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Stability of a Mixed Type Additive and Quartic Functional Equation

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Abstract. In this paper we obtain the general solution of a mixed additive and quartic functional equation. We also prove the Hyers-Ulam stability of this functional equation in random normed spaces.

1. Introduction

The problem of stability of functional equations was originally raised by S. M. Ulam [20] in 1940: given a group G, a metric group H with metric $d(\cdot,\cdot)$, and a $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $f : G \longrightarrow H$ satisfies $d(f(xy), f(x)f(y)) \leq \delta$ for all $x, y \in G$, then a homomorphism $g : G \longrightarrow H$ exists with $d(f(x), g(x)) \leq \epsilon$ for all $x \in G$? For Banach spaces the Ulam problem was first solved by D. H. Hyers [11] in 1941, which states that if $\delta > 0$ and $f : X \longrightarrow Y$ is a mapping with Banach spaces X and Y, so that

$$\|f(x + y) - f(x) - f(y)\| \leq \delta \tag{1}$$

for all $x, y \in X$, then there exists a unique additive mapping $T : X \longrightarrow Y$ such that $\|f(x) - T(x)\| \leq \delta$ for all $x, y \in X$. Due to this fact, the additive functional equation $f(x + y) = f(x) + f(y)$ is said to have the Hyers-Ulam stability property on X. This result was generalized by Aoki [1] for additive mappings and by Rassias [17] for linear mappings by considering an unbounded Cauchy difference. This terminology is also applied to other functional equations which has been studied by many authors (see, for example, [3], [5], [6], [8], [12] and [15]).

Rassias [16] investigated stability properties of the following quartic functional equation

$$f(x + 2y) + f(x - 2y) + 6f(x) = 4f(x + y) + 4f(x - y) + 24f(y).$$

This equation is equivalent to the following

$$f(x + 2y) + f(x - 2y) = 10f(x) + 24f(y) - f(2x) + 4f(x + y) + 4f(x - y). \tag{2}$$

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In [7], Chung and Sahoo determined the general solution of (2) without assuming any regularity conditions on the unknown function. Indeed, they proved that the function $f : \mathbb{R} \to \mathbb{R}$ is a solution of (2) if and only if $f(x) = Q(x, x, x, x)$ where the function $Q : \mathbb{R}^4 \to \mathbb{R}$ is symmetric and additive in each variable. The fact that every solution of (2) is even implies that it can be written as follows:

$$f(2x + y) + f(2x - y) = 24f(x) - 6f(y) + 4f(x + y) + 4f(x - y). \tag{3}$$

Lee et al. [13] obtained the general solution of (3) and proved the Hyers-Ulam stability of this equation.

In [9], Eshaghi Gordji introduced and obtained the general solution of the following mixed type additive and quartic functional equation

$$f(2x + y) + f(2x - y) = 4\{(f(x + y) + f(x - y)\} - \frac{3}{7}(f(2y) - 2f(y)) + 2f(2x) - 8f(x). \tag{4}$$

He also established the Hyers-Ulam Rassias stability of the above functional equation in real normed spaces. The stability of (4) in non-Archimedean orthogonality spaces is stadied in [14] (see also [4]).

In this paper, we present a new form of the functional equation (4) as follows:

$$f(x + 2y) - 4f(x + y) - 4f(x - y) + f(x - 2y) = \frac{12}{7}(f(2y) - 2f(y)) - 6f(x) \tag{5}$$

It is easily verified that the function $f(x) = \alpha x^4 + \beta x$ is a solution of the functional equation (5). We find out the general solution of (5) and investigate the Hyers-Ulam stability of this functional equation in random normed spaces which our way is different from [10].

2. General Solution of (5)

Lemma 2.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be real vector spaces.*

(i) If an odd mapping $f : \mathcal{X} \longrightarrow \mathcal{Y}$ satisfies the functional equation (5), *then f is additive;*

(ii) If an even mapping $f : \mathcal{X} \longrightarrow \mathcal{Y}$ satisfies the functional equation (5), *then f is quartic.*

