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Oscillation Behavior of Third-Order Nonlinear Neutral Dynamic Equations on Time Scales with Distributed Deviating Arguments

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Abstract. The aim of this paper is to give oscillation criteria for the third-order neutral dynamic equations with distributed deviating arguments of the form

$$\left[r(t)\Big([x(t)+p(t)x(\tau(t))]^{\Delta\Delta}\Big)^{\gamma}\right]^{\Delta}+\int_c^d f(t,x[\phi(t,\xi)])\Delta\xi=0,$$

where $\gamma>0$ is the quotient of odd positive integers with $r(t)$ and $p(t)$ real-valued rd-continuous positive functions defined on $\mathbb{T}$. By using a generalized Riccati transformation and integral averaging technique, we establish some new sufficient conditions which ensure that every solution of this equation oscillates or converges to zero.

1. Introduction

In this paper, we deal with the oscillatory behavior of all solutions of the third-order neutral dynamic equation

$$\left[r(t)\Big([x(t)+p(t)x(\tau(t))]^{\Delta\Delta}\Big)^{\gamma}\right]^{\Delta}+\int_c^d f(t,x[\phi(t,\xi)])\Delta\xi=0,\quad t\in\mathbb{T},\quad t\geq t_0. \tag{1}$$

Define the function by

$$z(t)=x(t)+p(t)x(\tau(t)). \tag{2}$$

Furthermore, the equation (1) can be written as

$$\left[r(t)\Big([z(t)]^{\Delta\Delta}\Big)^{\gamma}\right]^{\Delta}+\int_c^d f(t,x[\phi(t,\xi)])\Delta\xi=0. \tag{3}$$

In this paper, we will assume the following hypotheses:

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(h1) $\gamma > 0$ is the ratio of positive odd integers;
(h2) $r : \mathbb{T} \to (0, \infty)$ is a real valued rd-continous function on $\mathbb{T}$ and

$$\int_{t_0}^{\infty} \left(\frac{1}{r(t)}\right)^{\frac{1}{\gamma}} \Delta t = \infty, \quad t_0 \in \mathbb{T}; \tag{4}$$

(h3) $p(t)$ is real valued rd-continuous positive function on $\mathbb{T}$, $0 \leq p(t) \leq P < 1$;
(h4) $\tau : \mathbb{T} \to \mathbb{T}$ is strictly increasing and differentiable function such that $\tau(t) \leq t$ and $\lim_{t\to\infty} \tau(t) = \infty$;
(h5) $\phi(t, \xi) \in C_{rd}([t_0, \infty) \times [c, d], \mathbb{T})$ is not decreasing function for ξ and such that $\phi(t, \xi) \leq t$ and $\lim_{t\to\infty} \min_{\xi\in[c,d]} \phi(t, \xi) = \infty$;
(h6) The function $f \in C(\mathbb{T} \times \mathbb{R}, \mathbb{R})$ is assumed to satisfy $uf(t, u) > 0$ and there exists a positive rd-continuous function $\delta(t)$ on $\mathbb{T}$ such that $\frac{f(t,u)}{u^{\gamma}} \geq \delta(t)$, for $u \neq 0$.

A solution $x(t)$ of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is non-oscillatory. The theory of time scales, which has recently received a lot of attention, was introduced by Hilger [1], in order to unify continuous and discrete analysis. Since then, several authors have expounded on various aspects of this new theory; see the survey paper by R.P. Agarwal, M. Bohner, D. O'Regan and A. Peterson [2]. A book on the subject of time scales by M. Bohner and A. Peterson [3] also summarizes and organizes much of the time scale calculus. In the recent years, there has been increasing interest in obtaining sufficient conditions for the oscillation and non-oscillation of solutions of various equations on time-scales; we refer the reader to the papers [4-14]. T. Candan [15] considered second order nonlinear neutral dynamic equation with distributed deviating arguments

$$(r(t)((y(t) + p(t)y(\tau(t)))^{\Delta})^{\gamma})^{\Delta} + \int_{c}^{d} f(t, y(\theta(t, \xi)))\Delta\xi = 0,$$

where $\gamma > 0$ is the ratio of odd positive integers.
To the best of our knowledge, it seems to have few oscillation results for the oscillation of third-order dynamic equations. L. Erbe, A. Peterson, S.H. Saker [19] considered third-order nonlinear dynamic equation

$$\left(c(t)(a(t)x^{\Delta}(t))^{\Delta}\right)^{\Delta} + q(t)f(x(t)) = 0,$$

on a time scale $\mathbb{T}$.
Li, Han, Sun, Zhao [17] considered third-order nonlinear delay dynamic equation

$$(a(t)([r(t)x^{\Delta}(t)]^{\Delta})^{\gamma})^{\Delta} + f(t, x(\tau(t))) = 0,$$

on a time scale $\mathbb{T}$, where $\gamma > 0$ is quotient of odd positive integers. Grace, Graef, El-Beltagy [18] considered third-order neutral delay dynamic equation

$$(r(t)(x(t) - a(t)x(\tau(t)))^{\Delta\Delta})^{\Delta} + p(t)x^{\gamma}(\delta(t)) = 0,$$

on a time scale $\mathbb{T}$.
Z. H. Yu , Q. R. Wang [16] studied asymptotic behavior of solutions to third-order nonlinear dynamic equations on time scales of the form

$$\left(\frac{1}{a_2(t)}\left((\frac{1}{a_1(t)}(x^{\Delta}(t))^{\alpha_1})^{\Delta}\right)^{\alpha_2}\right)^{\Delta} + q(t)f(x(t)) = 0.$$

In this paper, we consider third-order nonlinear neutral dynamic equation with distributed deviating arguments on time scales which is not in literature. We obtain some conclusions which contribute to oscillation theory of third order neutral dynamic equations.

