Filomat 28:6 (2014), 1271–1279
DOI 10.2298/FIL1406271E

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On Pointwise $\lambda-$Statistical Convergence of Order α of Sequences of Fuzzy Mappings

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Abstract. In this study, we introduce the concepts of pointwise $\lambda-$statistical convergence of order α of sequences of fuzzy mappings and strongly pointwise $(V, \lambda, p)-$summable of order α of sequences of fuzzy mappings. Some relations between pointwise $\lambda-$statistical convergence of order α and strongly pointwise $(V, \lambda, p)-$summable of order α of sequences of fuzzy mappings are given.

1. Introduction

The idea of statistical convergence was given by Zygmund [41] in the first edition of his monograph published in Warsaw in 1935. The concept of statistical convergence was introduced by Steinhaus [36] and Fast [16] and later reintroduced by Schoenberg [35] independently. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory, number theory, measure theory, trigonometric series, turnpike theory and Banach spaces. Later on it was further investigated from the sequence space point of view and linked with summability theory by Alotaibi and Alroqi [1], Connor [10], Et *et al.* ([12],[14]), Fridy [17], Güngor *et al.* ([20],[21]), Işık [22], Kolk [23], Mohiuddine *et al.* ([26],[28]) Mursaleen *et al.* ([27],[29],[30]), Salat [32], Ozarslan *et al.* [31], Srivastava *et al.* ([13],[33]) Tripathy [38] and many others.

The existing literature on statistical convergence appears to have been restricted to real or complex sequences, but Altin *et al.*([2],[3]) Altinok *et al.* [4], Burgin [5], Colak *et al.* [6], Et *et al.* [15], Gökhan *et al.* [19], Savaş [34], Talo and Başar [37], Tripathy and Dutta [39] extended the idea to apply to sequences of fuzzy numbers.

In the present paper, we introduce pointwise $\lambda-$statistical convergence of order α and strongly pointwise $(V, \lambda, p)-$summable of order α of sequences of fuzzy mappings. In section 2 we give a brief overview about statistical convergence, $p-$Cesàro summability and fuzzy numbers. In section 3 we introduce the concepts of pointwise $\lambda-$statistical convergence of order α and strongly pointwise $(V, \lambda, p)-$summable of order α of sequences of fuzzy mappings. We also establish some inclusion relations between $w^{\alpha}_{\lambda p}(F)$ and $S^{\alpha}_{\lambda}(F)$ for different $\alpha' s$ and $\mu' s$.

2010 *Mathematics Subject Classification*. 40A05; 40C05; 46A45; 03E72
Keywords. Pointwise statistical convergence, Sequences of fuzzy mappings, Cesàro summability
Received: 06 June 2013; Accepted: 18 October 2013
Communicated by Hari Srivastava
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2. Definitions and Preliminaries

A sequence $x = (x_k)$ is said to be statistically convergent to L if for every $\varepsilon > 0$, $\delta(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}) = 0$. In this case we write $x_k \overset{\text{stat}}{\longrightarrow} L$ or $S - \lim x_k = L$. The set of all statistically convergent sequences will be denoted by S.

The order of statistical convergence of a sequence of numbers was given by Gadjiev and Orhan in [18] and after then statistical convergence of order α and strong $p-$Cesàro summability of order α was studied by Çolak ([7], [8]) and was generalized by Çolak and Asma [9].

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive real numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$. The generalized de la Vallée-Poussin mean is defined by $t_n(x) = \frac{1}{\lambda_n} \sum\limits_{k \in I_n} x_k$, where $I_n = [n - \lambda_n + 1, n]$ for $n = 1, 2, \ldots$. A sequence $x = (x_k)$ is said to be $(V, \lambda)-$summable to a number L if $t_n(x) \to L$ as $n \to \infty$.

By Λ we denote the class of all non-decreasing sequence of positive real numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$.

