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New Error Bounds for Gauss-Legendre Quadrature Rules

Mohammad Masjed-Jamei[a]

[a] *Department of Mathematics, K.N.Toosi University of Technology, P.O.Box 16315-1618, Tehran, Iran,*

Abstract. It is well-known that the remaining term of any n-point interpolatory quadrature rule such as Gauss-Legendre quadrature formula depends on at least an n-order derivative of the integrand function, which is of no use if the integrand is not smooth enough and requires a lot of differentiation for large n. In this paper, by defining a specific linear kernel, we resolve this problem and obtain new bounds for the error of Gauss-Legendre quadrature rules. The advantage of the obtained bounds is that they do not depend on the norms of the integrand function. Some illustrative examples are given in this direction.

1. Introduction

A general n-point quadrature formula is denoted by

$$\int_a^b f(x)dx = \sum_{k=1}^{n} w_{k,n} f(x_{k,n}) + R_n[f], \tag{1}$$

where $\{x_{k,n}\}_{k=1}^{n}$ and $\{w_{k,n}\}_{k=1}^{n}$ are respectively nodes and weight coefficients and $R_n[f]$ is the corresponding error [2, 9].

Let $\mathbf{\Pi}_d$ be the set of algebraic polynomials of degree at most d. The quadrature formula (1) has degree of exactness d if for every $p \in \mathbf{\Pi}_d$ we have $R_n[p] = 0$. In addition, if $R_n[p] \neq 0$ for some $\mathbf{\Pi}_{d+1}$, formula (1) has precise degree of exactness d.

The convergence order of quadrature rule (1) depends on the smoothness of the function f as well as on its degree of exactness. It is well known that for given mutually different nodes $\{x_{k,n}\}_{k=1}^{n}$ we can always achieve a degree of exactness $d = n-1$ by interpolating at these nodes and integrating the interpolated polynomial instead of f. Namely, taking the node polynomial

$$\Psi_n(x) = \prod_{k=1}^{n} (x - x_{k,n}),$$

by integrating the Lagrange interpolation formula

$$f(x) = \sum_{k=1}^{n} f(x_{k,n}) L(x; x_{k,n}) + r_n(f; x),$$

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Email address: mmjamei@kntu.ac.ir , mmjamei@yahoo.com (Mohammad Masjed-Jamei)

where

$$L(x;x_{k,n}) = \frac{\Psi_n(x)}{\Psi'_n(x_{k,n})(x-x_{k,n})} \quad (k=1,2,...,n),$$

we obtain (1), with

$$w_{k,n} = \frac{1}{\Psi'_n(x_{k,n})}\int_a^b \frac{\Psi_n(x)}{x-x_{k,n}}dx \quad (k=1,2,...,n),$$

and

$$R_n[f] = \int_a^b r_n(f;x)\,dx.$$

It is clear that if $f \in \mathbf{\Pi}_{n-1}$ then $r_n(f;x) = 0$ and therefore $R_n[f] = 0$.
Quadrature formulae obtained in this way are known as interpolatory [9, 16]. A main problem in an n-point interpolatory quadrature is that the remaining term depends on at least an n-order derivative of the integrand function which is of no use if the integrand is not smooth enough and requires a lot of differentiation and calculation for large n. Hence, many authors prefer to use other techniques including lower order derivatives [1, 3, 4, 6, 17, 23]. For example, if only $f \in C^1[a,b]$, then the error term would strongly tend to zero for a large class of quadrature rules [5, 7].
One of the important cases of interpolatory quadratures is the Gauss-Legendre formula which can be constructed via Hermite interpolation [16]. As we pointed out, since the remaining term of this formula depends on a high $2n$-order derivative of the integrand function [22], many people have used other techniques e.g. [8, 10, 19–21]. In[11] , Gautschi and Varga used the contour integral representation of the remainder term for analytic integrands. Hunter in [12] improved the results of [11] giving a sharper inequality. However, a main concern in their technique is that, in most cases, it is hard to obtain the explicit forms of the bounds due to much complexity [18, p. 803].
In this paper, we consider a new approach to estimate the error term of any arbitrary interpolatory quadrature, including Gauss-Legendre rule, which is based on defining a linear generating kernel and the known values of the nodes and weight coefficients when the first derivative of the integrand function is bounded between two real functions. In other words, by defining a specific linear kernel we obtain a unified error bound for any interpolatory quadrature rule and consider the Gauss-Legendre quadrature as a special case in this direction. The advantage of linear kernels is that one can easily and explicitly compute the values L^1-norm, L^∞-norm and also their maximum and minimum. In the next section, we describe some notations and integral inequalities, which are related to results of this paper. We also study the general properties of a linear kernel in a parametric form. In section 2, we prove three main theorems by which one can estimate the residue of an interpolatory quadrature rule, independent of the integrand function and its derivatives. Finally in section 3, the results of the presented theorems are employed on two particular cases of Gauss-Legendre rules for $n = 2, 3$.

1.1. Preliminaries and Notations

Let $L^p[a,b]$ $(1 \le p \le \infty)$ denote the space of p-power integrable functions on the interval $[a,b]$ with the standard norm

$$\left\|f\right\|_p = \left(\int_a^b \left|f(t)\right|^p dt\right)^{1/p},$$

and $L^\infty[a,b]$ the space of all essentially bounded functions on $[a,b]$ with the norm

$$\left\|f\right\|_\infty = \sup_{x\in[a,b]} \left|f(x)\right|.$$

If $h \in L^1[a,b]$ and $g \in L^\infty[a,b]$, then

$$\left| \int_a^b h(x)\, g(x)\, dx \right| \leq \| h \|_1 \, \| g \|_\infty .$$

Let $A = (a_1, a_2, \ldots, a_m)$ and $H = (h_1, h_2, \ldots, h_{m-1})$, where $\alpha < h_1 < h_2 < \ldots < h_{m-1} < \beta$, be two real sequences. The following piecewise continuous function is known as a linear generating kernel on the interval $[\alpha, \beta]$

$$\mathbf{K}_m\,(t; A, H) = \begin{cases} t - a_1 & t \in [\alpha, h_1]\,, \\ t - a_2 & t \in (h_1, h_2]\,, \\ \vdots & \vdots \\ t - a_m & t \in (h_{m-1}, \beta]\,. \end{cases} \tag{2}$$

