t) = \int_0^1 G\,(t,s)\,\phi_q\left(\int_0^s q(\tau)f\,(\tau,u(\tau))\,\triangle\tau\right)\triangle s + A(f)(b+at) + B(f)(d+c(1-t)),$$

i.e.,

$$\begin{aligned} u(t) &= \int_0^t \frac{1}{\rho}(b+a\sigma(s))(d+c(1-t))\phi_q\left(\int_0^s q(\tau)f\,(\tau,u(\tau))\,\triangle\tau\right)\triangle s \\ &+ \int_t^1 \frac{1}{\rho}(b+at)(d+c(1-\sigma(s)))\phi_q\left(\int_0^s q(\tau)f\,(\tau,u(\tau))\,\triangle\tau\right)\triangle s \\ &+ A(f)(b+at)+B(f)(d+c(1-t)), \end{aligned}$$

$$\begin{aligned} u^{\triangle}(t) &= -\int_0^t \frac{c}{\rho}(b+a\sigma(s))\phi_q\left(\int_0^s q(\tau)f\,(\tau,u(\tau))\,\triangle\tau\right)\triangle s \\ &+ \int_t^1 \frac{a}{\rho}\,(d+c(1-\sigma(s)))\,\phi_q\left(\int_0^s q(\tau)f\,(\tau,u(\tau))\,\triangle\tau\right)\triangle s + A(f)a - B(f)c. \end{aligned}$$

So that

$$\begin{aligned} u^{\triangle\triangle}(t) &= \frac{1}{\rho}\,(-c(b+a\sigma(t)) - a(d+c(1-\sigma(t))))\,\phi_q\left(\int_0^t q(\tau)f(\tau,u(\tau))\triangle\tau\right) \\ &= \frac{1}{\rho}\,(-(ad+ac+bc))\,\phi_q\left(\int_0^t q(\tau)f(\tau,u(\tau))\triangle\tau\right) \\ &= -\phi_q\left(\int_0^t q(\tau)f(\tau,u(\tau))\triangle\tau\right), \end{aligned}$$

$$\phi_p\,(u^{\triangle\triangle}(t)) = -\int_t^1 q(\tau)f(\tau,u(\tau))\triangle\tau,$$
$$\left(\phi_p\,(u^{\triangle\triangle}(t))\right)^{\triangle} = -q(t)f(t,u(t)),$$
$$\left(\phi_p\,(u^{\triangle\triangle}(t))\right)^{\triangle} + q(t)f(t,u(t)) = 0.$$

Since

$$\begin{aligned} u(0) &= \int_0^1 \frac{b}{\rho}\,(d+c(1-\sigma(s)))\,\phi_q\left(\int_0^s q(\tau)f\,(\tau,u(\tau))\,\triangle\tau\right)\triangle s + A(f)b + B(f)(d+c), \\ u^{\triangle}(0) &= \int_0^1 \frac{a}{\rho}\,(d+c(1-\sigma(s)))\,\phi_q\left(\int_0^s q(\tau)f\,(\tau,u(\tau))\,\triangle\tau\right)\triangle s + A(f)a - B(f)c, \end{aligned}$$

we have that

$$au(0) - bu^{\triangle}(0) = B(f)\rho$$

$$= \sum_{i=1}^{m-2} \alpha_i \left[ \int_0^1 G(\xi_i, s)\, \phi_q \left( \int_0^s q(\tau) f(\tau, u(\tau)) \triangle\tau \right) \triangle s + A(f)(b + a\xi_i) + B(f)(d + c(1 - \xi_i)) \right]. \tag{2.10}$$

Since

$$\begin{aligned} u(1) &= \int_0^1 \frac{d}{\rho}(b + a\sigma(s))\, \phi_q \left( \int_0^s q(\tau) f(\tau, u(\tau)) \triangle\tau \right) \triangle s + A(f)(b + a) + B(f)d, \\ u^{\triangle}(1) &= -\int_0^1 \frac{c}{\rho}(b + a\sigma(s))\, \phi_q \left( \int_0^s q(\tau) f(\tau, u(\tau)) \triangle\tau \right) \triangle s + A(f)a - B(f)c, \end{aligned}$$

we have that

$$cu(1) + du^{\triangle}(1) = A(f)\rho$$

$$= \sum_{i=1}^{m-2} \beta_i \left[ \int_0^1 G(\xi_i, s)\, \phi_q \left( \int_0^s q(\tau) f(\tau, u(\tau)) \triangle\tau \right) \triangle s + A(f)(b + a\xi_i) + B(f)(d + c(1 - \xi_i)) \right]. \tag{2.11}$$