Fuzzy sets are considered with respect to a nonempty base set X of elements of interest. The essential idea is that each element $x \in X$ is assigned a membership grade $u(x)$ taking values in $[0, 1]$, with $u(x) = 0$ corresponding to nonmembership, $0 < u(x) < 1$ to partial membership, and $u(x) = 1$ to full membership. According to Zadeh [40] a fuzzy subset of X is a nonempty subset $\{(x, u(x)) : x \in X\}$ of $X \times [0, 1]$ for some function $u : X \longrightarrow [0, 1]$. The function u itself is often used for the fuzzy set.

Let $C(\mathbb{R}^n)$ denote the family of all nonempty, compact, convex subsets of $\mathbb{R}^n$. If $\alpha, \beta \in \mathbb{R}$ and $A, B \in C(\mathbb{R}^n)$, then

$$\alpha(A + B) = \alpha A + \alpha B, \qquad (\alpha\beta)A = \alpha(\beta A), \qquad 1A = A$$

and if $\alpha, \beta \geq 0$, then $(\alpha + \beta)A = \alpha A + \beta A$. The distance between A and B is defined by the Hausdorff metric

$$\delta_\infty(A, B) = \max\{\sup_{a \in A} \inf_{b \in B} \| a - b \|, \sup_{b \in B} \inf_{a \in A} \| a - b \|\},$$

where $\| \cdot \|$ denotes the usual Euclidean norm in $\mathbb{R}^n$. It is well known that $(C(\mathbb{R}^n), \delta_\infty)$ is a complete metric space.

Denote

$$L(\mathbb{R}^n) = \{u : \mathbb{R}^n \longrightarrow [0, 1] : u \text{ satisfies } (i) - (iv) \text{ below}\},$$

where

i) u is normal, that is, there exists an $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$;
ii) u is fuzzy convex, that is, for $x, y \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1$, $u(\lambda x + (1 - \lambda)y) \geq \min[u(x), u(y)]$;
iii) u is upper semicontinuous;
iv) the closure of $\{x \in \mathbb{R}^n : u(x) > 0\}$, denoted by $[u]^0$, is compact.

If $u \in L(\mathbb{R}^n)$, then u is called a fuzzy number, and $L(\mathbb{R}^n)$ is said to be a fuzzy number space.

For $0 < \alpha \leq 1$, the α-level set $[u]^\alpha$ is defined by

$$[u]^\alpha = \{x \in \mathbb{R}^n : u(x) \geq \alpha\}.$$

Then from $(i) - (iv)$, it follows that the α-level sets $[u]^\alpha \in C(\mathbb{R}^n)$.

Define, for each $1 \leq q < \infty$,

$$d_q(u, v) = \left(\int_0^1 [\delta_\infty([u]^\alpha, [v]^\alpha)]^q \, d\alpha \right)^{1/q}$$

and $d_\infty(u, v) = \sup\limits_{0 \leq \alpha \leq 1} \delta_\infty([u]^\alpha, [v]^\alpha)$, where δ_∞ is the Hausdorff metric. Clearly $d_\infty(u, v) = \lim\limits_{q \to \infty} d_q(u, v)$ with $d_q \leq d_s$ if $q \leq s$ ([11], [24]). For simplicity in notation, throughout the paper d will denote the notation d_q with $1 \leq q \leq \infty$.

A fuzzy mapping X is a mapping from a set $T\,(\subset \mathbb{R}^n)$ to the set of all fuzzy numbers. A sequence of fuzzy mappings is a function whose domain is the set of positive integers and whose range is a set of fuzzy mappings. We denote a sequence of fuzzy mappings by (X_k). If (X_k) is a sequence of fuzzy mappings then $(X_k(t))$ is a sequence of fuzzy numbers for every $t \in T$. Corresponding to a number t in the domain of each of terms of the sequence of fuzzy mappings (X_k), there is a sequence of fuzzy numbers $(X_k(t))$. If $(X_k(t))$ converges for each number t in a set T and we get $\lim_k X_k(t) = X(t)$, then we say that (X_k) converges pointwise to X on T [25]. By $B(A_F)$ we denote the class of all bounded sequences of fuzzy mappings on T.

Throughout the paper, unless stated otherwise, by "for all $n \in \mathbb{N}_{n_o}$" we mean "for all $n \in \mathbb{N}$ except finite numbers of positive integers" where $\mathbb{N}_{n_o} = \{n_o, n_o + 1, n_o + 2, ...\}$ for some $n_o \in \mathbb{N} = \{1, 2, 3, ...\}$.