The main advantage of the kernel (2) is that its L^1 -norm, L^∞-norm and its maximum and minimum are explicitly computable on $[\alpha, \beta]$ as follows

$$\begin{aligned} \| . \|_1 = \int_\alpha^\beta |\,\mathbf{K}_m\,(t; A, H)|\, dt &= \int_\alpha^{h_1} |\, t - a_1 |\, dt + \int_{h_1}^{h_2} |\, t - a_2 |\, dt + \ldots + \int_{h_{m-1}}^\beta |\, t - a_m |\, dt \\ &= \int_{\alpha - a_1}^{h_1 - a_1} |\, z |\, dz + \int_{h_1 - a_2}^{h_2 - a_2} |\, z |\, dz + \ldots + \int_{h_{m-1} - a_m}^{\beta - a_m} |\, z |\, dz\,. \end{aligned} \tag{3}$$

And by noting this general fact that

$$\int_p^q |\, z |\, dz = \begin{cases} (q^2 - p^2)/2 & \text{if} \ \ p, q > 0\,, \\ -(q^2 - p^2)/2 & \text{if} \ \ p, q < 0\,, \\ (q^2 + p^2)/2 & \text{if} \ \ p < 0 \ \& \ q > 0\,, \end{cases}$$

relation (3) can be computed easily. For L^∞-norm we have

$$\begin{aligned} \| . \|_\infty = \max_{t \in [\alpha, \beta]} |\,\mathbf{K}_m\,(t; A, H)| &= \max \left\{ \max_{t \in [\alpha, h_1]} |\, t - a_1 |\,,\ \max_{t \in (h_1, h_2]} |\, t - a_2 |\,,\ \ldots,\ \max_{t \in (h_{m-1}, \beta]} |\, t - a_m | \right\} \\ &= \max \left\{ |\,\alpha - a_1 |\,,\, |\, h_1 - a_1 |\,,\, |\, h_1 - a_2 |\,,\, |\, h_2 - a_2 |\,,\, \ldots,\, |\, h_{m-1} - a_m |\,,\, \left| \beta - a_m \right| \right\}. \end{aligned}$$

Finally the maximum and minimum of the kernel (2) can be computed in the forms

$$\begin{aligned} \max_{t \in [\alpha, \beta]} \mathbf{K}_m\,(t; A, H) &= \max \left\{ \max_{t \in [\alpha, h_1]} (t - a_1)\,,\ \max_{t \in (h_1, h_2]} (t - a_2)\,,\ \ldots,\ \max_{t \in (h_{m-1}, \beta]} (t - a_m) \right\} \\ &= \max \{ h_1 - a_1\,,\, h_2 - a_2\,,\, \ldots,\, \beta - a_m \}\,, \end{aligned}$$

and

$$\begin{aligned} \min_{t \in [\alpha, \beta]} \mathbf{K}_m\,(t; A, H) &= \min \left\{ \min_{t \in [\alpha, h_1]} (t - a_1)\,,\ \min_{t \in (h_1, h_2]} (t - a_2)\,,\ \ldots,\ \min_{t \in (h_{m-1}, \beta]} (t - a_m) \right\} \\ &= \min \{ \alpha - a_1\,,\, h_1 - a_2\,,\, \ldots,\, h_{m-1} - a_m \}\,. \end{aligned}$$

2. Error bounds for interpolatory quadrature rules

Let us consider the error functional of an interpolatory quadrature as

$$\left|R_n^*[f]\right| = \left|\int_a^b f(x)dx - \sum_{k=1}^{n} w_{k,n} f(x_{k,n})\right|, \tag{4}$$

where $a < x_{1,n} < x_{2,n} < \ldots < x_{n,n} < b$ and $\{w_{n,k}\}_{k=0}^{n}$ are solutions of the linear system

$$\sum_{k=1}^{n} x_{k,n}^{j} w_{k,n} = \frac{b^{j+1} - a^{j+1}}{j+1}, \quad (j = 0, 1, \ldots, n-1), \tag{5}$$

and then define a particular case of the linear kernel (2) on $[a, b]$ as

$$\mathbf{K}_{n+1}\left(t; \{x_{k,n}\}_{k=1}^{n}, \{w_{k,n}\}_{k=1}^{n}\right) = \begin{cases} u_0(t) = t - a & t \in [a, x_{1,n}]\,, \\ u_1(t) = t - (a + w_{1,n}) & t \in (x_{1,n}, x_{2,n}]\,, \\ \quad\vdots & \quad\vdots \\ u_{n-1}(t) = t - (a + \sum\limits_{k=1}^{n-1} w_{k,n}) & t \in (x_{n-1,n}, x_{n,n}]\,, \\ u_n(t) = t - b & t \in (x_{n,n}, b]\,. \end{cases} \tag{6}$$

From (6), the following identity can be directly verified

$$\left|\int_a^b \mathbf{K}_{n+1}\left(t; \{x_{k,n}\}_{k=1}^{n}, \{w_{k,n}\}_{k=1}^{n}\right) f'(t)\,dt\right| = \left|\int_a^b f(x)dx - \sum_{k=1}^{n} w_{k,n} f(x_{k,n})\right|. \tag{7}$$