From (2.10) and (2.11), we get that

$$\begin{cases} \left[ -\sum_{i=1}^{m-2} \alpha_i (b + a\xi_i) \right] A(f) + \left[ \rho - \sum_{i=1}^{m-2} \alpha_i (d + c(1 - \xi_i)) \right] B(f) \\ = \sum_{i=1}^{m-2} \alpha_i \left( \int_0^1 G(\xi_i, s)\, \phi_q \left( \int_0^s q(\tau) f(\tau, u(\tau)) \triangle\tau \right) \triangle s \right) \\ \\ \left[ \rho - \sum_{i=1}^{m-2} \beta_i (b + a\xi_i) \right] A(f) + \left[ -\sum_{i=1}^{m-2} \beta_i (d + c(1 - \xi_i)) \right] B(f) \\ = \sum_{i=1}^{m-2} \beta_i \left( \int_0^1 G(\xi_i, s)\, \phi_q \left( \int_0^s q(\tau) f(\tau, u(\tau)) \triangle\tau \right) \triangle s \right) \end{cases},$$

which implies that $A(f)$ and $B(f)$ satisfy (2.8) and (2.9), respectively. □

**Lemma 2.2.** *Let* $(C1) - (C3)$ *hold. Assume*

(C5) $\Delta < 0,\ \rho - \sum_{i=1}^{m-2} \beta_i (b + a\xi_i) > 0,\ a - \sum_{i=1}^{m-2} \alpha_i > 0.$

*Then for* $u \in C[0, 1]$, *the solution* $u$ *of the problem* (1.1) *satisfies* $u(t) \geq 0$ *for* $t \in [0, 1]_{\mathbb{T}}$.

*Proof.* It is an immediate subsequence of the facts that $G \geq 0$ on $[0, 1]_{\mathbb{T}} \times [0, 1]_{\mathbb{T}}$ and $A(f) \geq 0,\ B(f) \geq 0$. □

**Lemma 2.3.** *Let* $(C1) - (C3)$ *and* $(C5)$ *hold. Assume*

(C6) $c - \sum_{i=1}^{m-2} \beta_i < 0.$

*Then the solution* $u \in C[0, 1]$ *of the problem* (1.1) *satisfies* $u^{\triangle}(t) \geq 0$ *for* $t \in [0, 1]_{\mathbb{T}}$.

*Proof.* Assume that the inequality $u^{\triangle}(t) < 0$ holds. Since $u^{\triangle}(t)$ is nonincreasing on $[0,1]_{\mathbb{T}}$, one can verify that

$$u^{\triangle}(1) \leq u^{\triangle}(t),\ t \in [0,1]_{\mathbb{T}}.$$

From the boundary conditions of the problem (1.1), we have

$$-\frac{c}{d}u(1) + \frac{1}{d}\sum_{i=1}^{m-2}\beta_i u(\xi_i) \leq u^{\triangle}(t) < 0.$$

The last inequality yields

$$-cu(1) + \sum_{i=1}^{m-2}\beta_i u(\xi_i) < 0.$$

Therefore, we obtain that

$$\sum_{i=1}^{m-2}\beta_i u(1) < \sum_{i=1}^{m-2}\beta_i u(\xi_i) < cu(1),$$

i.e.,

$$\left(c - \sum_{i=1}^{m-2}\beta_i\right)u(1) > 0.$$

According to Lemma 2.2, we have that $u(1) \geq 0$. So, $c - \sum_{i=1}^{m-2}\beta_i > 0$. However, this contradicts to condition (C6). Consequently, $u^{\triangle}(t) \geq 0$ for $t \in [0,1]_{\mathbb{T}}$. □