Proof. (i) Letting $x = y = 0$ in (5), we have $f(0) = 0$. Once more, by putting $x = 0$ in (5), then by oddness of f, we get

$$f(2y) = 2f(y) \tag{6}$$

for all $y \in \mathcal{X}$. Hence (5) can be rewritten as

$$f(x + 2y) + f(x - 2y) = 4[f(x + y) + f(x - y)] - 6f(x) \tag{7}$$

for all $x, y \in \mathcal{X}$. Repalcing x by $2x$ in (7) and using (6), we have

$$2[f(2x + y) + f(2x - y)] = f(x + y) + f(x - y) + 6f(x) \tag{8}$$

for all $x, y \in \mathcal{X}$. Interchanging x into y in (7), we get

$$f(2x+y) - f(2x-y) = 4[f(x+y) - f(x-y)] - 6f(y) \tag{9}$$

for all $x, y \in \mathcal{X}$. Replacing y by $-y$ in (9), we have

$$f(2x-y) - f(2x+y) = 4[f(x-y) - f(x+y)] + 6f(y) \tag{10}$$

for all $x, y \in \mathcal{X}$. Plugging (8) into (10), we obtain

$$4f(2x+y) = 9f(x+y) - 7f(x-y) + 6f(x) - 12f(y) \tag{11}$$

for all $x, y \in \mathcal{X}$. Replacing y by $y - x$ in (11) and multiplying the result by $\frac{4}{7}$, we arrive at

$$4f(2x-y) = \frac{48}{7}f(x-y) - \frac{16}{7}f(x+y) + \frac{24}{7}f(x) + \frac{36}{7}f(y) \tag{12}$$

for all $x, y \in \mathcal{X}$. Combining the equations (8), (11) and (12) to obtain

$$33f(x+y) - 15f(x-y) = 18f(x) - 48f(y) \tag{13}$$

for all $x, y \in \mathcal{X}$. Substituting x by $2x$ in (8), we have

$$2[f(4x+y) + f(4x-y)] = f(2x+y) + f(2x-y) + 12f(x) \tag{14}$$

for all $x, y \in \mathcal{X}$. Replacing y by $y + 2x$ in (8), we get

$$4f(4x+y) - 4f(y) = 2f(3x+y) - 2f(x+y) + 12f(x) \tag{15}$$

for all $x, y \in \mathcal{X}$. Putting $-y$ instead of y in (15), we have

$$4f(4x-y) + 4f(y) = 2f(3x-y) - 2f(x-y) + 12f(x) \tag{16}$$

for all $x, y \in \mathcal{X}$. Adding (15) to (16), we deduce that

$$\begin{aligned} &4[f(4x+y) + f(4x-y)] \\ &= 2[f(3x+y) + f(3x-y)] - 2[f(x+y) + f(x-y)] + 24f(x) \end{aligned} \tag{17}$$

for all $x, y \in \mathcal{X}$. Replacing y by $x - y$ and $x + y$ in (8), respectively, and combining the results to obtain

$$\begin{aligned} &4[f(3x+y) + f(3x-y)] \\ &= -4[f(x+y) + f(x-y)] + 2[f(2x+y) + f(2x-y)] + 24f(x) \end{aligned} \tag{18}$$

for all $x, y \in \mathcal{X}$. Now, the relations (8) and (18) imply that

$$4[f(3x+y) + f(3x-y)] = -3[f(x+y) + f(x-y)] + 30f(x) \tag{19}$$

for all $x, y \in \mathcal{X}$. It follows from (14), (17) and (19) that

$$f(x+y) + f(x-y) = 2f(x) \tag{20}$$

for all $x, y \in \mathcal{X}$. Substituting x, y by y, x in (20), respectively, we obtain

$$f(x+y) - f(x-y) = 2f(y) \tag{21}$$

for all $x, y \in \mathcal{X}$. The equalities (20) and (21) show that

$$f(x+y) = f(x) + f(y) \qquad (x, y \in \mathcal{X}).$$

(ii) Similar to the part (i), we can show that $f(0) = 0$ and $f(2y) = 16f(y)$ for all $y \in \mathcal{X}$. These results imply that (5) is as

$$f(x+2y) + f(x-2y) = 4[f(x+y) + f(x-y)] + 24f(y) - 6f(x) \tag{22}$$

for all $x, y \in \mathcal{X}$. Replacing x by $2x$ in (22), we get

$$f(2x+y) + f(2x-y) = 4[f(x+y) + f(x-y)] + \frac{3}{2}f(2x) - 6f(y) \tag{23}$$

for all $x, y \in \mathcal{X}$. Since $f(2x) = 16f(x)$, the equation (23) is equivalent to the following equation

$$f(2x+y) + f(2x-y) = 4[f(x+y) + f(x-y)] + 24f(x) - 6f(y) \quad (x, y \in \mathcal{X}).$$

Therefore f satisfies (3) and so f is a quartic mapping. □

Throughout this paper, we use the abbreviation for the given mapping $f : \mathcal{X} \longrightarrow \mathcal{Y}$ as follows:

$$\mathcal{D}f(x,y) := f(x+2y) - 4f(x+y) - 4f(x-y) + f(x-2y) - \frac{12}{7}(f(2y) - 2f(y)) + 6f(x)$$

for all $x, y \in \mathcal{X}$.