Theorem 2.1. *Let $f \in C^1[a, b]$. If $\alpha(x) \le f'(x) \le \beta(x)$ for any $\alpha, \beta \in C[a, b]$, then by noting the elements of the kernel (6), the error functional (4) can be bounded as*

$$m_n^{(1)} \le \sum_{k=1}^{n} w_{k,n} f(x_{k,n}) - \int_a^b f(x)\,dx \ \le M_n^{(1)},$$

where

$$m_n^{(1)} = \sum_{k=0}^{n} \int_{x_{k,n}}^{x_{k+1,n}} \left(\frac{u_k(t) + |u_k(t)|}{2}\,\alpha(t) + \frac{u_k(t) - |u_k(t)|}{2}\,\beta(t)\right) dt$$

with $x_{0,n} = a, x_{n+1,n} = b$ and

$$M_n^{(1)} = \sum_{k=0}^{n} \int_{x_{k,n}}^{x_{k+1,n}} \left(\frac{u_k(t) - |u_k(t)|}{2}\,\alpha(t) + \frac{u_k(t) + |u_k(t)|}{2}\,\beta(t)\right) dt.$$

Proof. By referring to relation (4) and the identity (7) first we have

$$\int_a^b \mathbf{K}_{n+1}\left(t; \{x_{k,n}\}_{k=1}^n, \{w_{k,n}\}_{k=1}^n\right)\left(f'(t) - \tfrac{\alpha(t)+\beta(t)}{2}\right) dt$$
$$= R_n^*[f] - \tfrac{1}{2}\left(\int_a^b \mathbf{K}_{n+1}\left(t; \{x_{k,n}\}_{k=1}^n, \{w_{k,n}\}_{k=1}^n\right)(\alpha(t)+\beta(t))\,dt\right) = R_n^*[f] \tag{8}$$
$$-\tfrac{1}{2}\left(\int_a^{x_{1,n}} u_0(t)\,(\alpha(t)+\beta(t))\,dt + \int_{x_{1,n}}^{x_{2,n}} u_1(t)\,(\alpha(t)+\beta(t))\,dt + \dots + \int_{x_{n,n}}^b u_n(t)\,(\alpha(t)+\beta(t))\,dt\right).$$

Also, the assumption $\alpha(t) \le f'(t) \le \beta(t)$ gives

$$\left| f'(t) - \frac{\alpha(t)+\beta(t)}{2}\right| \le \frac{\beta(t)-\alpha(t)}{2}. \tag{9}$$

Hence, from (8) and (9) one can conclude that

$$\left| R_n^*[f] - \tfrac{1}{2}\left(\int_a^{x_{1,n}} u_0(t)\,(\alpha(t)+\beta(t))\,dt + \int_{x_{1,n}}^{x_{2,n}} u_1(t)\,(\alpha(t)+\beta(t))\,dt + \dots + \int_{x_{n,n}}^b u_n(t)\,(\alpha(t)+\beta(t))\,dt\right)\right|$$
$$= \left|\int_a^b \mathbf{K}_{n+1}\left(t; \{x_{k,n}\}_{k=1}^n, \{w_{k,n}\}_{k=1}^n\right)\left(f'(t) - \tfrac{\alpha(t)+\beta(t)}{2}\right) dt\right|$$
$$\le \int_a^b \left|\mathbf{K}_{n+1}\left(t; \{x_{k,n}\}_{k=1}^n, \{w_{k,n}\}_{k=1}^n\right)\right| \tfrac{\beta(t)-\alpha(t)}{2}\, dt \tag{10}$$
$$= \tfrac{1}{2}\left(\int_a^{x_{1,n}} |u_0(t)|\,(\alpha(t)+\beta(t))\,dt + \int_{x_{1,n}}^{x_{2,n}} |u_1(t)|\,(\alpha(t)+\beta(t))\,dt + \dots + \int_{x_{n,n}}^b |u_n(t)|\,(\alpha(t)+\beta(t))\,dt\right).$$

After re-arranging (10) we eventually obtain

$$m_n^{(1)} = \sum_{k=0}^{n} \int_{x_{k,n}}^{x_{k+1,n}} \left(\frac{u_k(t)+|u_k(t)|}{2}\,\alpha(t) + \frac{u_k(t)-|u_k(t)|}{2}\,\beta(t)\right) dt,$$

and

$$M_n^{(1)} = \sum_{k=0}^{n} \int_{x_{k,n}}^{x_{k+1,n}} \left(\frac{u_k(t)-|u_k(t)|}{2}\,\alpha(t) + \frac{u_k(t)+|u_k(t)|}{2}\,\beta(t)\right) dt.$$

Remark 2.2. *Although $\alpha(x) \le f'(x) \le \beta(x)$ is a straightforward condition in theorem 2.1, sometimes one might not be able to easily obtain both bounds of $\alpha(x)$ and $\beta(x)$ for f'. In this case, we can make use of two analogue theorems. The first one would be helpful when f' is unbounded from above and the second one would be helpful when f' is unbounded from below.*

Theorem 2.3. *Let $f \in C^1[a,b]$. If $\alpha(x) \le f'(x)$ for any $\alpha \in [a,b]$, then by noting the elements of the kernel (6), the error functional (4) can be bounded as*

$$m_n^{(2)} \le \sum_{k=1}^{n} w_{k,n} f(x_{k,n}) - \int_a^b f(x)\,dx \ \le M_n^{(2)}, \tag{11}$$

where

$$m_n^{(2)} = \sum_{k=0}^{n} \int_{x_{k,n}}^{x_{k+1,n}} u_k(t)\,\alpha(t)\,dt - \left(f(b) - f(a) - \int_a^b \alpha(t)\,dt\right)$$

$$\times \max \left\{ \begin{array}{l} \left|a - x_{1,n}\right|, \left|w_{1,n} + a - x_{1,n}\right|, \left|w_{1,n} + a - x_{2,n}\right| \ldots, \\ \left|\sum_{k=1}^{n-1} w_{k,n} + a - x_{n-1,n}\right|, \left|\sum_{k=1}^{n-1} w_{k,n} + a - x_{n,n}\right|, \left|b - x_{n,n}\right| \end{array} \right\},$$

with $x_{0,n} = a,\ x_{n+1,n} = b$ *and*

$$M_n^{(2)} = \sum_{k=0}^{n} \int_{x_{k,n}}^{x_{k+1,n}} u_k(t)\,\alpha(t)\,dt + \left(f(b) - f(a) - \int_a^b \alpha(t)\,dt\right)$$

$$\times \max \left\{ \begin{array}{l} \left|a - x_{1,n}\right|, \left|w_{1,n} + a - x_{1,n}\right|, \left|w_{1,n} + a - x_{2,n}\right| \ldots, \\ \left|\sum_{k=1}^{n-1} w_{k,n} + a - x_{n-1,n}\right|, \left|\sum_{k=1}^{n-1} w_{k,n} + a - x_{n,n}\right|, \left|b - x_{n,n}\right| \end{array} \right\}.$$