3. Main Results

In this section we give the main results of this paper. In Theorem 3.7 we give the inclusion relations between the sets of pointwise $\lambda-$statistical convergence of order α of sequences of fuzzy mappings for different $\alpha' s$ and $\mu' s$. In Theorem 3.10 we give the relationship between the sets of strongly pointwise $(V, \lambda, p)-$summable of order α of sequences of fuzzy mappings for different $\alpha' s$ and $\mu' s$. In Theorem 3.13 we give the relationship between pointwise $\lambda-$statistical convergence of order α and strongly pointwise $(V, \lambda, p)-$summable of order α of sequences of fuzzy mappings for different $\alpha' s$ and $\mu' s$.

Before giving the inculision relations we will give two new definitions.

Definition 3.1 Let the sequence $\lambda = (\lambda_n)$ be as above and $\alpha \in (0, 1]$ be any real number. A sequence of fuzzy mappings (X_k) is said to be pointwise $\lambda-$statistically convergent (or pointwise $S_\lambda^\alpha-$statistically convergent) of order α to X on a set T if, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} \left|\{k \in I_n : d(X_k(t), X(t)) \geq \varepsilon \text{ for every } t \in T\}\right| = 0,$$

where $I_n = [n - \lambda_n + 1, n]$ and λ_n^α denote the α^{th} power $(\lambda_n)^\alpha$ of λ_n, that is $\lambda^\alpha = (\lambda_n^\alpha) = \left(\lambda_1^\alpha,\ \lambda_2^\alpha,\ ...,\ \lambda_n^\alpha,\ ...\right)$. In this case we write $S_\lambda^\alpha - \lim X_k(t) = X(t)$ on T. $S_\lambda^\alpha - \lim X_k(t) = X(t)$ means that for every $\delta > 0$ and $0 < \alpha \leq 1$, there is an integer N such that

$$\frac{1}{\lambda_n^\alpha} \left|\{k \in I_n : d(X_k(t), X(t)) \geq \varepsilon \text{ for every } t \in T\}\right| < \delta,$$

for all $n > N\ (= N(\varepsilon, \delta, x))$ and for each $\varepsilon > 0$.

The set of all pointwise $\lambda-$statistically convergent sequences of fuzzy mappings of order α will be denoted by $S_\lambda^\alpha(F)$. For $\lambda_n = n$ for all $n \in \mathbb{N}$ and $\alpha = 1$, we shall write $S(F)$ instead of $S_\lambda^\alpha(F)$ and in the special case $X = \bar{0}$, we shall write $S_{0\lambda}^\alpha(F)$ instead of $S_\lambda^\alpha(F)$, where

$$\bar{0}(t) = \begin{cases} 1, & \text{for } t = (0, 0, 0, ..., 0) \\ 0, & \text{otherwise} \end{cases}$$

Definition 3.2 Let the sequence $\lambda = (\lambda_n)$ be as above, $\alpha \in (0, 1]$ be any real number and let p be a positive real number. A sequence of fuzzy mappings (X_k) is said to be strongly pointwise $(V, \lambda, p)-$summable of order α (or strongly pointwise $w_{\lambda p}^\alpha-$summable), if there is a function X such that

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} \sum_{k\in I_n, t\in T} (d(X_k(t), X(t)))^p = 0.$$

In this case we write $w_{\lambda p}^\alpha - \lim X_k(t) = X(t)$ on T. The set of all strongly pointwise $(V, \lambda, p)-$summable of order α of sequences of fuzzy mappings will be denoted by $w_{\lambda p}^\alpha(F)$. In the special case $X = \bar{0}$, we shall write $w_{\lambda p_0}^\alpha(F)$ instead of $w_{\lambda p}^\alpha(F)$.

Definition 3.3 A fuzzy sequence space $E(F)$ with metric d is said to be normal (or solid) if $(X_k) \in E(F)$ and (Y_k) is such that $d(Y_k, \bar{0}) \leq d(X_k, \bar{0})$ implies $(Y_k) \in E(F)$.