**Lemma 2.4.** *If $u \in \mathcal{P}$, then $u(t) \geq t\,\|u\|$ for $t \in [0,1]_{\mathbb{T}}$.*

*Proof.* Since $u \in \mathcal{P}$ nonnegative and nondecreasing, $\|u\| = \max_{t\in[0,1]_{\mathbb{T}}} |u(t)| = u(1)$. We have from the concavity of $u$,

$$\frac{u(1) - u(0)}{1} \geq \frac{u(1) - u(t)}{1 - t},$$

i.e.,

$$u(t) \geq (1-t)\,u(0) + tu\,(1)\,.$$

Since $u$ is nonnegative, we get

$$u(t) \geq tu\,(1) = t\,\|u\|\,.$$

The proof is complete. □

Now define an operator $T : \mathcal{P} \longrightarrow \mathbb{B}$ by

$$(Tu)(t) = \int_0^1 G\,(t,s)\,\phi_q\left(\int_0^s q(\tau)f\,(\tau,u(\tau))\,\triangle\tau\right)\triangle s + A(f)(b+at) + B(f)(d + c(1-t)), \tag{2.12}$$

where $G$, $A(f)$ and $B(f)$ are defined as in (2.7), (2.8) and (2.9) respectively.

**Lemma 2.5.** *Let* $(C1)-(C6)$ *hold. Then $T : \mathcal{P} \to \mathcal{P}$ is completely continuous.*

*Proof.* By Arzela-Ascoli theorem, we can easily prove that operator $T$ is completely continuous. □

For a nonnegative continuous functional $\gamma$ on a cone $\mathcal{P}$ in a real Banach space $\mathbb{B}$, and each $l > 0$, we set

$$\mathcal{P}(\gamma, l) = \{u \in \mathcal{P} | \gamma(u) < l\}.$$

The following fixed point theorem [4] is fundamental and important to the proof of our main result.

**Theorem 2.6.** *(Double Fixed Point Theorem) [4] Let $\mathcal{P}$ be a cone in a real Banach space $\mathbb{B}$. Let $\alpha$ and $\gamma$ be increasing, nonnegative, continuous functionals on $\mathcal{P}$, and let $\beta$ be a nonnegative, continuous functional on $\mathcal{P}$ with $\beta(0) = 0$ such that, for some $l > 0$ and $M > 0$,*

$$\gamma(u) \leq \beta(u) \leq \alpha(u) \text{ and } \|u\| \leq M\gamma(u)$$

*for all $u \in \overline{\mathcal{P}(\gamma, l)}$. Suppose that there exist positive numbers $j$ and $k$ with $j < k < l$ such that*

$$\beta(\lambda u) \leq \lambda\beta(u), \quad for \quad 0 \leq \lambda \leq 1 \quad and \quad u \in \partial\mathcal{P}(\beta, k)$$

*and*

$$T : \overline{\mathcal{P}(\gamma, l)} \to \mathcal{P}$$

*is a completely continuous operator such that:*

*(i)* $\gamma(Tu) > l$, *for all* $u \in \partial\mathcal{P}(\gamma, l)$;

*(ii)* $\beta(Tu) < k$, *for all* $u \in \partial\mathcal{P}(\beta, k)$;

*(iii)* $\mathcal{P}(\alpha, j) \neq \emptyset$, *and* $\alpha(Tu) > j$, *for all* $u \in \partial\mathcal{P}(\alpha, j)$.

*Then $T$ has at least two fixed points, $u_1$ and $u_2$ belonging to $\overline{\mathcal{P}(\gamma, l)}$ such that*

$$j < \alpha(u_1), \quad with \quad \theta(u_1) < k,$$

*and*

$$k < \theta(u_2), \quad with \quad \gamma(u_2) < l.$$

## 3. Main Results

In this section, we impose growth conditions on $f$ and then apply Theorem 2.6 to establish the existence of at least two positive solutions for the BVP (1.1).

Let us define the increasing, nonnegative, continuous functionals $\alpha$, $\beta$, and $\gamma$ on $\mathcal{P}$ by

$$\begin{aligned}
\alpha(u) &= \max_{t \in [0, \xi_{m-2}]_{\mathbb{T}}} u(t) = u(\xi_{m-2}), \\
\beta(u) &= \max_{t \in [0, \xi_1]_{\mathbb{T}}} u(t) = u(\xi_1), \\
\gamma(u) &= \min_{t \in [\xi_1, \xi_{m-2}]_{\mathbb{T}}} u(t) = u(\xi_1).
\end{aligned}$$