Proof. By referring to relation (4) and the identity (7) first we have

$$\int_a^b \mathbf{K}_{n+1}\left(t; \{x_{k,n}\}_{k=1}^n, \{w_{k,n}\}_{k=1}^n\right) \left(f'(t) - \alpha(t)\right)\,dt = R_n^*[f] - \int_a^b \mathbf{K}_{n+1}\left(t; \{x_{k,n}\}_{k=1}^n, \{w_{k,n}\}_{k=1}^n\right) \alpha(t)\,dt$$

$$= R_n^*[f] - \left(\int_a^{x_{1,n}} u_0(t)\,\alpha(t)\,dt + \int_{x_{1,n}}^{x_{2,n}} u_1(t)\,\alpha(t)\,dt + \ldots + \int_{x_{n,n}}^{b} u_n(t)\,\alpha(t)\,dt\right).$$

Therefore

$$\begin{aligned} &\left|R_n^*[f] - \left(\int_a^{x_{1,n}} u_0(t)\,\alpha(t)\,dt + \int_{x_{1,n}}^{x_{2,n}} u_1(t)\,\alpha(t)\,dt + \ldots + \int_{x_{n,n}}^{b} u_n(t)\,\alpha(t)\,dt\right)\right| \\ &= \left|\int_a^b \mathbf{K}_{n+1}\left(t; \{x_{k,n}\}_{k=1}^n, \{w_{k,n}\}_{k=1}^n\right) \left(f'(t) - \alpha(t)\right)\,dt\right| \\ &\le \int_a^b \left|\mathbf{K}_{n+1}\left(t; \{x_{k,n}\}_{k=1}^n, \{w_{k,n}\}_{k=1}^n\right)\right| \left(f'(t) - \alpha(t)\right)\,dt \\ &\le \max_{t\in[a,b]} \left|\mathbf{K}_{n+1}\left(t; \{x_{k,n}\}_{k=1}^n, \{w_{k,n}\}_{k=1}^n\right)\right| \int_a^b \left(f'(t) - \alpha(t)\right)\,dt \\ &= \max_{t\in[a,b]} \left|\mathbf{K}_{n+1}\left(t; \{x_{k,n}\}_{k=1}^n, \{w_{k,n}\}_{k=1}^n\right)\right| \left(f(b) - f(a) - \int_a^b \alpha(t)\,dt\right). \end{aligned} \tag{12}$$

After re-arranging (12), the main inequality (11) will be derived. ■

Theorem 2.4. *Let* $f \in C^1[a,b]$. *If* $f'(x) \le \beta(x)$ *for any* $\beta \in C[a,b]$, *then by noting the elements of the kernel (6), the error functional (4) can be bounded as*

$$m_n^{(3)} \le \sum_{k=1}^{n} w_{k,n} f(x_{k,n}) - \int_a^b f(x)\,dx \ \le M_n^{(3)}, \tag{13}$$

where

$$m_n^{(3)} = \sum_{k=0}^{n} \int_{x_{k,n}}^{x_{k+1,n}} u_k(t)\,\beta(t)\,dt - \left(\int_a^b \beta(t)\,dt - f(b) + f(a)\right)$$

$$\times \max \left\{ \begin{array}{l} \left|a - x_{1,n}\right|,\ \left|w_{1,n} + a - x_{1,n}\right|,\ \left|w_{1,n} + a - x_{2,n}\right| \ldots, \\ \left|\sum_{k=1}^{n-1} w_{k,n} + a - x_{n-1,n}\right|,\ \left|\sum_{k=1}^{n-1} w_{k,n} + a - x_{n,n}\right|,\ \left|b - x_{n,n}\right| \end{array} \right\},$$

with $x_{0,n} = a,\ x_{n+1,n} = b$ *and*

$$m_n^{(3)} = \sum_{k=0}^{n} \int_{x_{k,n}}^{x_{k+1,n}} u_k(t)\,\beta(t)\,dt - \left(\int_a^b \beta(t)\,dt - f(b) + f(a)\right)$$

$$\times \max \left\{ \begin{array}{l} \left|a - x_{1,n}\right|,\ \left|w_{1,n} + a - x_{1,n}\right|,\ \left|w_{1,n} + a - x_{2,n}\right| \ldots, \\ \left|\sum_{k=1}^{n-1} w_{k,n} + a - x_{n-1,n}\right|,\ \left|\sum_{k=1}^{n-1} w_{k,n} + a - x_{n,n}\right|,\ \left|b - x_{n,n}\right| \end{array} \right\},$$

Proof. Since

$$\int_a^b \mathbf{K}_{n+1}\left(t; \{x_{k,n}\}_{k=1}^n, \{w_{k,n}\}_{k=1}^n\right)(f'(t) - \beta(t))\,dt = R_n^*[f] - \int_a^b \mathbf{K}_{n+1}\left(t; \{x_{k,n}\}_{k=1}^n, \{w_{k,n}\}_{k=1}^n\right)\beta(t)\,dt$$

$$= R_n^*[f] - \left(\int_a^{x_{1,n}} u_0(t)\,\beta(t)\,dt + \int_{x_{1,n}}^{x_{2,n}} u_1(t)\,\beta(t)\,dt + \ldots + \int_{x_{n,n}}^{b} u_n(t)\,\beta(t)\,dt\right),$$

