Coupled Fixed Point Theorems for New Contractive Mixed Monotone Mappings and Applications to Integral Equations

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Abstract. The aim of this paper is to extend the results of Bhaskar and Lakshmikantham and some other authors and to prove some new coupled fixed point theorems for mappings having a mixed monotone property in a complete metric space endowed with a partial order. Our theorems can be used to investigate a large class of nonlinear problems. As an application, we discuss the existence and uniqueness for a solution of a nonlinear integral equation.

1. Introduction and Preliminaries

Let F be a function which maps an arbitrary nonempty set X into itself; i.e. $F : X \to X$. A fixed point of the mapping F is an element x belonging to X such that $Fx = x$. Fixed points are of interest in themselves but they also provide a way to establish the existence of a solution to a set of equations. Fixed point theory is a very useful tool in various fields of mathematics, game theory, mathematical economics, statistics, biology, chemistry, engineering, computer science and economics in dealing with problems arising in approximation theory, theory of differential equations, theory of matrix equations etc. (see, [1-7]). For example, in theoretical economics, such as general equilibrium theory, there comes at point where one needs to know whether the solution to a system of equations necessarily exists; or, more specifically, under which conditions will a solution necessarily exist (see, [1]). The mathematical analysis of this question usually relies on fixed point theorems.

The Banach contraction principle [8] is one of the pivotal results in fixed point theory. It guarantees the existence and uniqueness of fixed points of certain self-maps of metric spaces, and provides a constructive method to find those fixed points. Also its significance lies in its vast applicability in a number of branches of mathematics. This principle has been generalized by many authors to mappings that satisfy much weaker conditions (see [9-13]).

The existence of fixed points of nonlinear contraction mappings in metric spaces endowed with a partial ordering has been considered recently by Ran and Reurings [14] in order to obtain a solution of a matrix equation in 2004. Fixed point theorems in partially ordered metric spaces have been studied

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by some authors since 2004 (see [15-22]). Nieto and Lopez [15] extended the results in [14] by removing the continuity condition of the mapping. They applied their result to get a solution of a boundary value problem. The efficiency of these kind of extensions of fixed point theorems in such kind of problems, as it is well known, is due to the fact that most real valued function spaces are partially ordered metric spaces.

The concept of coupled fixed point theorem was introduced by Guo and Lakshmikantham [23]. Subsequently, Bhaskar and Lakshmikantham [24] introduced the notion of the mixed monotone property of a given mapping in 2006. Furthermore, they proved some coupled fixed point theorems for mappings which satisfy the mixed monotone property and discussed the existence and uniqueness of a solution for a periodic boundary value problem.

Definition 1.1. *Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$. We say that F has the mixed monotone property if $F(x, y)$ is monotone nondecreasing in x and is monotone nonincreasing in y, that is, for any $x, y \in X$,*

$$x_1, x_2 \in X, \quad x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad y_1 \leq y_2 \Rightarrow F(x, y_1) \geq F(x, y_2).$$

Definition 1.2. *An element $(x, y) \in X \times X$ is said to be a coupled fixed point of the mapping $F : X \times X \to X$ if*

$$x = F(x, y) \quad and \quad y = F(y, x).$$

Throughout the rest of this paper, we denote by $(X, \leq, d)$ a complete partially ordered metric space, i.e., $\leq$ is a partial order on the set X and d is a complete metric on X. Further, we consider in the product space $X \times X$ the following partial order:

$$\text{if } (x, y), (u, v) \in X \times X, \quad (x, y) \leq (u, v) \Leftrightarrow x \leq u \text{ and } y \geq v.$$

The main theoretical results of Bhaskar and Lakshmikantham in [24] are the following coupled fixed point theorems.

Theorem 1.3. *Let $(X, \leq, d)$ be a complete partially ordered metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X and assume that there exists $k \in [0, 1)$ with*

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)], \quad for \ any \ x \geq u \ and \ y \leq v.$$

If there exist $x_0, y_0 \in X$ such that

$$x_0 \leq F(x_0, y_0) \quad and \quad y_0 \geq F(y_0, x_0)$$

and we suppose that either F is continuous or X satisfies the following property:

if (x_n) is a nondecreasing sequence with $x_n \to x$ then $x_n \leq x$ for all n,
if (y_n) is a nonincreasing sequence with $y_n \to y$ then $y \leq y_n$ for all n (1)

then F has a coupled fixed point.

Because of the important role of Theorem 1.3 in nonlinear differential equations, nonlinear integral equations and differential inclusions, many authors have studied the existence of coupled fixed points of the given mappings in several spaces and applications (see [25–37]).

