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# Ćirić Type Generalized $F$-contractions on Complete Metric Spaces and Fixed Point Results

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**Abstract.** Recently, Wardowski [15] introduced the concept of $F$-contraction on complete metric space. This type contraction is proper generalization of ordinary contraction. In the present paper, we give some fixed point results for generalized $F$-contractions including Ćirić type generalized $F$-contraction and almost $F$-contraction on complete metric space. Also, we give some illustrative examples.

## 1. Introduction and Preliminaries

Fixed point theory contains many different fields of mathematics, such as nonlinear functional analysis, mathematical analysis, operator theory and general topology. The fixed point theory is divided into two major areas: One is the fixed point theory on contraction or contraction type mappings on complete metric spaces and the second is the fixed point theory on continuous operators on compact and convex subsets of a normed space. The beginning of fixed point theory in normed space is attributed to the work of Brouwer in 1910, who proved that any continuous self-map of the closed unit ball of $\mathbb{R}^n$ has a fixed point. The beginning of fixed point theory on complete metric space is related to Banach Contraction Principle, published in 1922. Let $(X, d)$ be a metric space and $T : X \to X$ be a mapping. Then $T$ is said to be a contraction mapping if there exists a constant $L \in [0, 1)$, called a contraction factor, such that

$$d(Tx, Ty) \leq Ld(x, y) \text{ for all } x, y \in X. \tag{1}$$

Banach Contraction Principle says that any contraction self-mappings on a complete metric space has a unique fixed point. This principle is one of a very power test for existence and uniqueness of the solution of considerable problems arising in mathematics. Because of its importance for mathematical theory, Banach Contraction Principle has been extended and generalized in many directions (see[1–3, 6, 7, 9–14, 16]). One of the most interesting generalization of it was given by Wardowski [15]. First we recall the concept of $F$-contraction, which was introduced by Wardowski [15], later we will mention his result.

Let $\mathcal{F}$ be the set of all functions $F : (0, \infty) \to \mathbb{R}$ satisfying the following conditions:

($F1$) $F$ is strictly increasing, i.e., for all $\alpha, \beta \in (0, \infty)$ such that $\alpha < \beta, F(\alpha) < F(\beta)$,

($F2$) For each sequence $\{\alpha_n\}$ of positive numbers $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} F(a_n) = -\infty$

($F3$) There exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

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**Definition 1.1 ([15]).** *Let $(X, d)$ be a metric space and $T : X \to X$ be a mapping. Then $T$ is said to be an $F$-contraction if $F \in \mathcal{F}$ and there exists $\tau > 0$ such that*

$$\forall x, y \in X \; [d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y))]. \tag{2}$$

When Wardowski considered in (2) the different type of the mapping $F$ then we obtain the variety of contractions, some of them are of a type known in the literature. We can examine the following examples:

**Example 1.2 ([15]).** *Let $F_1 : (0, \infty) \to \mathbb{R}$ be given by the formulae $F_1(\alpha) = \ln \alpha$. It is clear that $F_1 \in \mathcal{F}$. Then each self mappings $T$ on a metric space $(X, d)$ satisfying (2) is an $F_1$-contraction such that*

$$d(Tx, Ty) \leq e^{-\tau} d(x, y), \text{ for all } x, y \in X, Tx \neq Ty. \tag{3}$$

It is clear that for $x, y \in X$ such that $Tx = Ty$ the inequality $d(Tx, Ty) \leq e^{-\tau} d(x, y)$ also holds. Therefore $T$ satisfies (1) with $L = e^{-\tau}$, thus $T$ is a contraction.

**Example 1.3 ([15]).** *Let $F_2 : (0, \infty) \to \mathbb{R}$ be given by the formulae $F_2(\alpha) = \alpha + \ln \alpha$. It is clear that $F_2 \in \mathcal{F}$. Then each self mappings $T$ on a metric space $(X, d)$ satisfying (2) is an $F_2$-contraction such that*

$$\frac{d(Tx, Ty)}{d(x, y)} e^{d(Tx,Ty) - d(x,y)} \leq e^{-\tau}, \text{ for all } x, y \in X, Tx \neq Ty. \tag{4}$$

We can find some different examples for the function $F$ belonging to $\mathcal{F}$ in [15]. In addition, Wardowski concluded that every $F$-contraction $T$ is a contractive mapping, i.e.,

$$d(Tx, Ty) < d(x, y), \text{ for all } x, y \in X, Tx \neq Ty.$$

Thus, every $F$-contraction is a continuous mapping.