3. Stability of (5)

In this section, we state the usual terminology, notations and conventions of the theory of random normed spaces, as in [18] and [19]. The set of all probability distribution functions is denoted by

$$\Delta^+ := \{F : \mathbb{R} \cup \{-\infty, \infty\} \longrightarrow [0,1] | F \text{ is left-continuous and nondecreasing on } \mathbb{R}; \text{ where } F(0) = 0 \text{ and } F(+\infty) = 1\}.$$

Let us define $D^+ := \{F \in \Delta^+ | l^- F(+\infty) = 1\}$, where $l^- F(x)$ denotes the left limit of the function f at the point x. The set Δ^+ is partially ordered by the usual pointwise ordering of functions, that is, $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function ϵ_0 given by

$$\epsilon_0(t) = \begin{cases} 0, & \text{if } t \leq 0 \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 3.1. *([18]) A mapping $\tau : [0,1] \times [0,1] \longrightarrow [0,1]$ is said to be a continuous triangular norm (briefly, a continuous t-norm) if τ satisfies the following conditions:*

(i) τ is commutative and associative;

(ii) τ is continuous;

(iii) $\tau(a, 1) = a$ for all $a \in [0,1]$;

(iv) $\tau(a,b) \leq \tau(c,d)$ *whenever* $a \leq c$ *and* $b \leq d$ *for all* $a,b,c,d \in [0,1]$.

Typical examples of continuous t-norms are $\tau_P(a,b) = ab$, $\tau_M(a,b)$=min{a,b} and $\tau_L(a,b)$=max$\{a+b-1,0\}$.

Definition 3.2. *([19]) A random normed space (briefly, RN-space) is a triple* $(\mathcal{X},\mu,\tau)$, *where* $\mathcal{X}$ *is a vector space,* τ *is a continuous t-norm, and* μ *is a mapping from* $\mathcal{X}$ *into* D^+ *such that the following conditions hold:*

(RN1) $\mu_x(t) = \epsilon_0(t)$ *for all* $t > 0$ *if and only if* $x = 0$;

(RN2) $\mu_{\alpha x}(t) = \mu_x(t/|\alpha|)$ *for all* $x \in \mathcal{X}$, $\alpha \neq 0$ *and all* $t \geq 0$;

(RN3) $\mu_{x+y}(t+s) \geq \tau(\mu_x(t),\mu_y(s))$ *for all* $x,y \in \mathcal{X}$ *and all* $t,s \geq 0$.

Let $(\mathcal{X},\|\cdot\|)$ be a normed space. Define the mapping $\mu : \mathcal{X} \longrightarrow D^+$ via $\mu_x(t) = \frac{t}{t+\|x\|}$ for all $x \in \mathcal{X}$ and all $t \geq 0$. Then $(\mathcal{X},\mu,\tau_M)$ is a random normed space.

Definition 3.3. *Let* $(\mathcal{X},\mu,\tau)$ *be an RN-space.*

(1) A sequence $\{x_n\}$ *in* $\mathcal{X}$ *is said to be convergent to a point* $x \in \mathcal{X}$ *if, for every* $t > 0$ *and* $\epsilon > 0$, *there exists a positive integer N such that* $\mu_{x_n - x}(t) > 1-\epsilon$ *whenever* $n \geq N$;

(2) A sequence $\{x_n\}$ *in* $\mathcal{X}$ *is called a Cauchy sequence if, for every* $t > 0$ *and* $\epsilon > 0$, *there exists a positive integer N such that* $\mu_{x_n - x_m}(t) > 1-\epsilon$ *whenever* $n \geq m \geq N$;

(3) An RN-space $(\mathcal{X},\mu,\tau)$ *is said to be complete if and only if every Cauchy sequence in* $\mathcal{X}$ *is convergent to a point in* $\mathcal{X}$.

Theorem 3.4. *([18]) If* $(\mathcal{X},\mu,\tau)$ *is an RN-space and* $\{x_n\}$ *is a sequence such that* $x_n \to x$, *then* $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$.

For a t-norm τ and a given sequence $\{a_n\}$ in $[0,1]$, we define $\tau_{j=1}^{n} a_j$ recursively by $\tau_{j=1}^{1} a_j = a_1$ and $\tau_{j=1}^{n} a_j = \tau(\tau_{j=1}^{n-1} a_j, a_n)$ for all $n \geq 2$. We now prove the stability of the functional equation (5) in the setting of random normed spaces.