Definition 3.4 A fuzzy sequence space $E(F)$ is said to be monotone if $X = (X_k) \in E(F)$ implies $\chi.X \subset E(F)$ where χ is the class of all sequences of zeros and ones. The product considered is the term product.

Remark 1. From the above definitions it follows that if a fuzzy sequence space $E(F)$ is solid, then it is monotone.

Definition 3.5 A fuzzy sequence space $E(F)$ is said to be symmetric if $(X_k) \in E(F)$ implies $(X_{\pi(k)}) \in E(F)$, where π is a permutation of $\mathbb{N}$.

The proof of the following theorem is easy, so we state without proof.

Theorem 3.6 Let $0 < \alpha \leq 1$ and (X_k) and (Y_k) be two sequences of fuzzy mappings.

(i) If $S_\lambda^\alpha - \lim X_k(t) = X_0(t)$ and $c \in \mathbb{R}$, then $S_\lambda^\alpha - \lim cX_k(t) = cX_0(t)$,

(ii) If $S_\lambda^\alpha - \lim X_k(t) = X_0(t)$ and $S_\lambda^\alpha - \lim Y_k(t) = Y_0(t)$, then $S_\lambda^\alpha - lim\,(X_k(t) + Y_k(t)) = X_0(t) + Y_0(t)$.

Theorem 3.7 Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}_{n_o}$, $(0 < \alpha \leq \beta \leq 1)$ and (X_k) be a sequence of fuzzy mappings.

(i) If

$$\liminf_{n\to\infty} \frac{\lambda_n^\alpha}{\mu_n^\beta} > 0 \tag{1}$$

then $S_\mu^\beta(F) \subseteq S_\lambda^\alpha(F)$,

(ii) If

$$\lim_{n\to\infty} \frac{\mu_n}{\lambda_n^\beta} = 1 \tag{2}$$

then $S_\lambda^\alpha(F) \subseteq S_\mu^\beta(F)$.

Proof (i) Suppose that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}_{n_o}$ and let (1) be satisfied. Then $I_n \subset J_n$ and so that $\varepsilon > 0$ we may write

$$\{k \in J_n : d(X_k(t), X(t)) \geq \varepsilon,\ \text{for every } x \in T\} \supset \{k \in I_n : d(X_k(t), X(t)) \geq \varepsilon,\ \text{for every } t \in T\}$$

and so

$$\frac{1}{\mu_n^\beta}\left|\{k \in J_n : d(X_k(t), X(t)) \geq \varepsilon,\ \text{for every } x \in T\}\right| \geq \frac{\lambda_n^\alpha}{\mu_n^\beta}\frac{1}{\lambda_n^\alpha}\left|\left\{\begin{array}{c} k \in I_n : d(X_k(t), X(t)) \geq \varepsilon, \\ \text{for every } t \in T \end{array}\right\}\right|$$

for all $n \in \mathbb{N}_{n_o}$, where $J_n = [n - \mu_n + 1, n]$. Now taking the limit as $n \to \infty$ in the last inequality and using (1) we get $S_\mu^\beta(F) \subseteq S_\lambda^\alpha(F)$.

(ii) Let $S_\lambda^\alpha - \lim X_k(t) = X(t)$ on T and (2) be satisfied. Since $I_n \subset J_n$, for $\varepsilon > 0$ we may write