Therefore

$$\begin{aligned} &\left|R_n^*[f] - \left(\int_a^{x_{1,n}} u_0(t)\,\beta(t)\,dt + \int_{x_{1,n}}^{x_{2,n}} u_1(t)\,\beta(t)\,dt + \ldots + \int_{x_{n,n}}^{b} u_n(t)\,\beta(t)\,dt\right)\right| \\ &= \left|\int_a^b \mathbf{K}_{n+1}\left(t; \{x_{k,n}\}_{k=1}^n, \{w_{k,n}\}_{k=1}^n\right)(f'(t) - \beta(t))\,dt\right| \\ &\le \int_a^b \left|\mathbf{K}_{n+1}\left(t; \{x_{k,n}\}_{k=1}^n, \{w_{k,n}\}_{k=1}^n\right)\right|(\beta(t) - f'(t))\,dt \qquad (14) \\ &\le \max_{t\in[a,b]} \left|\mathbf{K}_{n+1}\left(t; \{x_{k,n}\}_{k=1}^n, \{w_{k,n}\}_{k=1}^n\right)\right| \int_a^b (\beta(t) - f'(t))\,dt \\ &= \max_{t\in[a,b]} \left|\mathbf{K}_{n+1}\left(t; \{x_{k,n}\}_{k=1}^n, \{w_{k,n}\}_{k=1}^n\right)\right| \left(\int_a^b \beta(t)\,dt - f(b) + f(a)\right). \end{aligned}$$

After re-arranging (14), the main inequality (13) will be derived. ∎

Remark 2.5. *Recently in [15] we have applied the results of theorems 2.1, 2.3 and 2.4 for all closed and open Newton-Cotes quadrature rules, see also [13, 14]. For the closed type of Newton-Cotes formulae residue*

$$\left|\int_a^b f(x)\,dx - \sum_{i=0}^{n} w_{i,n}\,f(a + ih)\right|,$$

where $h = \frac{b-a}{n}$ *and* $\{w_{i,n}\}_{i=0}^n$ *are solutions of the linear system*

$$\sum_{k=0}^{n} \left(a + k\frac{b-a}{n}\right)^j w_{k,n} = \frac{b^{j+1} - a^{j+1}}{j+1}, \qquad (j = 0, 1, \ldots, n),$$

we defined a particular case of the linear kernel (2) on $[a, b]$ *as*

$$\mathbf{K}_n\left(t; \{w_{i,n}\}_{i=0}^{n}\right) = \begin{cases} t - (a + w_{0,n}) & t \in [a, a+h]\,, \\ t - (a + w_{0,n} + w_{1,n}) & t \in (a+h, a+2h]\,, \\ \vdots & \vdots \\ t - (a + \sum\limits_{i=0}^{n-2} w_{i,n}) & t \in (a+(n-2)h, a+(n-1)h]\,, \\ t - (a + \sum\limits_{i=0}^{n-1} w_{i,n}) & t \in (a+(n-1)h, b]\,. \end{cases}$$

And for the residue of the open type of Newton-Cotes formulae

$$\left| \int_a^b f(x)\,dx - \sum_{i=1}^{n-1} v_{i,n}\, f(a+ih) \right|$$

where $h = \frac{(b-a)}{n}$ *and* $\{v_{i,n}\}_{i=1}^{n-1}$ *are solutions of the linear system*

$$\sum_{i=1}^{n-1} \left(a + i\frac{b-a}{n}\right)^j v_{i,n} = \frac{b^{j+1} - a^{j+1}}{j+1}\,, \qquad (j = 0, 1, ..., n-2)\,,$$

we defined a particular case of the linear kernel (2) on $[a, b]$ *as*

$$\mathbf{K}_n\left(t; \{v_{i,n}\}_{i=1}^{n-1}\right) = \begin{cases} t - a & t \in [a, a+h]\,, \\ t - (a + v_{1,n}) & t \in (a+h, a+2h]\,, \\ t - (a + v_{1,n} + v_{2,n}) & t \in (a+2h, a+3h]\,, \\ \vdots & \vdots \\ t - (a + \sum\limits_{i=1}^{n-2} v_{i,n}) & t \in (a+(n-2)h, a+(n-1)h]\,, \\ t - b & t \in (a+(n-1)h, b]\,. \end{cases}$$

Remark 2.6. *An important advantage in all theorems 2.1, 2.3 and 2.4 is that necessary computations for obtaining the bounds* $\{m_n^{(i)}\}_{i=1}^3$ *and* $\{M_n^{(i)}\}_{i=1}^3$ *are just in terms of the two functions* $\alpha(x), \beta(x)$ *not in terms of* f *and its derivatives while other works ([1–7],[17, 23]) contain a variety of norms (like* $\|f'\|_1$ *,*$\|f'\|_2$ *and* $\|f'\|_\infty$ *), which are rather difficult to calculate.*

3. Error bounds for Gauss-Legendre quadrature rules

One of the important cases in interpolatory quadrature rules is the Gauss-Legendre quadrature formula which can be constructed via Hermit interpolation [16]. The Legendre polynomials

$$P_n(x) = \frac{1}{2^n} \sum_{k=1}^{[n/2]} (-1)^k \begin{pmatrix} n \\ k \end{pmatrix} \begin{pmatrix} 2n-2k \\ n \end{pmatrix} x^{n-2k},$$

orthogonal with respect to the constant weight function have n real and distinct zeros $\{x_{k,n}\}_{k=1}^n$ with increasing order on $[-1, 1]$. The general form of Gauss-Legendre quadrature is denoted by

$$\int_{-1}^{1} f(x)\,dx = \sum_{k=1}^{n} \frac{2}{(1-x_{k,n}^2)(P_n'(x_{k,n}))^2} f(x_{k,n}) + \frac{2^{2n+1}(n!)^4}{(2n+1)\,((2n)!)^3} f^{(2n)}(\xi) \qquad (-1 < \xi < 1)\,. \tag{15}$$