It is obvious that for each $u \in \mathcal{P}$,

$$\gamma(u) = \beta(u) \leq \alpha(u).$$

In addition, from by Lemma 2.4, for each $u \in \mathcal{P}$,

$$\|u\| \leq \frac{1}{\xi_1} u(\xi_1) = \frac{1}{\xi_1}\gamma(u).$$

Thus,

$$\|u\| \leq \frac{1}{\xi_1}\gamma(u), \ \ \forall u \in \mathcal{P}.$$

For the convenience, we denote

$$A = \frac{1}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2}\alpha_i\left(\int_0^1 G(\xi_i,s)\,\phi_q\left(\int_0^s q(\tau)\triangle\tau\right)\triangle s\right) & \rho-\sum_{i=1}^{m-2}\alpha_i\,(d+c(1-\xi_i)) \\ \sum_{i=1}^{m-2}\beta_i\left(\int_0^1 G(\xi_i,s)\,\phi_q\left(\int_0^s q(\tau)\triangle\tau\right)\triangle s\right) & -\sum_{i=1}^{m-2}\beta_i\,(d+c(1-\xi_i)) \end{vmatrix},$$

$$B = \frac{1}{\Delta}\begin{vmatrix} -\sum_{i=1}^{m-2}\alpha_i\,(b+a\xi_i) & \sum_{i=1}^{m-2}\alpha_i\left(\int_0^1 G(\xi_i,s)\,\phi_q\left(\int_0^s q(\tau)\triangle\tau\right)\triangle s\right) \\ \rho-\sum_{i=1}^{m-2}\beta_i\,(b+a\xi_i) & \sum_{i=1}^{m-2}\beta_i\left(\int_0^1 G(\xi_i,s)\,\phi_q\left(\int_0^s q(\tau)\triangle\tau\right)\triangle s\right) \end{vmatrix},$$

$$L = \int_{\xi_{m-2}}^1 G(\xi_1,s)\,\phi_q\left(\int_{\xi_{m-2}}^s q(\tau)\triangle\tau\right)\triangle s,$$

$$N = \int_0^1 G(\xi_1,s)\,\phi_q\left(\int_0^s q(\tau)\triangle\tau\right)\triangle s + (b+a\xi_1)\,A + (d+c(1-\xi_1))\,B.$$

**Theorem 3.1.** *Suppose that assumptions* $(C1)-(C6)$ *are satisfied. Let there exist positive numbers* $j<k<l$ *such that*

$$0<j<\frac{L}{N}k<\frac{L\xi_1}{N}l,$$

*and assume that* $f$ *satisfies the following conditions*

(C7) $f(t,u)>\phi_p\left(\frac{l}{L}\right)$, *for all* $(t,u)\in[\xi_1,1]_{\mathbb{T}}\times\left[l,\frac{l}{\xi_1}\right]$,

(C8) $f(t,u)<\phi_p\left(\frac{k}{N}\right)$, *for all* $(t,u)\in[0,1]_{\mathbb{T}}\times\left[0,\frac{k}{\xi_1}\right]$,

(C9) $f(t,u)>\phi_p\left(\frac{j}{L}\right)$, *for all* $(t,u)\in[\xi_{m-2},1]_{\mathbb{T}}\times\left[j,\frac{j}{\xi_1}\right]$.

*Then the boundary value problem* (1.1) *has at least two positive solutions* $u_1$ *and* $u_2$ *satisfying*

$$j<\alpha(u_1) \text{ with } \beta(u_1)<k, \quad k<\beta(u_2) \text{ with } \gamma(u_2)<l.$$

*Proof.* We define the completely continuous operator $T$ by (2.12). So, it is easy to check that $T:\overline{\mathcal{P}(\gamma,l)}\to\mathcal{P}$. We now show that all the conditions of Theorem 2.6 are satisfied. In order to show that condition $(i)$ of Theorem 2.6, we choose $u\in\partial\mathcal{P}(\gamma,l)$. Then $\gamma(u)=\min_{t\in[\xi_1,\xi_{m-2}]_{\mathbb{T}}}u(t)=u(\xi_1)=l$, this implies that $u(t)\geq l$ for $t\in[\xi_1,1]_{\mathbb{T}}$. Recalling that $\|u\|\leq\frac{1}{\xi_1}\gamma(u)=\frac{1}{\xi_1}l$, we get

$$l\leq u(t)\leq\frac{l}{\xi_1},\ t\in[\xi_1,1]_{\mathbb{T}}.$$

Then assumption $(C7)$ implies

$$f(t,u)>\phi_p\left(\frac{l}{L}\right), \quad \text{for all} \quad (t,u)\in[\xi_1,1]_{\mathbb{T}}\times\left[l,\frac{l}{\xi_1}\right].$$