$$\begin{aligned}
&\frac{1}{\mu_n^\beta}\left|\{k \in J_n : d(X_k(t), X(t)) \geq \varepsilon,\ \text{for every } t \in T\}\right| \\
&= \frac{1}{\mu_n^\beta}\left|\{n - \mu_n + 1 \leq k \leq n - \lambda_n : d(X_k(t), X(t)) \geq \varepsilon,\ \text{for every } t \in T\}\right| \\
&+ \frac{1}{\mu_n^\beta}\left|\{k \in I_n : d(X_k(t), X(t)) \geq \varepsilon,\ \text{for every } t \in T\}\right| \\
&\leq \frac{\mu_n - \lambda_n}{\mu_n^\beta} + \frac{1}{\mu_n^\beta}\left|\{k \in I_n : d(X_k(t), X(t)) \geq \varepsilon,\ \text{for every } x \in T\}\right| \\
&\leq \frac{\mu_n - \lambda_n^\beta}{\lambda_n^\beta} + \frac{1}{\mu_n^\beta}\left|\{k \in I_n : d(X_k(t), X(t)) \geq \varepsilon,\ \text{for every } t \in T\}\right| \\
&\leq \left(\frac{\mu_n}{\lambda_n^\beta} - 1\right) + \frac{1}{\lambda_n^\alpha}\left|\{k \in I_n : d(X_k(t), X(t)) \geq \varepsilon,\ \text{for every } t \in T\}\right|
\end{aligned}$$

for all $n \in \mathbb{N}_{n_o}$. Since $\lim_n \frac{\mu_n}{\lambda_n^\beta} = 1$ and by (2) the first term and since $S_\lambda^\alpha - \lim X_k(t) = X(t)$ on T, the second term of right hand side of above inequality tend to 0 as $n \to \infty$ (Note that $\left(\frac{\mu_n}{\lambda_n^\beta} - 1\right) \geq 0$ for all $n \in \mathbb{N}_{n_o}$). This implies that $S_\lambda^\alpha(F) \subseteq S_\lambda^\beta(F)$.

From Theorem 3.7 we have the following.

Corollary 3.8 Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}_{n_o}$ and (X_k) be a sequence of fuzzy mappings. If (1) holds then,

i) $S_\mu^\alpha(F) \subseteq S_\lambda^\alpha(F)$ for each $\alpha \in (0,1]$ and for all $t \in T$,

ii) $S_\mu(F) \subseteq S_\lambda^\alpha(F)$ for each $\alpha \in (0,1]$ and for all $t \in T$,

iii) $S_\mu(F) \subseteq S_\lambda(F)$ for all $t \in T$.

Corollary 3.9 Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}_{n_o}$ and (X_k) be a sequence of fuzzy mappings. If (2) holds then,

i) $S_\lambda^\alpha(F) \subseteq S_\mu^\alpha(F)$ for each $\alpha \in (0,1]$ and for all $t \in T$,

ii) $S_\lambda^\alpha(F) \subseteq S_\mu(F)$ for each $\alpha \in (0,1]$ and for all $t \in T$,

iii) $S_\lambda(F) \subseteq S_\mu(F)$ for all $t \in T$.

Theorem 3.10 Given for $\lambda = (\lambda_n)$, $\mu = (\mu_n) \in \Lambda$ suppose that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}_{n_o}$, $(0 < \alpha \leq \beta \leq 1)$ and (X_k) be a sequence of fuzzy mappings. Then

(*i*) If (1) holds then $w_{\mu p}^\beta(F) \subset w_{\lambda p}^\alpha(F)$ for all $t \in T$,

(*ii*) If (2) holds then $B(A_F) \cap w_{\lambda p}^\alpha(F) \subset w_{\mu p}^\beta(F)$ for all $t \in T$.

Proof (*i*) Omitted.

(*ii*) Let $(X_k) \in B(A_F) \cap w_{\lambda p}^\alpha(F)$ and suppose that (2) holds. Since $(X_k) \in B(A_F)$ then there exists some $M > 0$ such that $d(X_k(t), X(t)) \leq M$ for all $k \in \mathbb{N}$ and $t \in T$. Now, since $\lambda_n \leq \mu_n$ and $I_n \subset J_n$ for all $n \in \mathbb{N}_{n_o}$, we may write