Also, Wardowski concluded that if $F_1, F_2 \in \mathcal{F}$ with $F_1(\alpha) \leq F_2(\alpha)$ for all $\alpha > 0$ and $G = F_2 - F_1$ is nondecreasing, then every $F_1$-contraction $T$ is an $F_2$-contraction.

He noted that for the mappings $F_1(\alpha) = \ln \alpha$ and $F_2(\alpha) = \alpha + \ln \alpha$, $F_1 < F_2$ and a mapping $F_2 - F_1$ is strictly increasing. Hence, it obtained that every Banach contraction (3) satisfies the contractive condition (4). On the other side, Example 2.5 in [15] shows that the mapping $T$ which is not $F_1$-contraction (Banach contraction), but still is an $F_2$-contraction. Thus, the following theorem, which was given by Wardowski, is a proper generalization of Banach Contraction Principle.

**Theorem 1.4 ([15]).** *Let $(X, d)$ be a complete metric space and let $T : X \to X$ be an $F$-contraction. Then $T$ has a unique fixed point in $X$.*

The aim of this paper is to introduce the generalized $F$-contractions, by combining the ideas of Wardowski [15] and Ćirić [8], also to introduce the almost $F$-contractions, by combining the ideas of Wardowski [15] and Berinde [2], and give some fixed point result for these type mappings on complete metric space.

## 2. The Results

**Definition 2.1.** *Let $(X, d)$ be a metric space and $T : X \to X$ be a mapping. Then $T$ is said to be a Ćirić type generalized $F$-contraction if $F \in \mathcal{F}$ and there exists $\tau > 0$ such that*

$$\forall x, y \in X \; [d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(M(x, y))], \tag{5}$$

*where*

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\}.$$

By the considering $F(\alpha) = \ln \alpha$,we can say that every Ćirić type generalized contraction is also Ćirić type generalized $F$-contraction.

One of our main result is as follows:

**Theorem 2.2.** *Let $(X, d)$ be a complete metric space and $T : X \to X$ be a Ćirić type generalized $F$-contraction. If $T$ or $F$ is continuous, then $T$ has a unique fixed point in $X$.*

*Proof.* Let $x_0 \in X$ be an arbitrary point and define a sequence $\{x_n\}$ in $X$ by $x_n = Tx_{n-1}$ for $n \in \{1, 2, \cdots\}$. If $x_{n_0+1} = x_{n_0}$ for some $n_0 \in \{0, 1, \cdots\}$, then $Tx_{n_0} = x_{n_0}$, and so $T$ has a fixed point. Now let $x_{n+1} \neq x_n$ for every $n \in \{0, 1, \cdots\}$ and let $\gamma_n = d(x_{n+1}, x_n)$ for $n \in \{0, 1, \cdots\}$. Then $\gamma_n > 0$ for all $n \in \{0, 1, \cdots\}$. Now using (5), we have

$$\begin{aligned} F(\gamma_n) &= F(d(x_{n+1}, x_n)) = F(d(Tx_n, Tx_{n-1})) \\ &\leq F(M(x_n, x_{n-1})) - \tau \\ &= F(\max\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\}) - \tau \\ &= F(\max\{\gamma_{n-1}, \gamma_n\}) - \tau. \end{aligned} \tag{6}$$

If $\gamma_n \geq \gamma_{n-1}$ for some $n \in \{1, 2, \cdots\}$, then from (6) we have $F(\gamma_n) \leq F(\gamma_n) - \tau$, which is a contradiction since $\tau > 0$. Thus $\gamma_n < \gamma_{n-1}$ for all $n \in \{1, 2, \cdots\}$ and so from (6) we have

$$F(\gamma_n) \leq F(\gamma_{n-1}) - \tau.$$

Therefore we obtain

$$F(\gamma_n) \leq F(\gamma_{n-1}) - \tau \leq F(\gamma_{n-2}) - 2\tau \leq \cdots \leq F(\gamma_0) - n\tau. \tag{7}$$