2. Several Lemmas

Before stating our main results, we begin with the following lemmas which play an important role in the proof of the main results. Throughout this paper, we let

$\eta_+(t) := \max\{0, \eta(t)\},\ \eta_-(t) := \max\{0, -\eta(t)\},$

and

$\beta(t) := \frac{t}{\sigma(t)},\ 0 < \gamma \leq 1,\ \ \beta(t) := (\frac{t}{\sigma(t)})^\gamma,\ \gamma > 1,$

$R(t, t_*) := \int_{t_*}^{t} \left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}} \Delta s,$

where, for sufficiently large $t_* \in [t_0, \infty)_{\mathbb{T}}$.

Lemma 2.1. Let $x(t)$ be a positive solution of (1), $z(t)$ is defined as in (2). Then $z(t)$ has only one of the following two properties:

(I) $z(t) > 0, z^{\Delta}(t) > 0, z^{\Delta\Delta}(t) > 0,$
(II) $z(t) > 0, z^{\Delta}(t) < 0, z^{\Delta\Delta}(t) > 0,$
where $t \geq t_1,\ t_1$ sufficiently large.
Proof. Let $x(t)$ be a positive solution of (1) on $[t_0, \infty)$, so that $z(t) > x(t) > 0$, and

$$\left[r(t)(z^{\Delta\Delta}(t))^\gamma\right]^\Delta = -\int_c^d f(t, x[\phi(t, \xi)])\Delta\xi < 0.$$

Then $r(t)([z(t)]^{\Delta\Delta})^\gamma$ is a decreasing function and therefore eventually of one sign, so $z^{\Delta\Delta}(t)$ is either eventually positive or eventually negative on $t \geq t_1 \geq t_0$. We assert that $z^{\Delta\Delta}(t) > 0$ on $t \geq t_1 \geq t_0$. Otherwise, assume that $z^{\Delta\Delta}(t) < 0$, then there exists a constant $M > 0$, such that

$$r(t)(z^{\Delta\Delta}(t))^\gamma \leq -M < 0.$$

By integrating the last inequality from t_1 to t, we obtain

$$z^{\Delta}(t) \leq z^{\Delta}(t_1) - M^{\frac{1}{\gamma}} \int_{t_1}^{t} (\frac{1}{r(s)})^{\frac{1}{\gamma}} \Delta s.$$

Let $t \to \infty$. Then from (4) , we have $z^{\Delta}(t) \to -\infty$, and therefore eventually $z^{\Delta}(t) < 0$.
Since $z^{\Delta\Delta}(t) < 0$ and $z^{\Delta}(t) < 0$, we have $z(t) < 0$, which contradicts our assumption $z(t) > 0$. Therefore, z(t) has only one of the two properties (I) and (II).
This completes the proof.

Lemma 2.2. Let $x(t)$ be an eventually positive solution of (1), correspondingly $z(t)$ has the property (II). Assume that (4) and

$$\int_{t_0}^{\infty} \int_{v}^{\infty} \left[\frac{1}{r(u)} \int_{u}^{\infty} \delta(s)\Delta s\right]^{\frac{1}{\gamma}} \Delta u \Delta v = \infty \tag{5}$$

hold. Then $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} z(t) = 0.$

Proof. Let $x(t)$ be an eventually positive solution of (1). Since $z(t)$ has the property (II), then there exists finite $\lim_{t\to\infty} z(t) = I$. We shall prove that $I = 0$. Assume that $I > 0$, then for any $\epsilon > 0$, we have $I + \epsilon > z(t) > I$, eventually. Choosing $0 < \epsilon < \frac{I(1-P)}{P}$ and using (h3), we obtain

$$x(t) = z(t) - p(t)x(\tau(t)) > I - p(t)z(\tau(t)) > I - P(I+\epsilon) = k(I+\epsilon) > kz(t),$$

where $k = \frac{I-P(I+\epsilon)}{I+\epsilon} > 0$. Using the above inequality and (h6), we obtain from (3)

$$\begin{aligned}[r(t)(z^{\Delta\Delta}(t))^{\gamma}]^{\Delta} &= -\int_c^d f(t, x[\phi(t,\xi)])\Delta\xi \\ &\le -\int_c^d (x[\phi(t,\xi)])^{\gamma}\delta(t)\Delta\xi \\ &\le -k^{\gamma}\delta(t)\int_c^d (z[\phi(t,\xi)])^{\gamma}\Delta\xi.\end{aligned}$$