$$\begin{aligned}
\frac{1}{\mu_n^\beta} \sum_{k \in J_n, t \in T} (d(X_k(t), X(t)))^p &= \frac{1}{\mu_n^\beta} \sum_{k \in J_n - I_n, t \in T} (d(X_k(t), X(t)))^p \\
&\quad + \frac{1}{\mu_n^\beta} \sum_{k \in I_n, t \in T} (d(X_k(t), X(t)))^p \\
&\leq \left(\frac{\mu_n - \lambda_n}{\mu_n^\beta}\right) M^p + \frac{1}{\mu_n^\beta} \sum_{k \in I_n, t \in T} (d(X_k(t), X(t)))^p \\
&\leq \left(\frac{\mu_n - \lambda_n^\beta}{\lambda_n^\beta}\right) M^p + \frac{1}{\mu_n^\beta} \sum_{k \in I_n, t \in T} (d(X_k(t), X(t)))^p \\
&\leq \left(\frac{\mu_n}{\lambda_n^\beta} - 1\right) M^p + \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n, t \in T} (d(X_k(t), X(t)))^p
\end{aligned}$$

for every $n \in \mathbb{N}_{n_o}$. Therefore $B(A_F) \cap w_{\lambda p}^\alpha(F) \subset w_{\mu p}^\beta(F)$.

Corollary 3.11 Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}_{n_o}$ and (X_k) be a sequence of fuzzy mappings. If (1) holds then,

i) $w^{\alpha}_{\mu p}(F) \subset w^{\alpha}_{\lambda p}(F)$ for each $\alpha \in (0, 1]$ and for all $t \in T$,

ii) $w_{\mu p}(F) \subset w^{\alpha}_{\lambda p}(F)$ for each $\alpha \in (0, 1]$ and for all $t \in T$,

iii) $w_{\mu p}(F) \subset w_{\lambda p}(F)$ for all $t \in T$.

Corollary 3.12 Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}_{n_o}$ and (X_k) be a sequence of fuzzy mappings. If (2) holds then,

i) $B(A_F) \cap w^{\alpha}_{\lambda p}(F) \subset w^{\alpha}_{\mu p}(F)$ for each $\alpha \in (0, 1]$ and for all $t \in T$,

ii) $B(A_F) \cap w^{\alpha}_{\lambda p}(F) \subset w_{\mu p}(F)$ for each $\alpha \in (0, 1]$ and for all $t \in T$,

iii) $B(A_F) \cap w_{\lambda p}(F) \subset w_{\mu p}(F)$ for all $t \in T$.

Theorem 3.13 Let α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$, $0 < p < \infty$ and $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}_{n_o}$ and (X_k) be a sequence of fuzzy mappings. Then

(*i*) Let (1) holds, if a sequence of fuzzy mappings is strongly $w^{\beta}_{\mu p}(F)-$summable to X then it is $S^{\alpha}_{\lambda}(F)-$statistically convergent to X,

(*ii*) Let (2) holds and (X_k) be a bounded sequence of fuzzy mappings, then if a sequence of fuzzy mappings is $S^{\alpha}_{\lambda}(F)-$statistically convergent to X then it is strongly $w^{\beta}_{\mu p}(F)-$summable to X.

Proof. (*i*) For any sequence of fuzzy mappings (X_k) and $\varepsilon > 0$, we have

$$\begin{aligned}
\sum_{k\in J_n, t\in T} (d(X_k(t), X(t)))^p &\geq \sum_{\substack{k\in I_n \\ d(X_k(t),X(t))\geq\varepsilon, t\in T}} (d(X_k(t), X(t)))^p \\
&\quad + \sum_{\substack{k\in I_n \\ d(X_k(t),X(t))<\varepsilon, t\in T}} (d(X_k(t), X(t)))^p \\
&\geq \sum_{\substack{k\in I_n \\ d(X_k(t),X(t))\geq\varepsilon, t\in T}} (d(X_k(t), X(t)))^p \\
&\geq |\{k \in I_n : d(X_k(t), X(t)) \geq \varepsilon\}| \, \varepsilon^p
\end{aligned}$$

and so that

$$\begin{aligned}
\frac{1}{\mu_n^{\beta}} \sum_{k\in J_n, t\in T} (d(X_k(t), X(t)))^p &\geq \frac{1}{\mu_n^{\beta}} |\{k \in I_n : d(X_k(t), X(t)) \geq \varepsilon\}| \, \varepsilon^p \\
&\geq \frac{\lambda_n^{\alpha}}{\mu_n^{\beta}} \frac{1}{\lambda_n^{\alpha}} |\{k \in I_n : d(X_k(t), X(t)) \geq \varepsilon\}| \, \varepsilon^p .
\end{aligned}$$

Since (1) holds it follows that if (X_k) is strongly $w^{\beta}_{\mu p}(F)-$summable to X then it is $S^{\alpha}_{\lambda}(X)-$statistically convergent to X.