From (7), we get $\lim_{n\to\infty} F(\gamma_n) = -\infty$.Thus, from (F2), we have

$$\lim_{n\to\infty} \gamma_n = 0.$$

From (F3) there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} \gamma_n^k F(\gamma_n) = 0.$$

By (7), the following holds for all $n \in \{1, 2, \cdots\}$

$$\gamma_n^k F(\gamma_n) - \gamma_n^k F(\gamma_0) \leq -\gamma_n^k n\tau \leq 0. \tag{8}$$

Letting $n \to \infty$ in (8), we obtain that

$$\lim_{n\to\infty} n\gamma_n^k = 0. \tag{9}$$

From (9), there exits $n_1 \in \{1, 2, \cdots\}$ such that $n\gamma_n^k \leq 1$ for all $n \geq n_1$. So, we have, for all $n \geq n_1$

$$\gamma_n \leq \frac{1}{n^{1/k}}. \tag{10}$$

In order to show that $\{x_n\}$ is a Cauchy sequence consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$.Using the triangular inequality for the metric and from (10), we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= \gamma_n + \gamma_{n+1} + \cdots + \gamma_{m-1} \\ &= \sum_{i=n}^{m-1} \gamma_i \\ &\leq \sum_{i=n}^{\infty} \gamma_i \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}} \end{aligned}$$

By the convergence of the series $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$, passing to limit $n \to \infty$, we get $d(x_n, x_m) \to 0$. This yields that $\{x_n\}$ is a Cauchy sequence in $(X, d)$. Since $(X, d)$ is a complete metric space, the sequence $\{x_n\}$ converges to some point $z \in X$, that is,$\lim_{n\to\infty} x_n = z$.

Now, if $T$ is continuous, then we have $z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T(\lim_{n\to\infty} x_n) = Tz$ and so $z$ is a fixed point of $T$.

Now, suppose $F$ is continuous. In this case, we claim that $z = Tz$. Assume the contrary, that is, $z \neq Tz$. In this case, there exist an $n_0 \in \mathbb{N}$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $d(Tx_{n_k}, Tz) > 0$ for all $n_k \geq n_0$. (Otherwise, there exists $n_1 \in \mathbb{N}$ such that $x_n = Tz$ for all $n \geq n_1$, which implies that $x_n \to Tz$. This is a contradiction, since $z \neq Tz$.) Since $d(Tx_{n_k}, Tz) > 0$ for all $n_k \geq n_0$, then from (5), we have

$$\begin{aligned} \tau + F(d(x_{n_k+1}, Tz)) &= \tau + F(d(Tx_{n_k}, Tz)) \\ &\leq F(M(x_{n_k}, z)) \\ &\leq F(\max\{d(x_{n_k}, z), d(x_{n_k}, x_{n_k+1}), d(z, Tz), \frac{1}{2}[d(x_{n_k}, Tz) + d(z, x_{n_k+1})]\}). \end{aligned}$$

Taking the limit $k \to \infty$ and using the continuity of $F$ we have $\tau + F(d(z, Tz)) \leq F(d(z, Tz))$, which is a contradiction. Therefore we claim is true, that is $z = Tz$.

The uniqueness of the fixed follows easily from (5). □

**Example 2.3.** *Let $X = \{\frac{1}{n^2} : n \in \mathbb{N}\} \cup \{0\}$ and $d(x, y) = \left|x - y\right|$, then $(X, d)$ is complete metric space. Define a map $T : X \to X$,*

$$Tx = \begin{cases} \frac{1}{(n+1)^2} &, \quad x = \frac{1}{n^2} \\ 0 &, \quad x = 0 \end{cases}.$$