Since $z^{\Delta}(t) < 0$, we have

$$[r(t)(z^{\Delta\Delta}(t))^{\gamma}]^{\Delta} \le -k^{\gamma}.\delta(t).(z[\phi(t,d)])^{\gamma} = -k^{\gamma}.\delta(t)z^{\gamma}(\phi_1(t)), \tag{6}$$

where $\phi_1(t) = \phi(t,d)$. Integrating inequality (6) from t to ∞, we obtain

$$r(t)(z^{\Delta\Delta}(t))^{\gamma} \ge k^{\gamma}\int_t^{\infty} \delta(s)z^{\gamma}(\phi_1(s))\Delta s.$$

Using $z^{\gamma}(\phi_1(s)) \ge I^{\gamma}$, we obtain

$$z^{\Delta\Delta}(t) \ge \frac{kI}{r^{\frac{1}{\gamma}}}\Big[\int_t^{\infty} \delta(s)\Big]^{\frac{1}{\gamma}}\Delta(s). \tag{7}$$

Integrating inequality (7) from t to ∞, we have

$$-z^{\Delta}(t) \ge kI\int_t^{\infty} [\frac{1}{r(u)}\int_u^{\infty} \delta(s)\Delta(s)]^{\frac{1}{\gamma}}\Delta u.$$

Integrating the last inequality from t_1 to ∞, we obtain

$$z(t_1) \ge kI\int_{t_1}^{\infty}\int_v^{\infty} [\frac{1}{r(u)}\int_u^{\infty} \delta(s)\Delta(s)]^{\frac{1}{\gamma}}\Delta u\Delta v. \tag{8}$$

The last inequality contradict (5), we have $I = 0$. And since $0 \le x(t) \le z(t)$, then $\lim_{t\to\infty} x(t) = 0$. This completes the proof.

Lemma 2.3. Assume that $x(t)$ is a positive solution of equation (1), $z(t)$ is defined as in (2) such that $z^{\Delta\Delta}(t) > 0, z^{\Delta}(t) > 0$, on $[t_*, \infty)_{\mathbb{T}},\ t_* \ge 0$. Then

$$z^{\Delta}(t) \ge R(t,t_*)r^{\frac{1}{\gamma}}(t)z^{\Delta\Delta}(t). \tag{9}$$

Proof. Since $r(t)(z^{\Delta\Delta}(t))^{\gamma}$ is strictly decreasing on $[t_*, \infty)_{\mathbb{T}}$, we get for $t \in [t_*, \infty)_{\mathbb{T}}$

$$\begin{aligned} z^{\Delta}(t) &> z^{\Delta}(t) - z^{\Delta}(t_*) \\ &= \int_{t_*}^{t} \frac{(r(s)(z^{\Delta\Delta}(t))^{\gamma})^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(s)} \Delta s \\ &\geq (r(t)(z^{\Delta\Delta}(t))^{\gamma})^{\frac{1}{\gamma}} \int_{t_*}^{t} \left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}} \Delta s \end{aligned}$$

and, hence

$z^{\Delta}(t) > R(t, t_*) r^{\frac{1}{\gamma}}(t) z^{\Delta\Delta}(t)$ on $[t_*, \infty)_{\mathbb{T}}$.

Lemma 2.4. Assume that $x(t)$ is a positive solution of equation (1), correspondingly $z(t)$ has the property (I). Such that $z^{\Delta}(t) > 0$, $z^{\Delta\Delta}(t) > 0$, on $[t_*, \infty)_{\mathbb{T}}$, $t_* \geq t_0$. Furthermore,

$$\int_{t_2}^{\infty} q(s)\phi_2^{\gamma}(s)\Delta s = \infty, \tag{10}$$

where $q(t) = \delta(t)(1 - p(t))^{\gamma}$, $\phi_2(t) = \phi(t, c)$. Then there exists a $T \in [t_*, \infty)_{\mathbb{T}}$, sufficiently large, so that

$$z(t) > tz^{\Delta}(t),$$

$z(t)/t$ is strictly decreasing, $t \in [T, \infty)_{\mathbb{T}}$.

Proof. Let $U(t) = z(t) - tz^{\Delta}(t)$. Hence $U^{\Delta}(t) = -\sigma(t)z^{\Delta\Delta}(t) < 0$. We claim there exists a $t_1 \in [t_*, \infty)_{\mathbb{T}}$ such that $U(t) > 0$, $z(\phi(t, \xi)) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. Assume not. Then $U(t) < 0$ on $[t_1, \infty)_{\mathbb{T}}$. Therefore,

$$\left(\frac{z(t)}{t}\right)^{\Delta} = \frac{tz^{\Delta}(t) - z(t)}{t\sigma(t)} = -\frac{U(t)}{t\sigma(t)} > 0, \ t \in [t_1, \infty)_{\mathbb{T}},$$

which implies that $z(t)/t$ is strictly increasing on $[t_1, \infty)_{\mathbb{T}}$. Pick $t_2 \in [t_1, \infty)_{\mathbb{T}}$ so that $\phi(t, \xi) \geq \phi(t_1, \xi)$, for $t \geq t_2$. Then

$$\frac{z(\phi(t, \xi))}{\phi(t, \xi)} \geq \frac{z(\phi(t_1, \xi))}{\phi(t_1, \xi)} = d > 0,$$

so that $z(\phi(t, \xi)) > d\phi(t, \xi)$, for $t \geq t_2$. By (2), we obtain

$$\begin{aligned} x(t) &= z(t) - p(t)x(\tau(t)) > z(t) - p(t)z(\tau(t)) \\ &\geq (1 - p(t))z(t). \end{aligned} \tag{11}$$