In [22] it is pointed out that the error in (15) is unsuitable in practice as the derivative of order $2n$ may be difficult to calculate and the actual error may be much less than a bound established by a high derivative of fixed order $2n$.
By noting (15), the linear kernel corresponding to Gauss-Legendre quadratures takes the form

$$\mathbf{K}_{n+1}\left(t; \{x_{k,n}\}_{k=1}^n, \{w_{k,n}\}_{k=1}^n\right) = \begin{cases} u_0(t) = t+1 & t \in [-1, x_{1,n}]\,, \\ u_1(t) = t+1-\frac{2}{(1-x_{1,n}^2)(P_n'(x_{1,n}))^2} & t \in (x_{1,n}, x_{2,n}]\,, \\ \vdots & \vdots \\ u_{n-1}(t) = t+1-\sum\limits_{k=1}^{n-1} \frac{2}{(1-x_{k,n}^2)(P_n'(x_{k,n}))^2}) & t \in (x_{n-1,n}, x_{n,n}]\,, \\ u_n(t) = t-1 & t \in (x_{n,n}, 1]\,. \end{cases} \tag{16}$$

In this section, we consider the results of theorems 2.1, 2.3 and 2.4 for two and three point Gauss-Legendre rules.

Example 3.1. *Error bounds for two point Gauss-Legendre rule.*

For $n = 2$, the residue of 2-point Gauss-Legendre formula is as

$$R_2^{GL}(f) = f(\frac{\sqrt{3}}{3}) + f(-\frac{\sqrt{3}}{3}) - \int_{-1}^{1} f(t)\,dt,$$

and replacing its values in (16) gives the corresponding linear kernel as

$$\mathbf{K}_3\left(t; \{x_{k,2}\}_{k=1}^2, \{w_{k,2}\}_{k=1}^2\right) = \begin{cases} t+1 & t \in [-1, -\sqrt{3}/3]\,, \\ t & t \in (-\sqrt{3}/3,\ \sqrt{3}/3]\,, \\ t-1 & t \in (\sqrt{3}/3, 1]\,. \end{cases} \tag{17}$$

Hence, by applying theorem 2.1 for (17) we obtain

$$m_{2,1}^{GL} \le R_2^{GL}(f) \ \le M_{2,1}^{GL},$$

where

$$m_{2,1}^{GL} = \int_{-1}^{-\frac{\sqrt{3}}{3}} (t+1)\,\alpha(t)\,dt + \int_{0}^{\frac{\sqrt{3}}{3}} t\alpha(t)\,dt + \int_{-\frac{\sqrt{3}}{3}}^{0} t\beta(t)\,dt + \int_{\frac{\sqrt{3}}{3}}^{1} (t-1)\,\beta(t)\,dt\,,$$

and

$$M_{2,1}^{GL} = \int_{-1}^{-\frac{\sqrt{3}}{3}} (t+1)\,\beta(t)\,dt + \int_{0}^{\frac{\sqrt{3}}{3}} t\,\beta(t)\,dt + \int_{-\frac{\sqrt{3}}{3}}^{0} t\,\alpha(t)\,dt + \int_{\frac{\sqrt{3}}{3}}^{1} (t-1)\,\alpha(t)\,dt\,,$$

provided that $\alpha(x) \le f'(x) \le \beta(x)$ for $x \in [-1,1]$. Similarly, by computing

$$\max_{t\in[-1,1]} \left| \mathbf{K}_3\left(t; \{x_{k,2}\}_{k=1}^2, \{w_{k,2}\}_{k=1}^2\right) \right| = \frac{\sqrt{3}}{3},$$

and applying theorem 2.3 for (17) we get

$$m_{2,2}^{GL} \le R_2^{GL}(f) \le M_{2,2}^{GL},$$

where

$$m_{2,2}^{GL} = \int_{-1}^{1} t\alpha(t)\,dt + \int_{-1}^{-\frac{\sqrt{3}}{3}} \alpha(t)\,dt - \int_{\frac{\sqrt{3}}{3}}^{1} \alpha(t)\,dt - \frac{\sqrt{3}}{3}\left(f(1) - f(-1) - \int_{-1}^{1} \alpha(t)\,dt\right),$$

and

$$M_{2,2}^{GL} = \int_{-1}^{1} t\alpha(t)\,dt + \int_{-1}^{-\frac{\sqrt{3}}{3}} \alpha(t)\,dt - \int_{\frac{\sqrt{3}}{3}}^{1} \alpha(t)\,dt + \frac{\sqrt{3}}{3}\left(f(1) - f(-1) - \int_{-1}^{1} \alpha(t)\,dt\right),$$

provided that $\alpha(x) \le f'(x)$ for $x \in [-1,1]$. Finally applying theorem 3 for (17) yields

$$m_{2,3}^{GL} \le R_2^{GL}(f) \le M_{2,3}^{GL},$$

where

$$m_{2,3}^{GL} = \int_{-1}^{1} t\beta(t)\,dt + \int_{-1}^{-\frac{\sqrt{3}}{3}} \beta(t)\,dt - \int_{\frac{\sqrt{3}}{3}}^{1} \beta(t)\,dt - \frac{\sqrt{3}}{3}\left(\int_{-1}^{1} \beta(t)\,dt - f(1) + f(-1)\right),$$

and

$$M_{2,3}^{GL} = \int_{-1}^{1} t\beta(t)\,dt + \int_{-1}^{-\frac{\sqrt{3}}{3}} \beta(t)\,dt - \int_{\frac{\sqrt{3}}{3}}^{1} \beta(t)\,dt + \frac{\sqrt{3}}{3}\left(\int_{-1}^{1} \beta(t)\,dt - f(1) + f(-1)\right),$$

provided that $f'(x) \le \beta(x)$ for $x \in [-1,1]$.