In this paper, we establish the existence of a coupled fixed point theorems for a mixed monotone mapping in a partially ordered metric space which are generalizations of the results of Bhaskar and Lakshmikantham [24]. Our results improve and extend some coupled fixed point theorems of [24] and others. As an application, we give an existence and uniqueness for a solution of a nonlinear integral equation.

2. Main Results

Let Φ denote all functions $\varphi : [0, \infty) \to [0, \infty)$ which satisfy
(*i*) φ is continuous and non-decreasing,
(*ii*) $\varphi(t) = 0$ if and only if $t = 0$,
(*iii*) $\varphi(t + s) \leq \varphi(t) + \varphi(s), \forall t, s \in [0, \infty)$
and Ψ denote the set of all functions $\psi : [0, \infty) \to [0, \infty)$ which satisfy
(*iv*) ψ is a continuous function with the condition $\varphi(t) > \psi(t)$ for all $t > 0$.
Note that, by (*i*), (*ii*) and (*iv*) we have that $\psi(0) = 0$.

Theorem 2.1. *Let $(X, \leq, d)$ be a complete partially ordered metric space. Let $F : X \times X \to X$ be a mixed monotone mapping for which there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that for all $x, y, u, v \in X$ with $x \geq u, y \leq v$,*

$$\varphi(d(F(x, y), F(u, v))) \leq \frac{1}{2}\psi(d(x, u) + d(y, v)) \tag{2}$$

Suppose either
   (a) F is continuous or
   (b) X satisfies property (1).
If there exist $x_0, y_0 \in X$ with $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then F has a coupled fixed point.

Proof. Since $x_0 \leq F(x_0, y_0) = x_1$ (say) and $y_0 \geq F(y_0, x_0) = y_1$ (say), letting $x_2 = F(x_1, y_1)$ and $y_2 = F(y_1, x_1)$, we denote

$$\begin{aligned} F^2(x_0, y_0) &= F(F(x_0, y_0), F(y_0, x_0)) = F(x_1, y_1) = x_2 \\ F^2(y_0, x_0) &= F(F(y_0, x_0), F(x_0, y_0)) = F(y_1, x_1) = y_2. \end{aligned}$$

With this notation, we now have, due to the mixed monotone property of F,

$$x_2 = F(x_1, y_1) \geq F(x_0, y_0) = x_1 \quad \text{and} \quad y_2 = F(y_1, x_1) \leq F(y_0, x_0) = y_1.$$

Further, for $n = 1, 2, \ldots$, we let,

$$\begin{aligned} x_{n+1} &= F^{n+1}(x_0, y_0) = F(F^n(x_0, y_0), F^n(y_0, x_0)), \\ y_{n+1} &= F^{n+1}(y_0, x_0) = F(F^n(y_0, x_0), F^n(x_0, y_0)). \end{aligned}$$

We can easily verify that

$$\begin{aligned} x_0 &\leq F(x_0, y_0) = x_1 \leq F^2(x_0, y_0) = x_2 \leq \cdots \leq F^{n+1}(x_0, y_0) = x_{n+1}, \\ y_0 &\geq F(y_0, x_0) = y_1 \geq F^2(y_0, x_0) = y_2 \geq \cdots \geq F^{n+1}(y_0, x_0) = y_{n+1}. \end{aligned}$$

Since $x_n \geq x_{n-1}$ and $y_n \leq y_{n-1}$, from (2) we have

$$\begin{aligned} \varphi(d(x_{n+1}, x_n)) &= \varphi(d(F(x_n, y_n), F(x_{n-1}, y_{n-1}))) \\ &\leq \frac{1}{2}\psi(d(x_n, x_{n-1}) + d(y_n, y_{n-1})) \end{aligned} \tag{3}$$

Similarly, since $y_{n-1} \geq y_n$ and $x_{n-1} \leq x_n$, from (2), we also have

$$\begin{aligned} \varphi(d(y_{n+1}, y_n)) &= \varphi(d(F(y_n, x_n), F(y_{n-1}, x_{n-1}))) \\ &\leq \frac{1}{2}\psi(d(y_n, y_{n-1}) + d(x_n, x_{n-1})) \end{aligned} \tag{4}$$

From (3) and (4), we get

$$\varphi(d(x_{n+1}, x_n)) + \varphi(d(y_{n+1}, y_n)) \leq \psi(d(x_n, x_{n-1}) + d(y_n, y_{n-1})). \tag{5}$$

By property (*iii*) of φ, we have

$$\varphi(d(x_{n+1}, x_n) + d(y_{n+1}, y_n)) \leq \psi(d(x_n, x_{n-1}) + d(y_n, y_{n-1})). \tag{6}$$