*First, let us consider the mapping $F_1$ defined by $F_1(\alpha) = \ln\alpha$. Then $T$ is not generalized $F_1$-contraction (which actually means that $T$ is not the generalized contraction of Ćirić type (see page 69 of [8])). Indeed, we get*

$$\sup_{x,y\in X, x\neq y} \frac{d(Tx, Ty)}{M(x, y)} = 1.$$

*On the other side, taking $F_2$ with*

$$F_2(\alpha) = \begin{cases} \frac{\ln\alpha}{\sqrt{\alpha}} &, \quad 0 < \alpha < e^2 \\ \alpha - e^2 + \frac{2}{e} &, \quad \alpha \geq e^2 \end{cases}.$$

*It is easy to see that the conditions (F1), (F2) and (F3) (for $k = \frac{2}{3}$) are satisfied (Also note that $F$ is continuous). We obtain that $T$ is generalized $F_2$-contraction with $\tau = \ln 2$. To see this, let us consider the following calculations: Note that $\sup_{x,y\in X} d(x, y) = 1 < e^2$.*

*$T$ is generalized $F_2$-contraction with $\tau = \ln 2$ if and only if*

$$\forall x, y \in X[d(Tx, Ty) > 0 \Rightarrow \ln 2 + F_2(d(Tx, Ty)) \leq F_2(M(x, y))]. \tag{11}$$

*To see (11), it is sufficient to show that (by the $(F_1)$)*

$$\forall x, y \in X[d(Tx, Ty) > 0 \Rightarrow \ln 2 + F_2(d(Tx, Ty)) \leq F_2(d(x, y))].$$

$$\Leftrightarrow \forall x, y \in X[d(Tx, Ty) > 0 \Rightarrow d(Tx, Ty)^{\frac{1}{\sqrt{d(Tx,Ty)}}} d(x, y)^{-\frac{1}{\sqrt{d(x,y)}}} \leq \tfrac{1}{2}]$$

$$\Leftrightarrow \forall x, y \in X[\left|Tx - Ty\right| > 0 \Rightarrow \left|Tx - Ty\right|^{\frac{1}{\sqrt{|Tx-Ty|}}} \left|x - y\right|^{-\frac{1}{\sqrt{|x-y|}}} \leq \tfrac{1}{2}].$$

*Now if* $x = \frac{1}{n^2}$ *and* $y = \frac{1}{m^2}$ *with* $m > n$, *then*

$$\begin{aligned}
\left|Tx - Ty\right|^{\frac{1}{\sqrt{|Tx-Ty|}}} \left|x - y\right|^{-\frac{1}{\sqrt{|x-y|}}} &= \left(\frac{1}{(n+1)^2} - \frac{1}{(m+1)^2}\right)^{\frac{1}{\sqrt{\frac{1}{(n+1)^2} - \frac{1}{(m+1)^2}}}} \left(\frac{1}{n^2} - \frac{1}{m^2}\right)^{-\frac{1}{\sqrt{\frac{1}{n^2} - \frac{1}{m^2}}}} \\
&= \left(\frac{(m+1)^2 - (n+1)^2}{(n+1)^2(m+1)^2}\right)^{\frac{(n+1)(m+1)}{\sqrt{(m+1)^2-(n+1)^2}}} \left(\frac{m^2 - n^2}{n^2 m^2}\right)^{-\frac{nm}{\sqrt{m^2-n^2}}} \\
&= \left(\frac{(m+1)^2 - (n+1)^2}{(n+1)^2(m+1)^2}\right)^{\frac{(n+1)(m+1)}{\sqrt{(m+1)^2-(n+1)^2}}} \left(\frac{m^2 - n^2}{n^2 m^2} \frac{m+n+2}{m+n+2}\right)^{-\frac{nm}{\sqrt{m^2-n^2}}} \\
&= \left(\frac{(m+1)^2 - (n+1)^2}{(n+1)^2(m+1)^2}\right)^{\frac{(n+1)(m+1)}{\sqrt{(m+1)^2-(n+1)^2}}} \times \\
&\quad \left(\frac{(m+1)^2 - (n+1)^2}{(n+1)^2(m+1)^2} \frac{(m+n)(n+1)^2(m+1)^2}{(m+n+2)n^2m^2}\right)^{-\frac{nm}{\sqrt{m^2-n^2}}} \\
&= \left(\frac{(m+1)^2 - (n+1)^2}{(n+1)^2(m+1)^2}\right)^{\frac{(n+1)(m+1)}{\sqrt{(m+1)^2-(n+1)^2}} - \frac{nm}{\sqrt{m^2-n^2}}} \left(\frac{(m+n+2)n^2m^2}{(m+n)(n+1)^2(m+1)^2}\right)^{\frac{nm}{\sqrt{m^2-n^2}}}.
\end{aligned}$$