Theorem 3.5. *Let* $\mathcal{X}$ *be a linear space,* $(\mathcal{Z},\Lambda,\tau_M)$ *be an RN-space and* $(\mathcal{Y},\mu,\tau_M)$ *be a complete RN-space. Suppose that* $\psi : \mathcal{X}\times\mathcal{X} \longrightarrow \mathcal{Z}$ *is a mapping such that for some* $0 < \alpha < 16$,

$$\Lambda_{\psi(0,2x)}(t) \geq \Lambda_{\alpha\psi(0,x)}(t) \qquad (x \in \mathcal{X}, t > 0) \tag{24}$$

and

$$\lim_{n\to\infty} \Lambda_{\psi(2^n x, 2^n y)}(16^n t) = 1 \qquad (x,y \in \mathcal{X}, t > 0). \tag{25}$$

If $f : \mathcal{X} \longrightarrow \mathcal{Y}$ *is an even mapping with* $f(0) = 0$ *and*

$$\mu_{\mathcal{D}f(x,y)}(t) \geq \Lambda_{\psi(x,y)}(t) \tag{26}$$

for all $x,y \in \mathcal{X}$ *and all* $t > 0$, *then there exists a unique quartic mapping* $Q : \mathcal{X} \longrightarrow \mathcal{Y}$ *such that*

$$\mu_{f(x)-Q(x)}(t) \geq \Lambda_{\psi(0,x)}\left(\frac{2(16-\alpha)}{7}t\right) \tag{27}$$

for all $x \in \mathcal{X}$ *and all* $t > 0$.

Proof. Replacing (x, y) by $(0, x)$ in (26), then by eveness of f, we have

$$\mu_{\left(\frac{2}{7}f(2x)-\frac{32}{7}f(x)\right)}(t) \geq \Lambda_{\psi(0,x)}(t)$$

for all $x \in \mathcal{X}$. Thus

$$\mu_{\left(\frac{1}{16}f(2x)-f(x)\right)}(t) \geq \Lambda_{\psi(0,x)}\left(\frac{32}{7}t\right) \tag{28}$$

for all $x \in \mathcal{X}$. Substituting x by $2^n x$ in (28) and applying (24), we get

$$\begin{aligned}\mu_{\left(\frac{f(2^{n+1}x)}{16^{n+1}}-\frac{f(2^n x)}{16^n}\right)}(t) &\geq \Lambda_{\psi(0,2^n x)}\left(\frac{32}{7}16^n t\right)\\ &\geq \Lambda_{\alpha^n\psi(0,x)}\left(\frac{32}{7}16^n t\right)\\ &\geq \Lambda_{\psi(0,x)}\left(\frac{32}{7}\left(\frac{16}{\alpha}\right)^n t\right) \end{aligned}\tag{29}$$

for all $x \in \mathcal{X}$ and all non-negative integers n. Using the inequality (29), we obtain

$$\begin{aligned}\mu_{\left(\frac{f(2^n x)}{16^n}-f(x)\right)}\left(\frac{7t}{32}\sum_{j=0}^{n-1}\left(\frac{\alpha}{16}\right)^j\right) &= \mu_{\left(\sum_{j=0}^{n-1}\left(\frac{f(2^{j+1}x)}{16^{j+1}}-\frac{f(2^j x)}{16^j}\right)\right)}\left(\frac{7}{32}\sum_{j=0}^{n-1}\left(\frac{\alpha}{16}\right)^j t\right)\\ &\geq (\tau_M)_{j=0}^{n-1}\left(\mu_{\left(\frac{f(2^{j+1}x)}{16^{j+1}}-\frac{f(2^j x)}{16^j}\right)}\left(\frac{7}{32}\left(\frac{\alpha}{16}\right)^j t\right)\right)\\ &= \mu_{\left(\frac{1}{16}f(2x)-f(x)\right)}\left(\frac{7}{32}t\right)\\ &\geq \Lambda_{\psi(0,x)}(t)\end{aligned}$$

for all $x \in \mathcal{X}$ and all non-negative integers n. In other words,

$$\mu_{\left(\frac{f(2^n x)}{16^n}-f(x)\right)}(t) \geq \Lambda_{\psi(0,x)}\left(\frac{t}{\frac{7}{32}\sum_{j=0}^{n-1}\left(\frac{\alpha}{16}\right)^j}\right) \tag{30}$$