Using the properties of φ and ψ, we get

$$d(x_{n+1}, x_n) + d(y_{n+1}, y_n) \leq d(x_n, x_{n-1}) + d(y_n, y_{n-1}).$$

Set $r_n = d(x_{n+1}, x_n) + d(y_{n+1}, y_n)$ then sequence $\{r_n\}$ is decreasing. Therefore, there is some $r \geq 0$ such that

$$\lim_{n\to\infty} r_n = \lim_{n\to\infty} [d(x_{n+1}, x_n) + d(y_{n+1}, y_n)] = r. \tag{7}$$

Letting $n \to \infty$ in (6), we have

$$\varphi(r) \leq \psi(r).$$

By using the properties of φ and ψ, we have $r = 0$, and hence

$$\lim_{n\to\infty} r_n = \lim_{n\to\infty} [d(x_{n+1}, x_n) + d(y_{n+1}, y_n)] = 0. \tag{8}$$

In what follows, we shall prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Suppose, to the contrary, that at least of $\{x_n\}$ or $\{y_n\}$ is not Cauchy sequence. Then there exists an $\varepsilon > 0$ for which we can find subsequences $\{x_{m_k}\}, \{x_{n_k}\}$ of $\{x_n\}$ and $\{y_{m_k}\}, \{y_{n_k}\}$ of $\{y_n\}$ with $n_k > m_k > k$ such that

$$d(x_{n_k}, x_{m_k}) + d(y_{n_k}, y_{m_k}) \geq \varepsilon. \tag{9}$$

Further, corresponding to m_k, we can choose n_k in such a way that it is the smallest integer with $n_k > m_k$ and satisfying (9). Then

$$d(x_{n_{k-1}}, x_{m_k}) + d(y_{n_{k-1}}, y_{m_k}) < \varepsilon. \tag{10}$$

Using (9), (10) and the triangle inequality, we have

$$
\begin{aligned}
\varepsilon \quad \leq \quad & \delta_k := d(x_{n_k}, x_{m_k}) + d(y_{n_k}, y_{m_k}) \\
\leq \quad & d(x_{n_k}, x_{n_{k-1}}) + d(x_{n_{k-1}}, x_{m_k}) + d(y_{n_k}, y_{n_{k-1}}) + d(y_{n_{k-1}}, y_{m_k}) \\
\leq \quad & d(x_{n_k}, x_{n_{k-1}}) + d(y_{n_k}, y_{n_{k-1}}) + \varepsilon.
\end{aligned}
$$

Taking $k \to \infty$ in the above inequality and using (8), we get

$$\lim_{k\to\infty} \delta_k = \lim_{k\to\infty} [d(x_{n_k}, x_{m_k}) + d(y_{n_k}, y_{m_k})] = \varepsilon. \tag{11}$$

By the triangle inequality, we obtain

$$
\begin{aligned}
\delta_k \quad = \quad & d(x_{n_k}, x_{m_k}) + d(y_{n_k}, y_{m_k}) \\
\leq \quad & d(x_{n_k}, x_{n_{k+1}}) + d(x_{n_{k+1}}, x_{m_{k+1}}) + d(x_{m_{k+1}}, x_{m_k}) \\
& + d(y_{n_k}, y_{n_{k+1}}) + d(y_{n_{k+1}}, y_{m_{k+1}}) + d(y_{m_{k+1}}, y_{m_k}) \\
= \quad & r_{n_k} + r_{m_k} + d(x_{n_{k+1}}, x_{m_{k+1}}) + d(y_{n_{k+1}}, y_{m_{k+1}}).
\end{aligned}
$$

Using the property of φ, we have

$$
\begin{aligned}
\varphi(\delta_k) \quad \leq \quad & \varphi(r_{n_k} + r_{m_k} + d(x_{n_{k+1}}, x_{m_{k+1}}) + d(y_{n_{k+1}}, y_{m_{k+1}})) \\
\leq \quad & \varphi(r_{n_k} + r_{m_k}) + \varphi(d(x_{n_{k+1}}, x_{m_{k+1}})) + \varphi(d(y_{n_{k+1}}, y_{m_{k+1}})).
\end{aligned} \tag{12}
$$