(*ii*) Suppose that $S^{\alpha}_{\lambda} - \lim X_k(t) = X(t)$ on T and $(X_k) \in B(A_F)$. Then there exists some $M > 0$ such that

$d(X_k(t), X(t)) \leq M$ for all k, then for every $\varepsilon > 0$ we may write

$$\frac{1}{\mu_n^{\beta}} \sum_{k \in J_n, t \in T} (d(X_k(t), X(t)))^p = \frac{1}{\mu_n^{\beta}} \sum_{k \in J_n - I_n, t \in T} (d(X_k(t), X(t)))^p + \frac{1}{\mu_n^{\beta}} \sum_{k \in I_n,} (d(X_k(t), X(t)))^p$$

$$\leq \left(\frac{\mu_n - \lambda_n}{\mu_n^{\beta}}\right) M^p + \frac{1}{\mu_n^{\beta}} \sum_{k \in I_n, t \in T} (d(X_k(t), X(t)))^p$$

$$\leq \left(\frac{\mu_n - \lambda_n^{\beta}}{\lambda_n^{\beta}}\right) M^p + \frac{1}{\mu_n^{\beta}} \sum_{k \in I_n, t \in T} (d(X_k(t), X(t)))^p$$

$$= \left(\frac{\mu_n}{\lambda_n^{\beta}} - 1\right) M^p + \frac{1}{\mu_n^{\beta}} \sum_{\substack{k \in I_n, t \in T \\ |d(X_k(t), X(t))| \geq \varepsilon}} (d(X_k(t), X(t)))^p$$

$$+ \frac{1}{\mu_n^{\beta}} \sum_{\substack{k \in I_n, t \in T \\ d(X_k(t), X(t)) < \varepsilon}} (d(X_k(t), X(t)))^p$$

$$\leq \left(\frac{\mu_n}{\lambda_n^{\beta}} - 1\right) M^p + \frac{M^p}{\lambda_n^{\alpha}} |\{k \in I_n : d(X_k(t), X(t)) \geq \varepsilon\}| + \varepsilon$$

for all $n \in \mathbb{N}_{n_o}$. Using (2) we obtain that $w_{\mu p}^{\beta}(F) - \lim X_k(t) = X(t)$ whenever $S_{\lambda}^{\alpha}(F) - \lim X_k(t) = X(t)$.

Corollary 3.14 Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}_{n_o}$. If (1) holds then,

i) If a sequence of fuzzy mappings is strongly $w_{\mu p}^{\alpha}(F)$ −summable to X then it is $S_{\lambda}^{\alpha}(F)$ −statistically convergent to X for each $\alpha \in (0, 1]$ and for all $t \in T$,

ii) If a sequence of fuzzy mappings is strongly $w_{\mu p}(F)$ −summable to X then it is $S_{\lambda}^{\alpha}(F)$ −statistically convergent to X for each $\alpha \in (0, 1]$ and for all $t \in T$,

iii) If a sequence of fuzzy mappings is strongly $w_{\mu p}(F)$ −summable to X then it is $S_{\lambda}(F)$ −statistically convergent to X for all $t \in T$.

Corollary 3.15 Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}_{n_o}$. If (2) holds then,

i) If a bounded sequence of fuzzy mappings is $S_{\lambda}^{\alpha}(F)$ −statistically convergent to X then it is strongly $w_{\mu p}^{\alpha}(F)$ −summable to X for each $\alpha \in (0, 1]$ and for all $t \in T$,

ii) If a bounded sequence of fuzzy mappings is $S_{\lambda}^{\alpha}(F)$ −statistically convergent to X then it is strongly $w_{\mu p}(F)$ −summable to X for each $\alpha \in (0, 1]$ and for all $t \in T$,

iii) If a bounded sequence of fuzzy mappings is $S_{\lambda}(F)$ −statistically convergent to X then it is strongly $w_{\mu p}(F)$ −summable to X for all $t \in T$,

Theorem 3.16 (*i*) The spaces $S_{\lambda 0}^{\alpha}(F)$ and $w_{\lambda p_0}^{\alpha}(F)$ are solid and monotone,

(*ii*) The spaces $S_{\lambda}^{\alpha}(F)$ and $w_{\lambda p}^{\alpha}(F)$ are neither solid nor monotone.