Intechanging x into $2^l x$ in (30), we have

$$\mu_{\left(\frac{f(2^{n+l}x)}{16^{n+l}}-\frac{f(2^l x)}{16^l}\right)}(t) \geq \Lambda_{\psi(0,x)}\left(\frac{t}{\left(\frac{7}{32}\sum_{j=l}^{l+n}\left(\frac{\alpha}{16}\right)^j\right)}\right) \tag{31}$$

for all $x \in \mathcal{X}$ and all integers $n \geq l \geq 0$. Due to the convegence of $\sum_{j=l}^{\infty}\left(\frac{\alpha}{16}\right)^j$, we see that $\Lambda_{\psi(0,x)}\left(\frac{t}{\left(\frac{7}{32}\sum_{j=l}^{l+n}\left(\frac{\alpha}{16}\right)^j\right)}\right)$ goes to 1 as l and n tend to infinity, and so $\left\{\frac{f(2^n x)}{16^n}\right\}$ is a Cauchy sequence in $(\mathcal{Y}, \mu, \tau_M)$. The completeness of $(\mathcal{Y}, \mu, \tau_M)$ as a *RN*-space implies that the mentioned sequence is converges to some point $Q(x) \in \mathcal{Y}$. It follows from (30) that for each $\epsilon > 0$

$$\begin{aligned}\mu_{(Q(x)-f(x))}(t+\epsilon) &\geq \tau_M\left(\mu_{\left(Q(x)-\frac{f(2^n x)}{16^n}\right)}(\epsilon), \mu_{\left(\frac{f(2^n x)}{16^n}-f(x)\right)}(t)\right)\\ &\geq \tau_M\left(\mu_{\left(Q(x)-\frac{f(2^n x)}{16^n}\right)}(\epsilon), \Lambda_{\psi(0,x)}\left(\frac{t}{\left(\frac{7}{32}\sum_{j=0}^{n-1}\left(\frac{\alpha}{16}\right)^j\right)}\right)\right)\end{aligned}$$

for all $x \in \mathcal{X}$. Taking n to infinity in the above inequality, we deduce that

$$\mu_{(Q(x)-f(x))}(t+\epsilon) \geq \Lambda_{\psi(0,x)}\left(\frac{2(16-\alpha)}{7}t\right) \tag{32}$$

Taking $\epsilon \to 0$ in (32), we get (27). Also, the inequality (26) implies that

$$\mu_{\frac{1}{16^n}\mathcal{D}f(2^n x,2^n y)}(t) \geq \Lambda_{\psi(2^n x,2^n y)}(16^n t) \tag{33}$$

for all $x, y \in \mathcal{X}$ and all $t > 0$. Letting n to infinity in (33), by (25), and the part (ii) of Lemma 2.1, we observe that the mapping Q is quartic. If $\mathcal{P} : \mathcal{X} \longrightarrow \mathcal{Y}$ is another quartic mapping satisfies (27), then

$$\begin{aligned}
\mu_{\left(\frac{\mathcal{P}(2^n x)}{16^n}-\frac{Q(2^n x)}{16^n}\right)}(t) &\geq \min\left\{\mu_{\left(\frac{f(2^n x)}{16^n}-\frac{Q(2^n x)}{16^n}\right)}\left(\frac{t}{2}\right), \mu_{\left(\frac{\mathcal{P}(2^n x)}{16^n}-\frac{f(2^n x)}{16^n}\right)}\left(\frac{t}{2}\right)\right\} \\
&\geq \Lambda_{(\psi(0,2^n x))}\left(16^n \frac{(16-\alpha)t}{7}\right) \\
&\geq \Lambda_{(\psi(0,x))}\left(\left(\frac{16}{\alpha}\right)^n \frac{(16-\alpha)t}{7}\right)
\end{aligned}$$

for all $x \in \mathcal{X}$. Therefore

$$\begin{aligned}
\mu_{\mathcal{P}(x)-Q(x)}(t) &= \lim_{n\to\infty} \mu_{\left(\frac{\mathcal{P}(2^n x)}{16^n}-\frac{Q(2^n x)}{16^n}\right)}(t) \\
&\geq \lim_{n\to\infty} \Lambda_{(\psi(0,x))}\left(\left(\frac{16}{\alpha}\right)^n \frac{(16-\alpha)t}{7}\right) = 1
\end{aligned}$$

The above relations show that $Q(x) = \mathcal{P}(x)$ for all $x \in \mathcal{X}$. This completes the proof. □