Since $n_k > m_k$, hence $x_{n_k} \geq x_{m_k}$ and $y_{n_k} \leq y_{m_k}$, from (2)

$$
\begin{aligned}
\varphi\left(d(x_{n_{k+1}}, x_{m_{k+1}})\right) &= \varphi\left(d(F\left(x_{n_k}, y_{n_k}\right), F\left(x_{m_k}, y_{m_k}\right))\right) \\
&\leq \frac{1}{2}\psi(d(x_{n_k}, x_{m_k}) + d(y_{n_k}, y_{m_k})) \\
&= \frac{1}{2}\psi\left(\delta_k\right).
\end{aligned} \tag{13}
$$

Similarly, we also have

$$
\begin{aligned}
\varphi\left(d(y_{m_{k+1}}, y_{n_{k+1}})\right) &= \varphi\left(d(F\left(y_{m_k}, x_{m_k}\right), F\left(y_{n_k}, x_{n_k}\right))\right) \\
&\leq \frac{1}{2}\psi(d(y_{m_k}, y_{n_k}) + d(x_{m_k}, x_{n_k})) \\
&= \frac{1}{2}\psi\left(\delta_k\right).
\end{aligned} \tag{14}
$$

From (12)-(14), we get

$$\varphi\left(\delta_k\right) \leq \varphi\left(r_{n_k} + r_{m_k}\right) + \psi\left(\delta_k\right).$$

Letting $k \to \infty$ and using (8) and (11), we have

$$\varphi\left(\varepsilon\right) \leq \varphi\left(0\right) + \psi\left(\varepsilon\right) = \psi\left(\varepsilon\right).$$

From the properties of φ and ψ, we get $\varepsilon = 0$, which is a contradiction. This shows that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Since X is a complete metric space, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} y_n = y.$$

Now, suppose that assumption (*a*) holds. Then

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} F\left(x_n, y_n\right) = F\left(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n\right) = F\left(x, y\right)$$

and

$$y = \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} F\left(y_n, x_n\right) = F\left(\lim_{n\to\infty} y_n, \lim_{n\to\infty} x_n\right) = F\left(y, x\right).$$

Therefore $x = F(x, y)$ and $y = F(y, x)$.

Suppose now assumption (*b*) holds. Since $\{x_n\}$ is a non-decreasing sequence that converges to x, we have that $x_n \leq x$ for all n. Similarly, $y_n \geq y$ for all n. Then

$$
\begin{aligned}
\varphi\left(d\left(x_{n+1}, F(x, y)\right)\right) &= \varphi\left(d\left(F\left(x_n, y_n\right), F(x, y)\right)\right) \\
&\leq \frac{1}{2}\psi(d(x_n, x) + d(y_n, y)).
\end{aligned}
$$

Letting $n \to \infty$ and using the property of φ, we have

$$\varphi\left(d\left(x, F(x, y)\right)\right) \leq \frac{1}{2}\psi(0) = 0$$

which implies $\varphi\left(d\left(x, F(x, y)\right)\right) = 0$. Thus $d\left(x, F(x, y)\right) = 0$ or equivalently, $x = F(x, y)$.
Similarly, one can show that $y = F(y, x)$. $\square$

If we take $\varphi\left(t\right) = t$ and $\psi\left(t\right) = kt$ in Theorem 2.1, we have the following corollary.

Corollary 2.2 (Bhaskar and Lakshmikantham [24]). *Let $(X, \leq, d)$ be a complete partially ordered metric space. Let $F : X \times X \to X$ be a mixed monotone mapping for which there exist $k \in [0, 1)$such that for all $x, y, u, v \in X$ with $x \geq u, y \leq v$,*

$$d(F(x, y), F(u, v)) \leq \frac{k}{2} [d(x, u) + d(y, v)]$$

Suppose either
(a) F is continuous or
(b) X satisfies property (1).
If there exist $x_0, y_0 \in X$ with $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then F has a coupled fixed point.

If we take $\psi(t) = \varphi(t) - \psi_1(t)$ in Theorem 2.1, we have the following corollary.

Corollary 2.3 (Luong and Thuan [25]). *Let $(X, \leq, d)$ be a complete partially ordered metric space. Let $F : X \times X \to X$ be a mixed monotone mapping for which there exist $\varphi \in \Phi$ and $\psi_1 \in \Psi$ such that for all $x, y, u, v \in X$ with $x \geq u$, $y \leq v$,*

$$\varphi(d(F(x, y), F(u, v))) \leq \frac{1}{2} \varphi(d(x, u) + d(y, v)) - \psi_1(\frac{d(x, u) + d(y, v)}{2})$$

Suppose either
(a) F is continuous or
(b) X satisfies property (1).
If there exist $x_0, y_0 \in X$ with $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then F has a coupled fixed point.

If we take $\varphi(t) = t$ in Corollary 2.3, we have the following corollary.