*On the other hand, since*

$$\frac{(m+1)^2 - (n+1)^2}{(n+1)^2(m+1)^2} \leq \frac{1}{2},$$

$$\frac{(n+1)(m+1)}{\sqrt{(m+1)^2 - (n+1)^2}} - \frac{nm}{\sqrt{m^2 - n^2}} \geq 1$$

*and*

$$\frac{(m+n+2)n^2m^2}{(m+n)(n+1)^2(m+1)^2} < 1$$

*then we have*

$$\left|Tx - Ty\right|^{\frac{1}{\sqrt{|Tx-Ty|}}} \left|x - y\right|^{-\frac{1}{\sqrt{|x-y|}}} \leq \frac{1}{2}.$$

*Therefore (11) is satisfied.*

*Now if* $x = \frac{1}{n^2}$ *and* $y = 0$, *then*

$$\begin{aligned}
\left|Tx - Ty\right|^{\frac{1}{\sqrt{|Tx-Ty|}}} \left|x - y\right|^{-\frac{1}{\sqrt{|x-y|}}} &= \left|\frac{1}{(n+1)^2}\right|^{\frac{1}{\sqrt{\left|\frac{1}{(n+1)^2}\right|}}} \left|\frac{1}{n^2}\right|^{-\frac{1}{\sqrt{\left|\frac{1}{n^2}\right|}}} \\
&= \frac{n^{2n}}{(n+1)^{2(n+1)}} \\
&= \frac{n^{2(n+1)}}{(n+1)^{2(n+1)}} \frac{1}{n^2} \\
&= \left(\frac{n}{n+1}\right)^{2(n+1)} \frac{1}{n^2} \\
&\leq \frac{1}{2}.
\end{aligned}$$

*Therefore (11) is satisfied. Thus all conditions of Theorem 2.2 are satisfied and so T has a unique fixed point in X.*

For the second result, we recall the concept of almost contraction on metric space (see [3] and [4] for detailed information).

**Definition 2.4.** *Let $(X, d)$ be a metric space and $T : X \to X$ is a self operator. $T$ is said to be an almost contraction (or $(\delta, L)$-weak contraction) if there exist a constant $\delta \in (0, 1)$ and some $L \geq 0$ such that*

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx) \tag{12}$$

*for all $x, y \in X$.*

Note that, by the symmetry property of the distance, the almost contraction condition implicitly includes the following dual one

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(x, Ty) \tag{13}$$

for all $x, y \in X$. So, in order to check the almost contractiveness of a mapping $T$, it is necessary to check both (12) and (13).

In [3] and [4], Berinde shows that any Banach, Kannan, Chatterjea and Zamfirescu mappings are almost contraction. Using the concept of almost contraction mappings, Berinde [4] proved the following fixed point theorem:

**Theorem 2.5.** *Let $(X, d)$ be a complete metric space and $T : X \to X$ an almost contraction, then $T$ has a fixed point.*

Also, Berinde shows that any almost contraction mapping is a Picard operator. Again, Berinde [5] introduced the nonlinear type almost contraction using a comparison function and obtained some fixed point results

In the parallel manner we introduce the following definition.