Corollary 3.6. *Let $\mathcal{X}$ be a linear space, $(\mathcal{Z}, \Lambda, \tau_M)$ be an RN-space and let $(\mathcal{Y}, \mu, \tau_M)$ be a complete RN-space. Let r, s be real numbers such that $r, s \in [0, 4)$ and $z_0 \in \mathcal{Z}$. If $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is an even mapping such that*

$$\mu_{\mathcal{D}f(x,y)}(t) \geq \Lambda_{(\|x\|^r+\|y\|^s)z_0}(t) \tag{34}$$

for all $x, y \in \mathcal{X}$ and all $t > 0$, then there exists a unique quartic mapping $Q : \mathcal{X} \longrightarrow \mathcal{Y}$ satisfying

$$\mu_{f(x)-Q(x)}(t) \geq \Lambda_{\|x\|^s z_0}\left(\frac{2(16-2^s)}{7}t\right)$$

for all $x \in \mathcal{X}$ and all $t > 0$.

Proof. Putting $x = y = 0$ in (34), we observe that $f(0) = 0$. Now, by defining $\psi(x, y) := (\|x\|^r + \|y\|^s)z_0$ and applying Theorem 3.5 when $\alpha = 2^s$, we get the desired result. □

Corollary 3.7. *Let $\mathcal{X}$ be a linear space, $(\mathcal{Z}, \Lambda, \tau_M)$ be an RN-space and let $(\mathcal{Y}, \mu, \tau_M)$ be a complete RN-space. Let $z_0 \in \mathcal{Z}$ and $\delta > 0$. If $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is an even mapping with $f(0) = 0$ such that*

$$\mu_{\mathcal{D}f(x,y)}(t) \geq \Lambda_{\delta z_0}(t)$$

for all $x, y \in \mathcal{X}$ and all $t > 0$, then there exists a unique quartic mapping $Q : \mathcal{X} \longrightarrow \mathcal{Y}$ satisfying

$$\mu_{f(x)-Q(x)}(t) \geq \Lambda_{\delta z_0}\left(\frac{1}{4}t\right)$$

for all $x \in \mathcal{X}$ and all $t > 0$.

Proof. The result follows from Theorem 3.5 if $\alpha = 2$ and $\psi(x, y) := \delta z_0$. □

In the upcoming result, we prove the superstability of the functional equation (5) under some conditions. Recall that a functional equation is called *superstable* if any approximate solution to the functional equation is a its exact solution.

Corollary 3.8. *Let $\mathcal{X}$ be a linear space, $(\mathcal{Z}, \Lambda, \tau_M)$ be an RN-space and $(\mathcal{Y}, \mu, \tau_M)$ be a complete RN-space. Let r, s be non-negative real numbers such that $r + s \neq 4$ and $z_0 \in \mathcal{Z}$. If $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is an even mapping such that*

$$\mu_{\mathcal{D}f(x,y)}(t) \geq \Lambda_{\|x\|^r\|y\|^s z_0}(t)$$

for all $x, y \in \mathcal{X}$ and all $t > 0$, then f is a quartic mapping.

Proof. Putting $\psi(x, y) := \|x\|^r\|y\|^s z_0$ in Theorem 3.5, we have $f(x) = \frac{f(2^n x)}{16^n}$ for all $n \in \mathbb{N}$. Now, by applying the same theorem, we obtain the desired result. □

We have the following result which is analogous to Theorem 3.5 when f is an odd mapping. The proof is similar but we bring it.

Theorem 3.9. *Let $\mathcal{X}$ be a linear space, $(\mathcal{Z}, \Lambda, \tau_M)$ be an RN-space and $(\mathcal{Y}, \mu, \tau_M)$ be a complete RN-space. Suppose that $\psi : \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{Z}$ is a mapping such that for some $0 < \alpha < 2$,*

$$\Lambda_{\psi(0,2x)}(t) \geq \Lambda_{\alpha\psi(0,x)}(t) \qquad (x \in \mathcal{X}, t > 0) \tag{35}$$

and

$$\lim_{n\to\infty} \Lambda_{\psi(2^n x, 2^n y)}(2^n t) = 1 \qquad (x, y \in \mathcal{X}, t > 0). \tag{36}$$

If $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is an odd mapping and

$$\mu_{\mathcal{D}f(x,y)}(t) \geq \Lambda_{\psi(x,y)}(t) \tag{37}$$

for all $x, y \in \mathcal{X}$ and all $t > 0$, then there exists a unique additive mapping $A : \mathcal{X} \longrightarrow \mathcal{Y}$ such that

$$\mu_{f(x)-A(x)}(t) \geq \Lambda_{\psi(0,x)}\left(\frac{12(2-\alpha)}{7}t\right) \tag{38}$$

for all $x \in \mathcal{X}$ and all $t > 0$.