Using (11), (h4) and (h5), we have

$$\begin{aligned} [r(t)([z(t)]^{\Delta\Delta})^{\gamma}]^{\Delta} &= -\int_{c}^{d} f(t, x[\phi(t, \xi)])\Delta\xi \\ &\leq -\delta(t)(1 - p(t))^{\gamma} \int_{c}^{d} z^{\gamma}(\phi(t, \xi))\Delta\xi \\ &\leq -\delta(t)(1 - p(t))^{\gamma} z^{\gamma}(\phi(t, c)) \end{aligned}$$

$$\leq \quad -q(t)z^{\gamma}(\phi_2(t)), \tag{12}$$

where $q(t) = \delta(t)(1 - p(t))^{\gamma}, \ \phi_2(t) = \phi(t, c)$.

Now by integrating both sides of last equation from t_2 to t, we have

$$r(t)(z^{\Delta\Delta}(t))^{\gamma} - r(t_2)(z^{\Delta\Delta}(t_2))^{\gamma} + \int_{t_2}^{t} q(s)z^{\gamma}(\phi_2(s))\Delta s \leq 0.$$

This implies that

$$r(t_2)(z^{\Delta\Delta}(t_2))^{\gamma} \geq \int_{t_2}^{t} q(s)(z(\phi_2(s)))^{\gamma}\Delta s \geq d^{\gamma} \int_{t_2}^{t} q(s)\phi_2^{\gamma}(s)\Delta s,$$

which contradicts (10). So $U(t) > 0$ on $t \in [t_1, \infty)_{\mathbb{T}}$ and consequently,

$$\left(\frac{z(t)}{t}\right)^{\Delta} = \frac{tz^{\Delta}(t) - z(t)}{t\sigma(t)} = -\frac{U(t)}{t\sigma(t)} < 0, \ t \in [t_1, \infty)_{\mathbb{T}},$$

and we have that $z(t)/t$ is strictly decreasing on $t \in [t_1, \infty)_{\mathbb{T}}$. The proof is now complete.

3. Main Results

In this section we give some new oscillation criteria for (1).

Theorem 3.1. Assume that (4), (5) and (10) hold and that, for all sufficiently large $T_1 \in [t_0, \infty)_{\mathbb{T}}$, there is a $T > T_1$ such that

$$\limsup_{t\to\infty} \int_T^t \Big[\rho^\sigma(s)q(s)\Big(\frac{\phi_2(s)}{\sigma(s)}\Big)^\gamma - \frac{((\rho^\Delta(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^\sigma(s)R(s,t_*))^\gamma}\Big]\Delta s = \infty, \tag{13}$$

where the function $\rho \in C^1_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$ is a positive function. Then every solution of equation (1) is either oscillatory or tends to zero.

Proof. Assume (1) has a nonoscillatory solution $x(t)$ on $[t_0,\infty)_{\mathbb{T}}$. Then, without loss of generality that $x(t) > 0$, $x(\tau(t)) > 0$ for $t \geq t_1$ and $x(\phi(t,\xi)) > 0$, $(t,\xi) \in [t_1,\infty) \times [c,d]$ for all $t_1 \in [t_0,\infty)_{\mathbb{T}}$. $z(t)$ is defined as in (2). We suppose that $z(t) > 0$. We shall consider only this case, since the proof when $z(t)$ is eventually negative is similar. By Lemmas 2.1 and 2.2, we have

$$\Big[r(t)([z(t)]^{\Delta\Delta})^\gamma\Big]^\Delta < 0, \ z^{\Delta\Delta}(t) > 0, \ t \in [t_1,\infty)_{\mathbb{T}},$$

and either $z^\Delta(t) > 0$ for $t \geq t_2 \geq t_1$ or $\lim_{t\to\infty} z(t) = \lim_{t\to\infty} x(t) = 0$. Let $z^\Delta(t) > 0$ on $[t_2,\infty)_{\mathbb{T}}$. By (11) and (12), we have

$$[r(t)([z(t)]^{\Delta\Delta})^\gamma]^\Delta \leq -q(t)z^\gamma(\phi_2(t)),$$

where $q(t) = \delta(t)(1-p(t))^\gamma$, $\phi_2(t) = \phi(t,c)$.