Example 3.2. *Error bounds for three point Gauss-Legendre rule.*

For $n = 3$, the residue of 3-point Gauss-Legendre formula is as

$$R_3^{GL}(f) = \frac{5}{9} f(-\frac{\sqrt{15}}{5}) + \frac{8}{9} f(0) + \frac{5}{9} f(\frac{\sqrt{15}}{5}) - \int_{-1}^{1} f(t)\,dt,$$

and the linear kernel as

$$\mathbf{K}_4\left(t; \{x_{k,3}\}_{k=1}^3, \{w_{k,3}\}_{k=1}^3\right) = \begin{cases} t+1 & t \in [-1, -\sqrt{15}/5], \\ t+4/9 & t \in (-\sqrt{15}/5, 0], \\ t-4/9 & t \in (0, \sqrt{15}/5], \\ t-1 & t \in (\sqrt{15}/5, 1]. \end{cases} \tag{18}$$

Hence, applying theorem 2.1 for (18) yields

$$m_{3,1}^{GL} \leq R_3^{GL}(f) \ \leq M_{3,1}^{GL},$$

where

$$m_{3,1}^{GL} = \int_{-1}^{-\frac{\sqrt{15}}{5}} (t+1)\,\alpha(t)\,dt + \int_{-\frac{4}{9}}^{0} (t+\tfrac{4}{9})\alpha(t)\,dt + \int_{\frac{4}{9}}^{\frac{\sqrt{15}}{5}} (t-\tfrac{4}{9})\alpha(t)\,dt$$
$$+ \int_{-\frac{\sqrt{15}}{5}}^{-\frac{4}{9}} (t+\tfrac{4}{9})\beta(t)\,dt + \int_{0}^{\frac{4}{9}} (t-\tfrac{4}{9})\beta(t)\,dt + \int_{\frac{\sqrt{15}}{5}}^{1} (t-1)\,\beta(t)\,dt\,,$$

and

$$M_{3,1}^{GL} = \int_{-1}^{-\frac{\sqrt{15}}{5}} (t+1)\,\beta(t)\,dt + \int_{-\frac{4}{9}}^{0} (t+\tfrac{4}{9})\beta(t)\,dt + \int_{\frac{4}{9}}^{\frac{\sqrt{15}}{5}} (t-\tfrac{4}{9})\beta(t)\,dt$$
$$+ \int_{-\frac{\sqrt{15}}{5}}^{-\frac{4}{9}} (t+\tfrac{4}{9})\alpha(t)\,dt + \int_{0}^{\frac{4}{9}} (t-\tfrac{4}{9})\alpha(t)\,dt + \int_{\frac{\sqrt{15}}{5}}^{1} (t-1)\,\alpha(t)\,dt\,,$$

provided that $\alpha(x) \leq f'(x) \leq \beta(x)$ for $x \in [-1,1]$. Similarly, by computing

$$\max_{t\in[-1,1]} \left| \mathbf{K}_4\left(t; \{x_{k,3}\}_{k=1}^3, \{w_{k,3}\}_{k=1}^3\right) \right| = \frac{4}{9},$$

and applying theorem 2.3 for (18) one gets

$$m_{3,2}^{GL} \leq R_3^{GL}(f) \ \leq M_{3,2}^{GL},$$

where

$$m_{3,2}^{GL} = \int_{-1}^{1} t\alpha(t)\,dt + \int_{-1}^{-\frac{\sqrt{15}}{5}} \alpha(t)\,dt + \tfrac{4}{9}\int_{-\frac{\sqrt{15}}{5}}^{0} \alpha(t)\,dt - \tfrac{4}{9}\int_{0}^{\frac{\sqrt{15}}{5}} \alpha(t)\,dt - \int_{\frac{\sqrt{15}}{5}}^{1} \alpha(t)\,dt$$
$$-\tfrac{4}{9}\left(f(1) - f(-1) - \int_{-1}^{1} \alpha(t)\,dt\right),$$

and

$$m_{3,2}^{GL} = \int_{-1}^{1} t\alpha(t)\,dt + \int_{-1}^{-\frac{\sqrt{15}}{5}} \alpha(t)\,dt + \tfrac{4}{9}\int_{-\frac{\sqrt{15}}{5}}^{0} \alpha(t)\,dt - \tfrac{4}{9}\int_{0}^{\frac{\sqrt{15}}{5}} \alpha(t)\,dt - \int_{\frac{\sqrt{15}}{5}}^{1} \alpha(t)\,dt$$
$$+\tfrac{4}{9}\left(f(1) - f(-1) - \int_{-1}^{1} \alpha(t)\,dt\right),$$

provided that $\alpha(x) \leq f'(x)$ for $x \in [-1,1]$. And eventually applying theorem2.4 for (18) yields

$$m_{3,3}^{GL} \leq R_3^{GL}(f) \ \leq M_{3,3}^{GL},$$

where

$$m_{3,2}^{GL} = \int_{-1}^{1} t\beta(t)\,dt + \int_{-1}^{-\frac{\sqrt{15}}{5}} \beta(t)\,dt + \tfrac{4}{9}\int_{-\frac{\sqrt{15}}{5}}^{0} \beta(t)\,dt - \tfrac{4}{9}\int_{0}^{\frac{\sqrt{15}}{5}} \beta(t)\,dt - \int_{\frac{\sqrt{15}}{5}}^{1} \beta(t)\,dt$$
$$-\tfrac{4}{9}\left(\int_{-1}^{1} \beta(t)\,dt - f(1) + f(-1)\right),$$

and

$$M_{3,2}^{GL} = \int_{-1}^{1} t\beta(t)\,dt + \int_{-1}^{-\frac{\sqrt{15}}{5}} \beta(t)\,dt + \frac{4}{9}\int_{-\frac{\sqrt{15}}{5}}^{0} \beta(t)\,dt - \frac{4}{9}\int_{0}^{\frac{\sqrt{15}}{5}} \beta(t)\,dt - \int_{\frac{\sqrt{15}}{5}}^{1} \beta(t)\,dt$$

$$+\frac{4}{9}\left(\int_{-1}^{1} \beta(t)\,dt - f(1) + f(-1)\right),$$

provided that $f'(x) \leq \beta(x)$ for $x \in [-1,1]$.