Proof. Similar to the proof of Theorem 3.5, by replacing (x, y) with $(0, x)$ in (37) and using the oddness of f, we get

$$\mu_{\left(\frac{1}{2}f(2x)-f(x)\right)}(t) \geq \Lambda_{\psi(0,x)}\left(\frac{24}{7}t\right) \tag{39}$$

for all $x \in \mathcal{X}$. Replacing x by $2^n x$ in (39) and using (35), we obtain

$$\begin{aligned}\mu_{\left(\frac{f(2^{n+1}x)}{2^{n+1}}-\frac{f(2^n x)}{2^n}\right)}(t) &\geq \Lambda_{\psi(0,2^n x)}\left(\frac{24}{7}2^n t\right)\\ &\geq \Lambda_{\alpha^n\psi(0,x)}\left(\frac{24}{7}2^n t\right)\\ &\geq \Lambda_{\psi(0,x)}\left(\frac{24}{7}\left(\frac{2}{\alpha}\right)^n t\right)\end{aligned} \tag{40}$$

for all $x \in \mathcal{X}$ and all non-negative integers n. Applying the inequality (40), we have

$$\begin{aligned}\mu_{\left(\frac{f(2^n x)}{2^n}-f(x)\right)}\left(\frac{7t}{24}\sum_{j=0}^{n-1}\left(\frac{\alpha}{2}\right)^j\right) &= \mu_{\left(\sum_{j=0}^{n-1}\left(\frac{f(2^{j+1}x)}{2^{j+1}}-\frac{f(2^j x)}{2^j}\right)\right)}\left(\frac{7}{24}\sum_{j=0}^{n-1}\left(\frac{\alpha}{2}\right)^j t\right)\\ &\geq (\tau_M)_{j=0}^{n-1}\left(\mu_{\left(\frac{f(2^{j+1}x)}{2^{j+1}}-\frac{f(2^j x)}{2^j}\right)}\left(\frac{7}{24}\left(\frac{\alpha}{2}\right)^j t\right)\right)\\ &= \mu_{\left(\frac{1}{2}f(2x)-f(x)\right)}\left(\frac{7}{24}t\right)\\ &\geq \Lambda_{\psi(0,x)}(t)\end{aligned}$$

for all $x \in \mathcal{X}$ and all non-negative integers n. Hence

$$\mu_{\left(\frac{f(2^n x)}{2^n}-f(x)\right)}(t) \geq \Lambda_{\psi(0,x)}\left(\frac{t}{\frac{7}{24}\sum_{j=0}^{n-1}\left(\frac{2}{\alpha}\right)^j}\right) \tag{41}$$

for all $x \in \mathcal{X}$ and all non-negative integers n. Substituting x by $2^m x$ in (41), we obtain

$$\mu_{\left(\frac{f(2^{n+m}x)}{2^{n+m}}-\frac{f(2^m x)}{2^m}\right)}(t) \geq \Lambda_{\psi(0,x)}\left(\frac{t}{\left(\frac{7}{24}\sum_{j=m}^{m+n}\left(\frac{\alpha}{2}\right)^j\right)}\right) \tag{42}$$

for all $x \in \mathcal{X}$ and all integers $n \geq m \geq 0$. Since the above series is convergent, the sequence $\left\{\frac{f(2^n x)}{2^n}\right\}$ is Cauchy in $(\mathcal{Y}, \mu, \tau_M)$. Now, the completeness of $(\mathcal{Y}, \mu, \tau_M)$ as a *RN*-space implies that the mentioned sequence is converges to some point $A(x) \in \mathcal{Y}$. It follows from (41) that

$$\begin{aligned}\mu_{(A(x)-f(x))}(t+\epsilon) &\geq \tau_M\left(\mu_{\left(A(x)-\frac{f(2^n x)}{2^n}\right)}(\epsilon), \mu_{\left(\frac{f(2^n x)}{2^n}-f(x)\right)}(t)\right)\\ &\geq \tau_M\left(\mu_{\left(A(x)-\frac{f(2^n x)}{2^n}\right)}(\epsilon), \Lambda_{\psi(0,x)}\left(\frac{t}{\left(\frac{7}{24}\sum_{j=0}^{n-1}\left(\frac{\alpha}{2}\right)^j\right)}\right)\right)\end{aligned}$$

for all $x \in \mathcal{X}$ in which $\epsilon > 0$. Taking n to infinity in the above inequality, we have

$$\mu_{(A(x)-f(x))}(t+\epsilon) \geq \Lambda_{\psi(0,x)}\left(\frac{12(2-\alpha)}{7}t\right) \tag{43}$$

for all $x \in \mathcal{X}$. Taking $\epsilon \to 0$ in (43), we see that the inequality (38) holds. Also, the inequality (37) implies that

$$\mu_{\frac{1}{2^n}\mathcal{D}f(2^nx,2^ny)}(t) \geq \Lambda_{\psi(2^nx,2^ny)}(2^nt) \tag{44}$$

for all $x, y \in \mathcal{X}$ and all $t > 0$. Letting n to infinity in (44), by (36) and the part (i) of Lemma 2.1, we see that the mapping A is additive. Now, similar to the proof of Theorem 3.5, one can complete the rest of the proof. □