Corollary 2.4. *Let $(X, \leq, d)$ be a complete partially ordered metric space. Let $F : X \times X \to X$ be a mixed monotone mapping for which there exist $\psi_1 \in \Psi$ such that for all $x, y, u, v \in X$ with $x \geq u, y \leq v$,*

$$d(F(x, y), F(u, v)) \leq \frac{1}{2}(d(x, u) + d(y, v)) - \psi_1(\frac{d(x, u) + d(y, v)}{2})$$

Suppose either
(a) F is continuous or
(b) X satisfies property (1).
If there exist $x_0, y_0 \in X$ with $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then F has a coupled fixed point.

Next theorem gives a sufficient condition for the uniqueness of the coupled fixed point.

Theorem 2.5. *Let all the conditions of Theorem 2.1 be fulfilled and let the following condition be satisfied: for arbitrary points $(x, y), (u, v) \in X \times X$ there exists $(z, t) \in X \times X$ which is comparable with both (x, y) and (u, v). Then F has a unique coupled fixed point.*

Proof. From Theorem 2.1, the set of coupled fixed points of F is non-empty. Suppose (x, y) and (u, v) are coupled fixed points of F, that is, $x = F(x, y)$, $y = F(y, x)$, $u = F(u, v)$ and $v = F(v, u)$. We shall show that $x = u$ and $y = v$.

By assumption, there exists $(z, t) \in X \times X$ that is comparable to (x, y) and (u, v).
We define sequences $\{z_n\}$ and $\{t_n\}$ as follows

$$z_0 = z, \quad t_0 = t, \quad z_{n+1} = F(z_n, t_n) \quad \text{and} \quad t_{n+1} = F(t_n, z_n) \quad \text{for all } n.$$

Since (z, t) is comparable with (x, y), we may assume that $(x, y) \geq (z, t) = (z_0, t_0)$. By using the mathematical induction, it is easy to prove that $(x, y) \geq (z_n, t_n)$, for all n. Then by (2), we have

$$
\begin{aligned}
\varphi(d(x, z_{n+1})) &= \varphi(d(F(x, y), F(z_n, t_n))) \\
&\leq \frac{1}{2} \psi(d(x, z_n) + d(y, t_n)),
\end{aligned}
\tag{15}
$$

and

$$
\begin{aligned}
\varphi\left(d(t_{n+1},y)\right) &= \varphi\left(d(F\left(t_n,z_n\right),F(y,x))\right) \\
&\leq \frac{1}{2}\psi(d(t_n,y)+d(z_n,x)).
\end{aligned}
\tag{16}
$$

From (15), (16) and the property of φ, we get

$$
\begin{aligned}
\varphi\left(d(x,z_{n+1})+d(y,t_{n+1})\right) &\leq \varphi\left(d(x,z_{n+1})\right)+\varphi\left(d(y,t_{n+1})\right) \\
&\leq \psi(d(x,z_n)+d(y,t_n)).
\end{aligned}
\tag{17}
$$

Hence, φ is a nondecreasing function and the condition of $\varphi(t) > \psi(t)$ for $t > 0$. This gives us that $\{d(x,z_n)+d(y,t_n)\}$ is a nonnegative decreasing sequence, and consequently, there exists $\gamma \geq 0$ such that

$$
\lim_{n\to\infty}\left[d(x,z_n)+d(y,t_n)\right]=\gamma.
\tag{18}
$$

Suppose that $\gamma > 0$. Letting $n \to \infty$ in (17) and taking into account that φ and ψ are continuous functions, we obtain

$$
\varphi\left(\gamma\right)\leq\psi\left(\gamma\right)
$$

which implies, by the properties of φ and ψ, that $\psi\left(\gamma\right)=0$ and consequently, $\gamma = 0$. Therefore

$$
\lim_{n\to\infty}\left[d(x,z_n)+d(y,t_n)\right]=0.
$$

It follows that

$$
\lim_{n\to\infty}d(x,z_n)=\lim_{n\to\infty}d(y,t_n)=0.
$$

Similarly, one can show that

$$
\lim_{n\to\infty}d(u,z_n)=\lim_{n\to\infty}d(v,t_n)=0.
$$

From the triangle inequality, we have

$$
\begin{aligned}
d(x,u) &\leq d(x,z_n)+d(z_n,u), \\
d(y,v) &\leq d(y,t_n)+d(t_n,v).
\end{aligned}
$$

Taking the limit as $n \to \infty$ in the above inequality, we get

$$
d(x,u)=d(y,v)=0
$$

and hence $x = u$ and $y = v$. $\square$

Corollary 2.6. *Let all the conditions of Corollary 2.2 (resp. Corollary 2.3 and Corollary 2.4) be fulfilled and let the following condition be satisfied: for arbitrary points $(x,y),(u,v) \in X \times X$ there exists $(z,t) \in X \times X$ which is comparable with both (x,y) and (u,v). Then F has a unique coupled fixed point.*

An alternative uniqueness condition is given in the next theorem.