**Definition 2.6.** *Let $(X, d)$ be a metric space and $T : X \to X$ be a mapping. Then $T$ is said to be an almost $F$-contraction if $F \in \mathcal{F}$ and there exist $\tau > 0$ and $L \geq 0$ such that*

$$\forall x, y \in X \; [d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y) + Ld(y, Tx))]. \tag{14}$$

In order to check the almost $F$-contractiveness of a mapping $T$, it is necessary to check both (14) and

$$\forall x, y \in X \; [d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y) + Ld(x, Ty))]. \tag{15}$$

**Remark 2.7.** *By the considering $F(\alpha) = \ln \alpha$, we can say that every almost contraction is also almost $F$-contraction. But the converse may not be true. To see this, we can take $x = \frac{1}{n^2}$ and $y = \frac{1}{(n+1)^2}$ in Example 2.3, then we have $d(y, Tx) = 0$ and*

$$\sup_{n \in \mathbb{N}} \frac{d(T\frac{1}{n^2}, T\frac{1}{(n+1)^2})}{d(\frac{1}{n^2}, \frac{1}{(n+1)^2})} = 1.$$

*Thus we can not find $\delta \in (0, 1)$ and $L \geq 0$ satisfying (12). But, by the same example, $T$ is an almost $F$-contraction.*

Using the concept of almost $F$-contraction, we can give the following fixed point result. Note that, $T$ (or $F$) need not be continuous in the following theorem.

**Theorem 2.8.** *Let $(X, d)$ be a complete metric space and $T : X \to X$ be an almost $F$-contraction, then $T$ has a fixed point in $X$.*

*Proof.* Let $x_0 \in X$ be an arbitrary point and define a sequence $\{x_n\}$ in $X$ by $x_n = Tx_{n-1}$ for $n \in \{1, 2, \cdots\}$. If $x_{n_0+1} = x_{n_0}$ for some $n_0 \in \{0, 1, \cdots\}$, then $Tx_{n_0} = x_{n_0}$, and so $T$ has a fixed point. Now let $x_{n+1} \neq x_n$ for every $n \in \{0, 1, \cdots\}$ and let $\gamma_n = d(x_{n+1}, x_n)$ for $n \in \{0, 1, \cdots\}$. Then $\gamma_n > 0$ for all $n \in \{0, 1, \cdots\}$. Now using (14), we have

$$\begin{aligned} F(\gamma_n) &= F(d(x_n, x_{n+1})) \\ &= F(d(Tx_{n-1}, Tx_n)) \\ &\leq F(d(x_{n-1}, x_n)) - \tau \end{aligned}$$

Therefore we obtain

$$F(\gamma_n) \leq F(\gamma_{n-1}) - \tau \leq F(\gamma_{n-2}) - 2\tau \leq \cdots \leq F(\gamma_0) - n\tau.$$

and so we get $\lim_{n\to\infty} F(\gamma_n) = -\infty$.Thus, from $(F2)$, we have

$$\lim_{n\to\infty} \gamma_n = 0.$$

By the same way as in the proof of Theorem 2.2, we can show that $\{x_n\}$ is a Cauchy sequence in $(X, d)$. Since $(X, d)$ is a complete metric space, the sequence $\{x_n\}$ converges to some point $z \in X$, that is, $\lim_{n\to\infty} x_n = z$.

On the other hand, from $(F2)$ and (14), it is easy to conclude that

$$d(Tx, Ty) < d(x, y) + Ld(y, Tx)$$

for all $x, y \in X$ with $Tx \neq Ty$. Therefore, for all $x, y \in X$

$$d(Tx, Ty) \leq d(x, y) + Ld(y, Tx) \tag{16}$$

is satisfied. Thus, from (16),

$$\begin{aligned} d(Tz, x_{n+1})) &= d(Tz, Tx_n) \\ &\leq d(x_n, z) + Ld(z, Tx_n) \\ &= d(x_n, z) + Ld(z, x_{n+1}). \end{aligned}$$

Taking the $n \to \infty$ we have $d(z, Tz) = 0$ and so so $z = Tz$. □

In Remark 2.7 we give an example that the mapping $T$ is almost $F$-contraction but not almost contraction. Now we give an example showing that $T$ is almost $F$-contraction but not $F$-contraction. Therefore Theorem 2.8 can be applied to this example but Theorem 1.4 can not.