Define the function $w(t)$ by Riccati substitution

$$w(t) = \rho(t)\frac{r(t)([z(t)]^{\Delta\Delta})^\gamma}{z^\gamma(t)}. \tag{14}$$

Then

$$\begin{aligned} w^\Delta(t) &= \rho^\Delta(t)\frac{r(t)([z(t)]^{\Delta\Delta})^\gamma}{z^\gamma(t)} + \rho^\sigma(t)\Big[\frac{r(t)([z(t)]^{\Delta\Delta})^\gamma}{z^\gamma(t)}\Big]^\Delta \\ &= \rho^\Delta(t)\frac{r(t)([z(t)]^{\Delta\Delta})^\gamma}{z^\gamma(t)} + \rho^\sigma(t)\frac{[r(t)([z(t)]^{\Delta\Delta})^\gamma]^\Delta}{z^{\gamma\sigma}(t)} - \rho^\sigma(t)\frac{r(t)([z(t)]^{\Delta\Delta})^\gamma(z^\gamma(t))^\Delta}{z^\gamma(t)z^{\gamma\sigma}(t)}. \end{aligned}$$

¿From equation (1), the definition of $w(t)$ and using the fact $z(t)/t$ is strictly decreasing for $t \in [t_3,\infty)_{\mathbb{T}}$, $t_3 \geq t_2$ it follows that

$$w^\Delta(t) \leq \frac{\rho^\Delta(t)}{\rho(t)}w(t) - \rho^\sigma(t)q(t)\frac{z^\gamma(\phi_2(t))}{z^\gamma(\sigma(t))} - \rho^\sigma(t)\frac{r(t)([z(t)]^{\Delta\Delta})^\gamma(z^\gamma(t))^\Delta}{z^\gamma(t)z^{\gamma\sigma}(t)}$$

$$w^\Delta(t) \leq \frac{\rho^\Delta(t)}{\rho(t)}w(t) - \rho^\sigma(t)q(t)\Big(\frac{\phi_2(t)}{\sigma(t)}\Big)^\gamma - \rho^\sigma(t)\frac{r(t)([z(t)]^{\Delta\Delta})^\gamma(z^\gamma(t))^\Delta}{z^\gamma(t)z^{\gamma\sigma}(t)}. \tag{15}$$

Now we consider the following two cases: $0 < \gamma \leq 1$ and $\gamma > 1$. In the first case $0 < \gamma \leq 1$. Using the Keller's chain rule(see [3]), we have

$$(z^\gamma(t))^\Delta = \gamma\int_0^1 [hz^\sigma + (1-h)z]^{\gamma-1}z^\Delta(t)dh \geq \gamma(z^\sigma(t))^{\gamma-1}z^\Delta(t), \tag{16}$$

in view of (16), Lemma 2.2, Lemma 2.3 and (9), we have

$$\begin{aligned} w^{\Delta}(t) &\leq -\rho^{\sigma}(t)q(t)(\frac{\phi_2(t)}{\sigma(t)})^{\gamma} + \frac{(\rho^{\Delta}(t))_+}{\rho(t)}w(t) - \gamma\rho^{\sigma}(t)\frac{r(t)(z^{\Delta\Delta}(t))^{\gamma}z^{\Delta}(t)z(t)}{z^{\gamma+1}(t)z^{\sigma}(t)} \\ &\leq -\rho^{\sigma}(t)q(t)(\frac{\phi_2(t)}{\sigma(t)})^{\gamma} + \frac{(\rho^{\Delta}(t))_+}{\rho(t)}w(t) - \gamma\rho^{\sigma}(t)R(t,t_*)\frac{r^{\frac{\gamma+1}{\gamma}}(t)(z^{\Delta\Delta}(t))^{\gamma+1}z(t)}{z^{\gamma+1}(t)z(\sigma(t))} \end{aligned}$$

$$\leq -\rho^{\sigma}(t)q(t)(\frac{\phi_2(t)}{\sigma(t)})^{\gamma} + \frac{(\rho^{\Delta}(t))_+}{\rho(t)}w(t) - \gamma\rho^{\sigma}(t)R(t,t_*)\frac{t}{\sigma(t)}\frac{w^{\frac{\gamma+1}{\gamma}}(t)}{\rho^{\frac{\gamma+1}{\gamma}}(t)}. \tag{17}$$

In the second case $\gamma > 1$. Applying the Keller's chain rule , we have

$$(z^{\gamma}(t))^{\Delta} = \gamma\int_0^1 [hz^{\sigma} + (1-h)z]^{\gamma-1}z^{\Delta}(t)dh \geq \gamma(z(t))^{\gamma-1}z^{\Delta}(t), \tag{18}$$

in the view of (18), Lemma 2.2, Lemma 2.3 and (9), we have

$$w^{\Delta}(t) \leq -\rho^{\sigma}(t)q(t)(\frac{\phi_2(t)}{\sigma(t)})^{\gamma} + \frac{(\rho^{\Delta}(t))_+}{\rho(t)}w(t) - \gamma\rho^{\sigma}(t)\frac{r(t)[z(t)]^{\Delta\Delta})^{\gamma}z^{\Delta}(t)z^{\gamma}(t)}{z^{\gamma+1}(t)z^{\gamma\sigma}(t)}$$

$$w^{\Delta}(t) \leq -\rho^{\sigma}(t)q(t)(\frac{\phi_2(t)}{\sigma(t)})^{\gamma} + \frac{(\rho^{\Delta}(t))_+}{\rho(t)}w(t) - \gamma\rho^{\sigma}(t)(\frac{t}{\sigma(t)})^{\gamma}R(t,t_*)\frac{w^{\frac{\gamma+1}{\gamma}}(t)}{\rho^{\frac{\gamma+1}{\gamma}}(t)}. \tag{19}$$