Theorem 2.7. *In addition to hypotheses of Theorem 2.1, if x_0 and y_0 are comparable then $x = F(x,y) = F(y,x) = y$ where (x,y) is a coupled fixed point of F.*

Proof. Following the proof of Theorem 2.1, F has a coupled fixed point (x, y). We only have to show that $x = y$. Since x_0 and y_0 are comparable, we may assume that $x_0 \geq y_0$. By using the mathematical induction, one can show that $x_n \geq y_n$, where $x_{n+1} = F(x_n, y_n)$ and $y_{n+1} = F(y_n, x_n)$ for all $n \geq 0$.

By the triangle inequality, we obtain

$$\begin{aligned} d(x, y) &\leq d(x, x_{n+1}) + d(x_{n+1}, y_{n+1}) + d(y_{n+1}, y) \\ &= d(x, x_{n+1}) + d(y_{n+1}, y) + d\left(F(x_n, y_n), F(y_n, x_n)\right). \end{aligned}$$

Therefore, by (2) and the property of φ, we have

$$\begin{aligned} \varphi\left(d(x, y)\right) &\leq \varphi\left(d(x, x_{n+1}) + d(y_{n+1}, y)\right) + \varphi\left(d\left(F(x_n, y_n), F(y_n, x_n)\right)\right) \\ &\leq \varphi\left(d(x, x_{n+1}) + d(y_{n+1}, y)\right) + \frac{1}{2}\psi\left(d(x_n, y_n) + d(y_n, x_n)\right) \\ &\leq \varphi\left(d(x, x_{n+1}) + d(y_{n+1}, y)\right) + \psi\left(d(x_n, y_n)\right) \end{aligned} \tag{19}$$

Suppose $x \neq y$, that is, $d(x, y) > 0$, letting $n \to \infty$ in (19), we get

$$\varphi\left(d(x, y)\right) \leq \varphi(0) + \psi\left(d(x, y)\right)$$

which shows, by the properties of φ and ψ, that $d(x, y) = 0$ and so $x = y$. □

Corollary 2.8. *In addition to hypotheses of Corollary 2.2 (resp. Corollary 2.3 and Corollary 2.4), if x_0 and y_0 are comparable then $x = F(x, y) = F(y, x) = y$ where (x, y) is a coupled fixed point of F.*

3. Application to Integral Equations

In this section, we study the existence of a unique solution to a nonlinear integral equation, as an application to the fixed point theorem proved in Section 2.

Consider the following integral equation:

$$x(t) = \int_a^b (K_1(t, s) + K_2(t, s))(f(s, x(s)) + g(s, x(s)))ds + a(t), \quad t \in I = [a, b]. \tag{20}$$

We will analyze Eq. (20) under the following assumptions:

(*i*) $K_1, K_2 \in C(I \times I, \mathbb{R})$ and $K_1(t, s) \geq 0$ and $K_2(t, s) \leq 0$.

(*ii*) $a \in C(I, R)$.

(*iii*) $f, g \in C(I \times \mathbb{R}, \mathbb{R})$.

(*iv*) There exist constants $\lambda, \mu > 0$ such that for all $x, y \in \mathbb{R}$ and $x \geq y$

$$0 \leq f(t, x) - f(t, y) \leq \lambda \ln\left(\left|x - y\right| + 1\right)$$

and

$$-\mu \ln\left(\left|x - y\right| + 1\right) \leq g(t, x) - g(t, y) \leq 0.$$

(*v*) $4 \cdot \max(\lambda, \mu)\|K_1 - K_2\|_\infty \leq 1$, where

$$\|K_1 - K_2\|_\infty = sup\{(K_1(t, s) - K_2(t, s)) : t, s \in I\}.$$

(*vi*) There exist $(\alpha, \beta) \in C(I, \mathbb{R}) \times C(I, \mathbb{R})$ a coupled lower and upper solution of the integral equation (20) if $\alpha(t) \leq \beta(t)$ and

$$\alpha(t) \leq \int_a^b K_1(t, s)(f(s, \alpha(s)) + g(s, \beta(s)))ds + \int_a^b K_2(t, s)(f(s, \beta(s)) + g(s, \alpha(s)))ds + a(t)$$

and

$$\beta(t) \geq \int_a^b K_1(t,s)(f(s,\beta(s)) + g(s,\alpha(s)))ds + \int_a^b K_2(t,s)(f(s,\alpha(s)) + g(s,\beta(s)))ds + a(t)$$

for all $t \in I$.