Therefore,

$$\begin{aligned}\gamma(Tu) &= \min_{t\in[\xi_1,\xi_{m-2}]_{\mathbb{T}}}(Tu)(t) = (Tu)(\xi_1)\\ &= \int_0^1 G(\xi_1,s)\,\phi_q\left(\int_0^s q(\tau)f(\tau,u(\tau))\,\triangle\tau\right)\triangle s + A(f)(b+a\xi_1) + B(f)(d+c(1-\xi_1))\\ &\geq \int_0^1 G(\xi_1,s)\,\phi_q\left(\int_0^s q(\tau)f(\tau,u(\tau))\,\triangle\tau\right)\triangle s\\ &\geq \int_{\xi_{m-2}}^1 G(\xi_1,s)\,\phi_q\left(\int_{\xi_{m-2}}^s q(\tau)f(\tau,u(\tau))\,\triangle\tau\right)\triangle s\\ &> \frac{l}{L}\left(\int_{\xi_{m-2}}^1 G(\xi_1,s)\,\phi_q\left(\int_{\xi_{m-2}}^s q(\tau)\triangle\tau\right)\triangle s\right) = l.\end{aligned}$$

Hence, condition $(i)$ is satisfied.

Secondly, we show that $(ii)$ of Theorem 2.6 is satisfied. For this, we select $u \in \partial\mathcal{P}(\beta,k)$. Then, $\beta(u) = \max_{t\in[0,\xi_1]_{\mathbb{T}}} u(t) = u(\xi_1) = k$, this means $0 \leq u(t) \leq k$, for all $t \in [0,\xi_1]_{\mathbb{T}}$. Noticing that $\|u\| \leq \frac{1}{\xi_1}\gamma(u) = \frac{1}{\xi_1}\beta(u) = \frac{k}{\xi_1}$, we get

$$0 \leq u(t) \leq \frac{k}{\xi_1}, \text{ for } 0 \leq t \leq 1.$$

Then, assumption $(C8)$ implies

$$f(t,u) < \phi_p\left(\frac{k}{N}\right), \text{ for all } \quad (t,u) \in [0,1]_{\mathbb{T}} \times \left[0, \frac{k}{\xi_1}\right].$$

Therefore

$$\begin{aligned}\beta(Tu) &= \max_{t\in[0,\xi_1]_{\mathbb{T}}}(Tu)(t) = (Tu)(\xi_1)\\ &= \int_0^1 G(\xi_1,s)\,\phi_q\left(\int_0^s q(\tau)f(\tau,u(\tau))\,\triangle\tau\right)\triangle s + A(f)(b+a\xi_1) + B(f)(d+c(1-\xi_1))\\ &< \frac{k}{N}\left(\int_0^1 G(\xi_1,s)\,\phi_q\left(\int_0^s q(\tau)\triangle\tau\right)\triangle s + (b+a\xi_1)A + (d+c(1-\xi_1))B\right)\\ &= k.\end{aligned}$$

So, we get $\beta(Tu) < k$. Hence, condition $(ii)$ is satisfied.

Finally, we show that the condition $(iii)$ of Theorem 2.6 is satisfied. We note that $u(t) = \frac{j}{5}$, $0 \leq t \leq 1$ is a member of $\mathcal{P}(\alpha, j)$, and so $\mathcal{P}(\alpha, j) \neq \emptyset$.