By (17), (19) and the definition of $\beta(t)$, we have, for $\gamma > 0$,

$$w^{\Delta}(t) \leq -\rho^{\sigma}(t)q(t)(\frac{\phi_2(t)}{\sigma(t)})^{\gamma} + \frac{(\rho^{\Delta}(t))_+}{\rho(t)}w(t) - \gamma\rho^{\sigma}(t)\beta(t)R(t,t_*)\frac{w^{\lambda}(t)}{\rho^{\lambda}(t)}, \tag{20}$$

where $\lambda := \frac{\gamma+1}{\gamma}$. Define $A \geq 0$ and $B \geq 0$ by

$A^{\lambda} := \gamma\rho^{\sigma}(t)\beta(t)R(t,t_*)\frac{w^{\lambda}(t)}{\rho^{\lambda}(t)}$,

$B^{\lambda-1} := \frac{\rho^{\Delta}(t)}{\lambda(\gamma\rho^{\sigma}(t)\beta(t)R(t,t_*))^{\frac{1}{\lambda}}}$.

Then using the inequality [20]

$$\lambda AB^{\lambda-1} - A^{\lambda} \leq (\lambda-1)B^{\lambda} \tag{21}$$

which yields

$$\frac{(\rho^{\Delta}(t))_+}{\rho(t)}w(t) - \gamma\rho^{\sigma}(t)\beta(t)R(t,t_*)\frac{w^{\lambda}(t)}{\rho^{\lambda}(t)} \leq \frac{((\rho^{\Delta}(t))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(t)\rho^{\sigma}(t)R(t,t_*))^{\gamma}}.$$

¿From this last inequality and (20), we find

$$w^{\Delta}(t) \leq -\rho^{\sigma}(t)q(t)(\frac{\phi_2(t)}{\sigma(t)})^{\gamma} + \frac{((\rho^{\Delta}(t))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(t)\rho^{\sigma}(t)R(t,t_*))^{\gamma}}.$$

Integrating both sides from T to t , we get

$$\int_T^t [\rho^{\sigma}(s)q(s)(\frac{\phi_2(s)}{\sigma(s)})^{\gamma} - \frac{((\rho^{\Delta}(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^{\sigma}(s)R(s,t_*))^{\gamma}}]\Delta s \leq w(T) - w(t) \leq w(T),$$

which contradicts to assumption (13). This completes the proof of Theorem 3.1.

Theorem 3.2. Assume that (4), (5) and (10) hold. Furthermore, suppose that there exist functions $H, h \in C_{rd}(\mathbb{D}, \mathbb{R})$, where $\mathbb{D} \equiv (t,s) : t \geq s \geq t_0$ such that
$H(t,t) = 0,\ t \geq 0$
$H(t,s) > 0,\ t > s \geq t_0,$
and H has a nonpositive continous Δ-partial derivative $H^{\Delta s}(t,s)$ with respect to the second variable and satisfies

$$H^{\Delta s}(\sigma(t),s) + H(\sigma(t),\sigma(s))\frac{\rho^{\Delta}(s)}{\rho(s)} = -\frac{h(t,s)}{\rho(s)}H(\sigma(t),\sigma(s))^{\frac{\gamma}{\gamma+1}}, \tag{22}$$

and for all sufficiently large $T_1 \in [t_0,\infty)_{\mathbb{T}}$, there is a $T > T_1$ such that

$$\limsup_{t\to\infty} \frac{1}{H(\sigma(t),T)} \int_T^{\sigma(t)} \chi(t,s)\Delta s = \infty, \tag{23}$$

where ρ is a positive Δ-differentiable function and

$$\chi(t,s) = H(\sigma(t),\sigma(s))\rho^{\sigma}(s)q(s)(\frac{\phi_2(s)}{\sigma(s)})^{\gamma} - \frac{(h_-(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^{\sigma}(s)R(s,T_1))^{\gamma}}.$$

Then every solution of equation (1) is either oscillatory or tends to zero.

Proof. Suppose that $x(t)$ is a nonoscillatory solution of (1) and $z(t)$ is defined as in (2). Without loss of generality, we may assume that there is a $t_1 \in [t_0,\infty)_{\mathbb{T}}$ sufficiently large so that the conclusions of Lemma 2.1. hold and (22) holds for $t_2 > t_1$. If case (I) of Lemma 2.1. holds then proceeding as in the proof of Theorem 3.1. , we see that (20) holds for $t > t_2$. Multiply both sides of (20) by $H(\sigma(t),\sigma(s))$ and integrating from T to $\sigma(t)$, we get

$$\begin{aligned}\int_T^{\sigma(t)} H(\sigma(t),\sigma(s))\rho^{\sigma}(s)q(s)(\frac{\phi_2(s)}{\sigma(s)})^{\gamma}\Delta s \quad &\leq \quad -\int_T^{\sigma(t)} H(\sigma(t),\sigma(s))w^{\Delta}(s)\Delta s \\ &+ \quad \int_T^{\sigma(t)} H(\sigma(t),\sigma(s))\frac{\rho^{\Delta}(s)}{\rho(s)}w(s)\Delta s \\ &- \quad \int_T^{\sigma(t)} H(\sigma(t),\sigma(s))\gamma\rho^{\sigma}(s)\beta(s)R(s,T_1)\frac{w^{\lambda}(s)}{\rho^{\lambda}(s)}\Delta s,\ (\lambda = \frac{\gamma+1}{\gamma})\end{aligned} \tag{24}$$