Theorem 3.1. *Under assumptions* $(i) - (vi)$, *Eq. (20) has a unique solution in* $C(I, \mathbb{R})$.

Proof. Let $X := C(I, \mathbb{R})$. X is a partially ordered set if we define the following order relation in X:

$$x, y \in C(I, \mathbb{R}), \quad x \leq y \Leftrightarrow x(t) \leq y(t), \quad \forall t \in I.$$

And (X, d) is a complete metric space with metric

$$d(x, y) = \sup_{t \in I} |x(t) - y(t)|, \quad x, y \in C(I, \mathbb{R}).$$

Now define on $X \times X$ the following partial order: for $(x, y), (u, v) \in X \times X$,

$$(x, y) \leq (u, v) \Leftrightarrow x(t) \leq u(t) \ \text{ and } \ y(t) \geq v(t), \quad \forall t \in I.$$

Obviously, for any $(x, y) \in X \times X$, the functions $\max\{x, y\}, \min\{x, y\}$ are the upper and lower bounds of x, y, respectively.

Therefore, for every $(x, y), (u, v) \in X \times X$, there exists the element $(\max\{x, u\}, \min\{y, v\})$ which is comparable to (x, y) and (u, v).

Define now the mapping $F : X \times X \to X$ by

$$
\begin{aligned}
F(x, y)(t) \ &= \ \int_a^b K_1(t, s)\left(f(s, x(s)) + g(s, y(s))\right) ds \\
&\quad + \int_a^b K_2(t, s)(f(s, y(s)) + g(s, x(s)))ds + a(t), \quad \forall t \in I.
\end{aligned}
$$

Now we shall show that F has the mixed monotone property. Indeed, for $x_1 \leq x_2$ and $t \in I$, we have

$$
\begin{aligned}
F(x_1, y)(t) - F(x_2, y)(t) \ &= \ \int_a^b K_1(t, s)(f(s, x_1(s)) + g(s, y(s)))ds \\
&\quad + \int_a^b K_2(t, s)(f(s, y(s)) + g(s, x_1(s)))ds + a(t) \\
&\quad - \int_a^b K_1(t, s)(f(s, x_2(s)) + g(s, y(s)))ds \\
&\quad - \int_a^b K_2(t, s)(f(s, y(s)) + g(s, x_2(s)))ds - a(t) \\
&= \ \int_a^b K_1(t, s)(f(s, x_1(s)) - f(s, x_2(s)))ds \\
&\quad + \int_a^b K_2(t, s)(g(s, x_1(s)) - g(s, x_2(s)))ds \leq 0,
\end{aligned}
$$

by our assumptions. Hence $F(x_1, y)(t) \leq F(x_2, y)(t)$, $\forall t \in I$, that is, $F(x_1, y) \leq F(x_2, y)$.

Similarly, if $y_1 \geq y_2$ and $t \in I$, we have

$$
\begin{aligned}
F(x, y_1)(t) - F(x, y_2)(t) &= \int_a^b K_1(t,s)(f(s, x(s)) + g(s, y_1(s)))ds \\
&+ \int_a^b K_2(t,s)(f(s, y_1(s)) + g(s, x(s)))ds + a(t) \\
&- \int_a^b K_1(t,s)(f(s, x(s)) + g(s, y_2(s)))ds \\
&- \int_a^b K_2(t,s)(f(s, y_2(s)) + g(s, x(s)))ds - a(t) \\
&= \int_a^b K_1(t,s)(g(s, y_1(s)) - g(s, y_2(s)))ds \\
&+ \int_a^b K_2(t,s)(f(s, y_1(s)) - f(s, y_2(s)))ds \leq 0,
\end{aligned}
$$

by our assumptions. Hence $F(x, y_1)(t) \leq F(x, y_2)(t)$, $\forall t \in I$, that is, $F(x, y_1) \leq F(x, y_2)$.
Thus, $F(x, y)$ is monotone nondecreasing in x and monotone nonincreasing in y.
In what follows, we estimate $d(F(x, y), F(u, v))$ for $x \geq u$ and $y \leq v$.
Indeed, as F has the mixed monotone property, $F(x, y) \geq F(u, v)$ and we have