**Example 2.9.** *Let $X = [0, 1] \cup \{2, 3\}$ and $d(x, y) = \left|x - y\right|$, then $(X, d)$ is complete metric space. Define a map $T : X \to X$,*

$$Tx = \begin{cases} \frac{1-x}{2} & , \quad x \in [0, 1] \\ x & , \quad x \in \{2, 3\} \end{cases}.$$

*Since $d(T2, T3) = 1 = d(2, 3)$, then for all $F \in \mathcal{F}$ and $\tau > 0$ we have*

$$\tau + F(d(T2, T3)) > F(d(2, 3)).$$

*Therefore, $T$ is not $F$-contraction, and so Theorem 1.4 can not be applied to this example.*

*Now, let us consider the mapping $F$ defined by $F(\alpha) = \ln \alpha$. Then $T$ is almost $F$-contraction with $\tau = \ln 2$ and $L = 4$. Note that if $d(Tx, Ty) > 0$, then $x \neq y$, and so*

$$\forall x, y \in X[d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y) + Ld(y, Tx))]$$

*is equivalent to*

$$\forall x, y \in X[x \neq y \Rightarrow d(Tx, Ty) \leq e^{-\tau} d(x, y) + Le^{-\tau} d(y, Tx)]$$

*and so*

$$\forall x, y \in X[x \neq y \Rightarrow d(Tx, Ty) \leq \frac{1}{2} d(x, y) + 2d(y, Tx)] \tag{17}$$

*Now we consider the following cases:*

*Case 1. Let* $x, y \in [0, 1]$*, then* $d(Tx, Ty) = \frac{1}{2}\left|x - y\right|$, $d(x, y) = \left|x - y\right|$ *and* $d(y, Tx) = \left|\frac{2y+x-1}{2}\right|$*. It is clear that (17) is satisfied.*

*Case 2. Let* $x, y \in \{2, 3\}$*, then* $d(Tx, Ty) = \left|x - y\right|$, $d(x, y) = \left|x - y\right|$ *and* $d(y, Tx) = \left|x - y\right|$*. It is clear that (17) is satisfied.*

*Case 3. Let* $x \in [0, 1]$ *and* $y \in \{2, 3\}$*, then* $d(Tx, Ty) = \left|\frac{2y+x-1}{2}\right|$, $d(x, y) = \left|x - y\right|$ *and* $d(y, Tx) = \left|\frac{2y+x-1}{2}\right|$*. It is clear that (17) is satisfied.*

*Case 4. Let* $x \in \{2, 3\}$ *and* $y \in [0, 1]$*, then* $d(Tx, Ty) = \left|\frac{2x+y-1}{2}\right|$, $d(x, y) = \left|x - y\right|$ *and* $d(y, Tx) = \left|x - y\right|$*. Therefore* $d(Tx, Ty) = \frac{2x+y-1}{2}$ *and* $\frac{1}{2}d(x, y) + 2d(y, Tx) = \frac{5}{2}(x - y)$*. Since* $\frac{2x+y-1}{2} \leq \frac{5}{2}(x - y)$*, (17) is satisfied.*

In Theorem 2.8, we show that if $T$ is an almost $F$-contraction, then it has a fixed point. But in order to guarantee the uniqueness of the fixed point of $T$, we have to consider an additional condition, as in the following theorem.

**Theorem 2.10.** *Let* $(X, d)$ *be a complete metric space and* $T : X \to X$ *be an almost F-contraction. Suppose T also satisfies the following condition: there exist* $G \in \mathcal{F}$ *and some* $L_1 \geq 0$ *and* $\tau_1 > 1$ *such that*

$$\forall x, y \in X\ [d(Tx, Ty) > 0 \Rightarrow \tau_1 + G(d(Tx, Ty)) \leq G(d(x, y) + L_1 d(x, Tx))] \tag{18}$$

*holds. Then T has a unique fixed point in X.*

*Proof.* Suppose that, there are two fixed point $z$ and $w$ of $T$. If $d(z, w) = 0$, it is clear that $z = w$. Assume that $d(z, w) > 0$. By (18) with $x = z$ and $y = w$, we have

$$\begin{aligned} \tau_1 + G(d(z, w)) &= \tau_1 + G(d(Tz, Tw)) \\ &\leq G(d(z, w) + L_1 d(z, Tz)) \\ &= G(d(z, w)), \end{aligned}$$

which is a contradiction. Therefore $T$ has a unique fixed point. □

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