Integrating by parts and using $H(t,t) = 0$, we obtain

$$\int_T^{\sigma(t)} H(\sigma(t),\sigma(s))w^{\Delta}(s)\Delta s = -H(\sigma(t),T)w(T) - \int_T^{\sigma(t)} H^{\Delta s}(\sigma(t),s)w(s)\Delta s.$$

It then follows from (24) that

$$\int_T^{\sigma(t)} H(\sigma(t),\sigma(s))\rho^{\sigma}(s)q(s)(\frac{\phi_2(s)}{\sigma(s)})^{\gamma}\Delta s \leq H(\sigma(t),T)w(T) + \int_T^{\sigma(t)} H^{\Delta s}(\sigma(t),s)w(s)\Delta s$$

$$+ \int_T^{\sigma(t)} H(\sigma(t),\sigma(s))\frac{\rho^{\Delta}(s)}{\rho(s)}w(s)\Delta s - \int_T^{\sigma(t)} H(\sigma(t),\sigma(s))\gamma\rho^{\sigma}(s)\beta(s)R(s,T_1)\frac{w^{\lambda}(s)}{\rho^{\lambda}(s)}\Delta s.$$

Then, we have

$$\int_T^{\sigma(t)} H(\sigma(t),\sigma(s))\rho^{\sigma}(s)q(s)(\frac{\phi_2(s)}{\sigma(s)})^{\gamma}\Delta s \leq H(\sigma(t),T)w(T)$$

$$+\Big[\int_T^{\sigma(t)} H^{\Delta s}(\sigma(t),s) + H(\sigma(t),\sigma(s))\frac{\rho^{\Delta}(s)}{\rho(s)}\Big]w(s)\Delta s - \int_T^{\sigma(t)} H(\sigma(t),\sigma(s))\gamma\rho^{\sigma}(s)\beta(s)R(s,T_1)\frac{w^{\lambda}(s)}{\rho^{\lambda}(s)}\Delta s.$$

It then follows from (22) that

$$\begin{aligned}
\int_T^{\sigma(t)} H(\sigma(t),\sigma(s))\rho^{\sigma}(s)q(s)(\frac{\phi_2(s)}{\sigma(s)})^{\gamma}\Delta s \quad &\leq \quad H(\sigma(t),T)w(T) \\
+ \int_T^{\sigma(t)} \Big[-\frac{h(t,s)}{\rho(s)}H(\sigma(t),\sigma(s))^{\frac{\gamma}{\gamma+1}}\Big]w(s)\Delta s &- \int_T^{\sigma(t)} H(\sigma(t),\sigma(s))\gamma\rho^{\sigma}(s)\beta(s)R(s,T_1)\frac{w^{\lambda}(s)}{\rho^{\lambda}(s)}\Delta s. \\
\leq \quad H(\sigma(t),T)w(T) &+ \int_T^{\sigma(t)} \Big[\frac{h_-(t,s)}{\rho(s)}H(\sigma(t),\sigma(s))^{\frac{\gamma}{\gamma+1}}\Big]w(s)\Delta s \\
- \quad \int_T^{\sigma(t)} &H(\sigma(t),\sigma(s))\gamma\rho^{\sigma}(s)\beta(s)R(s,T_1)\frac{w^{\lambda}(s)}{\rho^{\lambda}(s)}\Delta s.
\end{aligned}$$

Therefore, as in Theorem 3.1. , by letting

$A^{\lambda} := H(\sigma(t),\sigma(s))\gamma\rho^{\sigma}(t)\beta(t)R(t,T_1)\frac{w^{\lambda}(t)}{\rho^{\lambda}(t)}$,

$B^{\lambda-1} := \frac{h_-(t,s)}{\lambda(\gamma\rho^{\sigma}(t)\beta(t)R(t,T_1))^{\frac{1}{\lambda}}}$.