Proof. (*i*) We shall prove only for $S_{\lambda 0}^{\alpha}(F)$ and the other can be treated similarly. Let (X_k) be a sequence of fuzzy mappings in $S_{\lambda 0}^{\alpha}(F)$. Let $d(Y_k(t), \bar{0}) \leq d(X_k(t), \bar{0})$ for all $k \in \mathbb{N}$. The proof follows from the following inclusion:

$$\{k \in \mathbb{N} : d(X_k(t), \bar{0}) \geq \varepsilon\} \supseteq \{k \in \mathbb{N} : d(Y_k(t), \bar{0}) \geq \varepsilon\}.$$

The rest of the proof follows from the Remark 1.

(ii)The proof follows from the following examples.

Example 1 Let $\alpha = 1$, $\lambda_n = n$ for all $n \in \mathbb{N}$ and consider the sequences (X_k) and (Y_k) defined as follows:

$$X_k(t) = \begin{cases} 1, & \text{for } t = (1,1,1,...,1) \\ 0, & \text{otherwise.} \end{cases}$$

$$Y_k(t) = \begin{cases} 1, & \text{for all } k \text{ odd and } t = (-1,-1,-1,...,-1) \\ 0, & \text{for all } k \text{ odd and } t \neq (-1,-1,-1,...,-1) \\ 1, & \text{for all } k \text{ even and } t = (1,1,1,...,1) \\ 1, & \text{for all } k \text{ even and } t \neq (1,1,1,...,1) \end{cases}$$

Then (X_k) belongs to both $S_\lambda^\alpha(F)$ and $w_{\lambda p_0}^\alpha(F)$ and but (Y_k) does not belong, hence the spaces are not solid.

Example 2 Let $\alpha = 1$, $\lambda_n = n$ for all $n \in \mathbb{N}$ and consider the sequence (X_k) defined by

$$X_k = \bar{1}, \text{for all } k \in \mathbb{N}.$$

Consider its J^{th} step space Z_J defined as $(Y_k) \in Z_J \Rightarrow Y_k = X_k$ for all $k = 2i+1, i \in \mathbb{N}$ and $Y_k = \bar{0}$, otherwise. Then $(X_k) \in S_\lambda^\alpha(F)$, but (Y_k) does not belong to $S_\lambda^\alpha(F)$. Hence the space $S_\lambda^\alpha(F)$ is not monotone.

Theorem 3.17 The spaces $S_{\lambda 0}^\alpha(F)$, $w_{\lambda p_0}^\alpha(F)$, $S_\lambda^\alpha(F)$ and $w_{\lambda p}^\alpha(F)$ are not symmetric.

Proof . The proof follows from the following example.

Example 3 Let $\alpha = 1$, $\lambda_n = n$ for all $n \in \mathbb{N}$ consider the sequence (X_k) defined by

$$X_k(t) = \begin{cases} 1, & k = i^3, i \in \mathbb{N} \text{ and } t = (1,1,1,...,1) \\ 0, & k = i^3, i \in \mathbb{N} \text{ and } t \neq (1,1,1,...,1) \\ 1, & k \neq i^3, \text{ for any and } t = (0,0,0,...,0) \\ 0, & k \neq i^3, \text{ for any and } t \neq (0,0,0,...,0) \end{cases}$$

Consider the rearranged sequence (Y_k) of (X_k), defined as follows :

$$(Y_k) = (X_1, X_2, X_8, X_3, X_{27}, X_4, X_{64}, X_5, X_{125}, X_6, X_{216}, X_7, X_{343}, X_9, ...)$$

Then (X_k) belongs to all the spaces, but (Y_k) does not belong to, hence the spaces are not symmetric.

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