$$
\begin{aligned}
d(F(x, y), F(u, v)) &= \sup_{t \in I} |F(x, y)(t) - F(u, v)(t)| \\
&= \sup_{t \in I} (F(x, y)(t) - F(u, v)(t)) \\
&= \sup_{t \in I}[\int_a^b K_1(t,s)(f(s, x(s)) + g(s, y(s)))ds + \int_a^b K_2(t,s)(f(s, y(s)) + g(s, x(s)))ds + a(t) \\
&- \int_a^b K_1(t,s)(f(s, u(s)) + g(s, v(s)))ds - \int_a^b K_2(t,s)(f(s, v(s)) + g(s, u(s)))ds - a(t)] \\
&= \sup_{t \in I}[\int_a^b K_1(t,s)[(f(s, x(s)) - f(s, u(s))) - (g(s, v(s)) - g(s, y(s)))]ds \\
&- \int_a^b K_2(t,s)[(f(s, v(s)) - f(s, y(s))) - (g(s, x(s)) - g(s, u(s)))]ds] \\
&\leq \sup_{t \in I}[\int_a^b K_1(t,s)[\lambda \ln(|x(s) - u(s)| + 1)) + \mu \ln(|y(s) - v(s)| + 1))]ds \\
&+ \int_a^b (-K_2(t,s))[\lambda \ln(|v(s) - y(s)| + 1)) + \mu \ln(|x(s) - u(s)| + 1))]ds] \\
&\leq \max(\lambda, \mu)\sup_{t \in I}[\int_a^b (K_1(t,s) - K_2(t,s)) \ln(|x(s) - u(s)| + 1)\, ds \\
&+ \int_a^b (K_1(t,s) - K_2(t,s)) \ln(|y(s) - v(s)| + 1)\, ds].
\end{aligned}
\tag{21}
$$

Defining

$$
(I) = \int_a^b (K_1(t,s) - K_2(t,s)) \ln(|x(s) - u(s)| + 1)\, ds
$$

$$
(II) = \int_a^b (K_1(t,s) - K_2(t,s)) \ln(|y(s) - v(s)| + 1)\, ds
$$

and using the Cauchy–Schwartz inequality in (I) we obtain

$$
\begin{aligned}
(I) \quad &\leq \quad \Big(\int_a^b (K_1(t,s) - K_2(t,s))^2 ds\Big)^{\frac{1}{2}} \cdot \Big(\int_a^b (\ln(|x(s) - u(s)| + 1))^2 ds\Big)^{\frac{1}{2}} \\
&\leq \quad \|K_1 - K_2\|_\infty \cdot (\ln(|x - u| + 1)) = \|K_1 - K_2\|_\infty \cdot (\ln(d(x,u) + 1)).
\end{aligned}
\tag{22}
$$

Similarly, we can obtain the following estimate for (II) :

$$
(II) \leq \|K_1 - K_2\|_\infty \cdot (\ln(d(y,v) + 1)).
\tag{23}
$$

By (21)–(23) and assumption (v), we get

$$
\begin{aligned}
d(F(x,y), F(u,v)) \quad &\leq \quad \max(\lambda, \mu)\|K_1 - K_2\|_\infty \left[\ln(d(x,u) + 1) + \ln(d(y,v) + 1)\right] \\
&\leq \quad \max(\lambda, \mu)\|K_1 - K_2\|_\infty \left[\ln(d(x,u) + d(y,v) + 1) + \ln(d(y,v) + d(x,u) + 1)\right] \\
&= \quad 2\max(\lambda, \mu)\|K_1 - K_2\|_\infty \left[\ln(d(x,u) + d(y,v) + 1)\right] \\
&\leq \quad \frac{1}{2}\ln(d(x,u) + d(y,v) + 1).
\end{aligned}
\tag{24}
$$

Put $\varphi(x) = x$ and $\psi(x) = \ln(x + 1)$. Obviously, $\varphi \in \Phi$ and $\psi \in \Psi$, and by (24) we have

$$
\varphi(d(F(x,y), F(u,v))) \leq \frac{1}{2}\psi(d(x,u) + d(y,v)).
$$

This proves that the operator F satisfies the contractive condition appearing in Theorem 2.1.

Finally, let (α, β) be a coupled lower and upper solution of the integral equation (20) then, by assumption (vi), we have $\alpha \leq F(\alpha, \beta) \leq F(\beta, \alpha) \leq \beta$. Theorem 2.5 gives us that F has a unique coupled fixed point $(x,y) \in X \times X$. Since $\alpha \leq \beta$, Theorem 2.7 says us that $x = y$ and this implies $x = F(x,x)$ and x is the unique solution of Eq. (20). $\quad\square$

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