Then using the inequality [20]

$$\lambda AB^{\lambda-1} - A^{\lambda} \leq (\lambda - 1)B^{\lambda}.$$

We have

$$\int_T^{\sigma(t)} \Big[\frac{h_-(t,s)}{\rho(s)} H(\sigma(t),\sigma(s))^{\frac{\gamma}{\gamma+1}}\Big] w(s)\Delta s - \int_T^{\sigma(t)} H(\sigma(t),\sigma(s))\gamma\rho^{\sigma}(s)\beta(s)R(s,T_1)\frac{w^{\lambda}(s)}{\rho^{\lambda}(s)}\Delta s$$

$$\leq \int_T^{\sigma(t)} \frac{(h_-(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^{\sigma}(s)R(t,T_1))^{\gamma}}\Delta s$$

$$\int_T^{\sigma(t)} H(\sigma(t),\sigma(s))\rho^{\sigma}(s)q(s)(\frac{\phi_2(s)}{\sigma(s)})^{\gamma}\Delta s$$

$$\leq H(\sigma(t),T)w(T) + \int_T^{\sigma(t)} \frac{(h_-(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^{\sigma}(s)R(t,T_1))^{\gamma}}\Delta s.$$

Then for $T > T_1$ we have

$$\int_T^{\sigma(t)} \Big[H(\sigma(t),\sigma(s))\rho^{\sigma}(s)q(s)(\frac{\phi_2(s)}{\sigma(s)})^{\gamma} - \frac{(h_-(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^{\sigma}(s)R(s,T_1))^{\gamma}}\Big]\Delta s \leq H(\sigma(t),T)w(T),$$

and this implies that

$$\frac{1}{H(\sigma(t),T)}\int_T^{\sigma(t)} \Big[H(\sigma(t),\sigma(s))\rho^{\sigma}(s)q(s)(\frac{\phi_2(s)}{\sigma(s)})^{\gamma} - \frac{(h_-(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^{\sigma}(s)R(s,T_1))^{\gamma}}\Big]\Delta s < w(T),$$

for all large T, which contradicts (23). This completes the proof of Theorem 3.2.

Remark 3.1. ¿From Theorem 3.1, we can obtain different conditions for oscillation of equation (1.1) with different choices of $\rho(t)$.

Remark 3.2. The conclusion of Theorem 3.1 remains intact if assumption (13) is replaced by the two conditions

$$\limsup_{t\to\infty} \int_T^t \rho^{\sigma}(s)q(s)(\frac{\phi_2(s)}{\sigma(s)})^{\gamma}\Delta s = \infty,$$

$$\limsup_{t\to\infty} \int_T^t \frac{((\rho^{\Delta}(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^{\sigma}(s)R(s,t_*))^{\gamma}}\Delta s < \infty.$$

Remark 3.3. The conclusion of Theorem 3.2 remains intact if assumption (23) is replaced by the two conditions

$$\limsup_{t\to\infty} \frac{1}{H(\sigma(t),T)} \int_T^{\sigma(t)} H(\sigma(t),\sigma(s))\rho^\sigma(s)q(s)\Big(\frac{\phi_2(s)}{\sigma(s)}\Big)^\gamma \Delta s = \infty,$$

$$\liminf_{t\to\infty} \frac{1}{H(\sigma(t),T)} \int_T^{\sigma(t)} \frac{(h_-(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^\sigma(s)R(s,T_1))^\gamma}\Delta s < \infty.$$

Example 3.1. Consider the following third order neutral dynamic equation $t \in [t_0,\infty)_{\mathbb{T}}$,

$$\left(t^3\left((x(t) + (\frac{t-c-1}{t-c})x(t-1))^{\Delta\Delta}\right)^3\right)^\Delta + \int_c^d \frac{\alpha}{t} x^3(t-\xi)\Delta\xi = 0, \tag{25}$$

where $\gamma = 3$, $r(t) = t^3$, $\tau(t) = t-1$, $\phi(t,\xi) = t-\xi$, $p(t) = \frac{t-c-1}{t-c}$, $\phi_2(t) = \phi(t,c) = t-c$, $\delta(t) = \frac{\alpha}{t}$, α is a positive constant.

It is clear that condition (4), (5) and (10) hold. Thus, we assume $\mathbb{T}$ is a time scale satisfying $\sigma(t) \le kt$, for some $k \ge 1$, $t \ge T_k > t_*$. We will use the formula

$$(x^\gamma(t))^\Delta = \gamma \int_0^1 [hx^\sigma + (1-h)x]^{\gamma-1} x^\Delta(t)dh, \tag{26}$$

where $x(t)$ is delta differentiable and eventually positive or eventually negative, which is a simple consequence of Keller's chain rule [3, Theorem 1.90]. When $\gamma \ge 1$, by Theorem 3.1, pick $\rho(t) = t^3$, by (26), we have that $(t^\gamma)^\Delta \le \gamma\sigma^{\gamma-1}(t)$. Therefore

$$\limsup_{t\to\infty} \int_T^t \left[\rho^\sigma(s)q(s)\Big(\frac{\phi_2(s)}{\sigma(s)}\Big)^\gamma - \frac{((\rho^\Delta(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^\sigma(s)R(s,t_*))^\gamma}\right]\Delta s$$

$$\ge \limsup_{t\to\infty} \int_T^t \left[\frac{\alpha}{s} - \Big(\frac{3}{4}\Big)^4 \frac{k^8}{s}\right]\Delta s$$

$$\ge \left(\alpha - \Big(\frac{3}{4}\Big)^4 k^8\right) \limsup_{t\to\infty} \int_T^t \frac{\Delta s}{s} = \infty,$$

if $\alpha > (\frac{3}{4})^4 k^8$. Hence, by Theorem 3.1. every solution of (25) is oscillatory or tends to zero if $\alpha > (\frac{3}{4})^4 k^8$.

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