Now, let $u \in \partial\mathcal{P}(\alpha, j)$. Then $\alpha(u) = \max_{t\in[0,\xi_{m-2}]_{\mathbb{T}}} u(t) = u(\xi_{m-2}) = j$. This implies that $u(t) \geq j$ for $t \in [\xi_{m-2}, 1]_{\mathbb{T}}$. Recalling that $\|u\| \leq \frac{1}{\xi_1}\gamma(u) \leq \frac{1}{\xi_1}\alpha(u) = \frac{j}{\xi_1}$, we get

$$j \leq u(t) \leq \frac{j}{\xi_1}, \; t \in [\xi_{m-2}, 1]_{\mathbb{T}}.$$

By assumption $(C9)$,

$$f(t,u) > \phi_p\left(\frac{j}{L}\right), \text{ for all } (t,u) \in [\xi_{m-2},1]_{\mathbb{T}} \times \left[j, \frac{j}{\xi_1}\right].$$

Then,

$$\begin{aligned}
\alpha(Tu) &= \max_{t\in[0,\xi_{m-2}]_{\mathbb{T}}}(Tu)(t) = (Tu)(\xi_{m-2})\\
&\geq \int_0^1 G(\xi_{m-2},s)\,\phi_q\left(\int_0^s q(\tau)f(\tau,u(\tau))\,\triangle\tau\right)\triangle s\\
&\geq \int_{\xi_{m-2}}^1 G(\xi_{m-2},s)\,\phi_q\left(\int_{\xi_{m-2}}^s q(\tau)f(\tau,u(\tau))\,\triangle\tau\right)\triangle s\\
&\geq \int_{\xi_{m-2}}^1 G(\xi_1,s)\,\phi_q\left(\int_{\xi_{m-2}}^s q(\tau)f(\tau,u(\tau))\,\triangle\tau\right)\triangle s\\
&> \frac{j}{L}\left(\int_{\xi_{m-2}}^1 G(\xi_1,s)\,\phi_q\left(\int_{\xi_{m-2}}^s q(\tau)\triangle\tau\right)\triangle s\right) = j.
\end{aligned}$$

So, we get $\alpha(Tu) > j$. Thus, (*iii*) of Theorem 2.6 is satisfied. Hence, the boundary value problem (1.1) has at least two positive solutions $u_1$ and $u_2$ satisfying

$$j < \alpha(u_1) \text{ with } \beta(u_1) < k,$$

and

$$k < \beta(u_2) \text{ with } \gamma(u_2) < l.$$

The proof is complete. □

## 4. An example

**Example 4.1** In BVP (1.1), suppose that $\mathbb{T} = \left[0,\frac{1}{4}\right]\cup\left[\frac{1}{2},1\right]$, $p = 2$, $m = 4$, $q(t) = 1$, $a = 3$, $b = d = 1$, $c = 2$, $\xi_1 = \frac{1}{4}$, $\xi_2 = \frac{1}{2}$, $\alpha_1 = \frac{1}{3}$, $\alpha_2 = \frac{2}{3}$, $\beta_1 = \frac{3}{2}$ and $\beta_2 = 1$ i.e.,

$$\begin{cases}
(u^{\triangle\triangle}(t))^{\triangle} + f(t,u(t)) = 0,\ t\in[0,1]_{\mathbb{T}},\\
3u(0) - u^{\triangle}(0) = \frac{1}{3}u\left(\frac{1}{4}\right) + \frac{2}{3}u\left(\frac{1}{2}\right),\\
2u(1) + u^{\triangle}(1) = \frac{3}{2}u\left(\frac{1}{4}\right) + u\left(\frac{1}{2}\right),\\
u^{\triangle\triangle}(0) = 0,
\end{cases}\tag{4.1}$$

where

$$f(t,u) = \begin{cases}
\frac{7}{40}u + 180, & u\in[0,400),\\
\frac{175}{2}(u-400) + 250, & u\geq 400.
\end{cases}$$

By simple calculation, we get $\rho = 11$, $\theta(t) = 1 + 3t$, $\varphi(t) = 3 - 2t$, $\Delta = -\frac{935}{24}$, $A = \frac{8749}{89760}$, $B = \frac{309}{7480}$, $L = \frac{119}{2112}$, $N = \frac{3187}{8160}$ and

$$G(t,s) = \frac{1}{11}\begin{cases} (1+3\sigma(s))(3-2t), & \sigma(s) \le t, \\ (1+3t)\,(3-2\sigma(s))\,, & t \le s. \end{cases}$$