Let us finally add that the residue of 4-point Gauss-Legendre formula is in the form

$$R_4^{GL}(f) = \frac{18-\sqrt{30}}{36} f(-\frac{\sqrt{525+70\sqrt{30}}}{35}) + \frac{18+\sqrt{30}}{36} f(-\frac{\sqrt{525-70\sqrt{30}}}{35})$$

$$+\frac{18+\sqrt{30}}{36} f(\frac{\sqrt{525-70\sqrt{30}}}{35}) + \frac{18-\sqrt{30}}{36} f(\frac{\sqrt{525+70\sqrt{30}}}{35}) - \int_{-1}^{1} f(t)\,dt\,,$$

and its linear kernel as

$$\mathbf{K}_5\left(t; \{x_{k,4}\}_{k=1}^4, \{w_{k,4}\}_{k=1}^4\right) = \begin{cases} t+1 & t \in [-1, -\frac{\sqrt{525+70\sqrt{30}}}{35}]\,, \\ t + \frac{18+\sqrt{30}}{36} & t \in (-\frac{\sqrt{525+70\sqrt{30}}}{35}, -\frac{\sqrt{525-70\sqrt{30}}}{35}]\,, \\ t & t \in (-\frac{\sqrt{525-70\sqrt{30}}}{35}, \frac{\sqrt{525-70\sqrt{30}}}{35}]\,, \\ t & t \in (\frac{\sqrt{525-70\sqrt{30}}}{35}, \frac{\sqrt{525+70\sqrt{30}}}{35}]\,, \\ t-1 & t \in (\frac{\sqrt{525+70\sqrt{30}}}{35}, 1]\,. \end{cases}$$

where

$$\max_{t\in[-1,1]} \left| \mathbf{K}_5\left(t; \{x_{k,4}\}_{k=1}^4, \{w_{k,4}\}_{k=1}^4\right) \right| = \frac{\sqrt{525 - 70\sqrt{30}}}{35}.$$

References

[1] P. Cerone, Three points rules in numerical integration, Nonlinear Anal. TMA 47 (4) (2001) 2341-2352.

[2] P. J. Davis and P. Rabinowitz. Methods of numerical integration. Computer Science and Applied Mathematics. Academic Press Inc., Orlando, FL, second edition, 1984.

[3] S.S. Dragomir, R.P. Agarwal, P. Cerone, On Simpsons inequality and applications, J. Inequal. Appl. 5 (2000) 533-579.

[4] S.S. Dragomir, J. Pecaric, S.Wang, The unified treatment of trapezoid, Simpson and Ostrowski type inequalities for monotonic mappings and applications, Math. Comput. Modelling, 31 (2000) 61-70.

[5] J. Engelbrecht, I. Fedotov, T. Fedotova, A. Harding. Error bounds for quadrature methods involving lower order derivatives. Internat. J. Math. Ed. Sci. Tech., 34 (6) (2003) 831-846.

[6] I. Fedotov, S.S. Dragomir, An inequality of Ostrowski type and its applications for Simpsons rule and special means, Math. Inequal. Appl. 2 (4) (1999) 491-499.

[7] I. Fedotov, S. S. Dragomir. On convergence of quadrature methods for the Lipschitz-continuous functions. Ital. J. Pure Appl. Math., 13 (2003) 91-103.

[8] K.-J. Forster and K. Petras. Error estimates in Gaussian quadrature for functions of bounded variation. SIAM J. Numer. Anal., 28 (3) (1991) 880-889.

[9] W. Gautschi, Numerical Analysis: An Introduction, Birkhauser, Boston, 1997.

[10] W. Gautschi, E. Tychopoulos, and R. S. Varga. A note on the contour integral representation of the remainder term for a Gauss-Chebyshev quadrature rule. SIAM J. Numer. Anal., 27 (1) (1990) 219-224.

[11] W. Gautschi and R. S. Varga. Error bounds for Gaussian quadrature of analytic functions. SIAM J. Numer. Anal., 20 (6) (1983) 1170-1186.

[12] D. B. Hunter. Some error expansions for Gaussian quadrature. BIT, 35 (1) (1995) 64-82.

[13] M. Masjed-Jamei, A linear constructive approximation for integrable functions and a parametric quadrature model based on a generalization of Ostrowski-Grss type inequalities, Elect. Trans. Numer. Anal., 38 (2011) 218-232.
[14] M. Masjed-Jamei, A certain class of weighted approximations for integrable functions and applications, Numer. Func. Anal. Optim., 34 (2013) 1224-1244.
[15] M. Masjed-Jamei, Unified error bounds for all Newton-Cotes quadrature rules, J. Numer. Math., To appear.
[16] G. Mastroianni, G.V. Milovanovic, Interpolation Processes: Basic Theory and Applications, Springer Monographs in Mathematics, Springer-Verlag, Berlin-Heidelberg, 2008.
[17] M. Matic, Improvement of some inequalities of Euler- Grss type, Comput. Math. Appl. 46 (2003) 1325-1336.
[18] G.V. Milovanovic, M.M. Spalevic, and M.S. Pranic. Error estimates for Gaussian quadratures of analytic functions. J. Comput. Appl. Math., 233 (3) (2009) 802-807.
[19] K. Petras. Error bounds for Gaussian and related quadrature and applications to R-convex functions. SIAM J. Numer. Anal., 29 (2) (1992) 578-585.
[20] R. Scherer and T. Schira. Estimating quadrature errors for analytic functions using kernel representations and biorthogonal systems. Numer. Math., 84 (3) (2000) 497-518.
[21] F. Stenger. Bounds on the error of Gauss-type quadratures. Numer. Math., 8 (1966) 150-160.
[22] J. Stoer and R. Bulirsch. Introduction to numerical analysis, volume 12 of Texts in Applied Mathem atics. Springer-Verlag, New York, third edition, 2002.
[23] N. Ujevic, New error bounds for the Simpsons quadrature rule and applications, Comput. Math. Appl. 53 (2007) 64-72.