Choose $j = 10$, $k = 100$ and $l = 500$, it is easy to check that

$$0 < j = 10 < \frac{L}{N}k = \frac{505750}{35057} < \frac{L\xi_1}{N}l = \frac{1264375}{70114},$$

and the conditions $(C1) - (C6)$ are satisfied. Now, we show that $(C7)$, $(C8)$ and $(C9)$ are satisfied:

$$f(t,u) \ge 9000 > \phi_2\left(\frac{l}{L}\right) = \frac{1056000}{119}, \text{ for all } (t,u) \in \left[\frac{1}{4}, 1\right]_{\mathbb{T}} \times [500, 2000]\,,$$

$$f(t,u) \le 250 < \phi_2\left(\frac{k}{N}\right) = \frac{816000}{3187}, \text{ for all } (t,u) \in [0,1]_{\mathbb{T}} \times [0, 400]\,,$$

$$f(t,u) \ge \frac{727}{4} > \phi_2\left(\frac{j}{L}\right) = \frac{21120}{119}, \text{ for all } (t,u) \in \left[\frac{1}{2}, 1\right]_{\mathbb{T}} \times [10, 40]\,.$$

So, all conditions of Theorem 3.1 hold. Thus by Theorem 3.1, the BVP (4.1) has at least two positive solutions $u_1$ and $u_2$ such that

$$10 < \alpha(u_1) \text{ with } \beta(u_1) < 100,$$

and

$$100 < \beta(u_2) \text{ with } \gamma(u_2) < 500.$$

## References


[1] R. P. Agarwal, M. Bohner, Basic calculus on time scales and some of its applications, Result Math. 35 (1999) 3–22.
[2] D. R. Anderson, I. Y. Karaca, Higher-order three-point boundary value problem on time scales, Comput. Math. Appl. 56 (2008) 2429–2443.
[3] F. M. Atici, G. Sh. Guseinov, On Greens functions and positive solutions for boundary value problems on time scales, J. Comput. Appl. Math. 141 (2002) 75–99.
[4] R. I. Avery, J. Henderson, Two positive fixed points of nonlinear operators on ordered Banach spaces, Comm. Appl. Nonlin. Anal. 8 (2001) 27–36.
[5] M. Bohner, A. Peterson, Dynamic Equations on Time Scales:An Introduction with Applications, Birkhäuser, Boston, 2001.
[6] M. Bohner, A. Peterson, Advances in Dynamic Equations on Time Scales, Birkhäuser, Boston, 2003.
[7] C. P. Gupta, Solvability of a three-point nonlinear boundary value problem for a second order ordinary differential equation, J. Math. Anal. Appl. 168 (1992) 540–551.
[8] W. Han, S. Kang, Multiple positive solutions of nonlinear third-order BVP for a class of $p$-laplacian dynamic equations on time scales, Math. Comput. Modelling 49 (2009) 527–535.
[9] S. Hilger, Ein Masskettenkalkül mit Anwendug auf Zentrumsmanningfaltigkeiten, Ph.D. Thesis, Universitat Würzburg, 1988.
[10] V. A. Il'in, E. I. Moiseev, A nonlocal boundary value problem of the first kind for the Sturm-Liouville operator in differential and difference interpretations, Differentsial'nye Uravneniya 23 (1987) 1198–1207.
[11] V. A. Il'in, E. I. Moiseev, A nonlocal boundary value problem of the second kind for the Sturm-Liouville operator, Differentsial'nye Uravneniya 23 (1987) 1422–1431.
[12] V. Lakshmikantham, S. Sivasundaram, B. Kaymakcalan, Dynamic Systems on Measure Chains, Kluwer Academic Publishers, Dordrecht, 1996.
[13] J. Liu, F. Xu, Triple positive solutions for third-order $m$-point boundary value problems on time scales, Discrete Dyn. Nat. Soc. Art. ID 123283 (2009) 15 pp.
[14] R. Ma, Multiple positive solutions for nonlinear $m$-point boundary value problems, Appl. Math. Comput. 148 (2004) 249–262.
[15] F. Tokmak, I. Y. Karaca, Existence of positive solutions for third-order $m$-point boundary value problems with sign-changing nonlinearity on time scales, Abstr. Appl. Anal. Art. ID 506716 (2012) 15 pp.
[16] F. Xu, Positive solutions for third-order nonlinear $p$-Laplacian $m$-point boundary value problems on time scales, Discrete Dyn. Nat. Soc. Art. ID 143040 (2008) 16 pp.
[17] C. Yang, J. Yan, Positive solutions for third-order Sturm-Liouville boundary value problems with $p$-Laplacian, Comput. Math. Appl. 59 (2010) 2059–2066.