Corollary 3.10. *Let $\mathcal{X}$ be a linear space, $(\mathcal{Z}, \Lambda, \tau_M)$ be an RN-space and let $(\mathcal{Y}, \mu, \tau_M)$ be a complete RN-space. Let r, s be real numbers such that $r, s \in [0, 1)$ and $z_0 \in \mathcal{Z}$. If $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is an odd mapping such that*

$$\mu_{\mathcal{D}f(x,y)}(t) \geq \Lambda_{(\|x\|^r+\|y\|^s)z_0}(t)$$

for all $x, y \in \mathcal{X}$ and all $t > 0$, then there exists a unique additive mapping $A : \mathcal{X} \longrightarrow \mathcal{Y}$ satisfying

$$\mu_{f(x)-Q(x)}(t) \geq \Lambda_{\|x\|^s z_0}\left(\frac{12(2-2^s)}{7}t\right)$$

for all $x \in \mathcal{X}$ and all $t > 0$.

Proof. Defining $\psi(x, y) := (\|x\|^r + \|y\|^s)z_0$ and using Theorem 3.9, we get the desired result. □

The following result analogous to Corollary 3.8 for additive functional equations. Since the proof is similar, it is omitted.

Corollary 3.11. *Let $\mathcal{X}$ be a linear space, $(\mathcal{Z}, \Lambda, \tau_M)$ be an RN-space and $(\mathcal{Y}, \mu, \tau_M)$ be a complete RN-space. Let r, s be non-negative real numbers such that $r + s \neq 1$ and $z_0 \in \mathcal{Z}$. If $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is an odd mapping such that*

$$\mu_{\mathcal{D}f(x,y)}(t) \geq \Lambda_{\|x\|^r\|y\|^s z_0}(t)$$

for all $x, y \in \mathcal{X}$ and all $t > 0$, then f is a additive mapping.

Theorem 3.12. *Let $\mathcal{X}$ be a linear space, $(\mathcal{Z}, \Lambda, \tau_M)$ be an RN-space and $(\mathcal{Y}, \mu, \tau_M)$ be a complete RN-space. Suppose that $\psi : \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{Z}$ is a mapping such that $\psi(x, y) = \psi(-x, -y)$ for all $x, y \in \mathcal{X}$ and for some $0 < \alpha < 2$,*

$$\Lambda_{\psi(0,2x)}(t) \geq \Lambda_{\alpha\psi(0,x)}(t) \qquad (x \in \mathcal{X}, t > 0) \tag{45}$$

and

$$\lim_{n\to\infty} \Lambda_{\psi(2^nx,2^ny)}(2^nt) = 1 \qquad (x, y \in \mathcal{X}, t > 0). \tag{46}$$

If $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is a mapping with $f(0) = 0$ and

$$\mu_{\mathcal{D}f(x,y)}(t) \geq \Lambda_{\psi(x,y)}(t) \tag{47}$$

for all $x, y \in \mathcal{X}$ and all $t > 0$, then there exists a unique additive mapping $A : \mathcal{X} \longrightarrow \mathcal{Y}$ and a unique quartic mapping $Q : \mathcal{X} \longrightarrow \mathcal{Y}$ such that

$$\mu_{f(x)-A(x)-Q(x)}(t) \geq \min\left\{\Lambda_{\psi(0,x)}\left(\frac{16-\alpha}{7}t\right), \Lambda_{\psi(0,x)}\left(\frac{6(2-\alpha)}{7}t\right)\right\} \tag{48}$$

for all $x \in \mathcal{X}$ and all $t > 0$.

Proof. We decompose f into the even part and odd part by setting

$$f_{\mathrm{e}}(x) = \frac{f(x)+f(-x)}{2}, \qquad f_{\mathrm{o}}(x) = \frac{f(x)-f(-x)}{2}.$$

for all $x \in \mathcal{X}$. Obviously, $f(x) = f_{\mathrm{e}}(x) + f_{\mathrm{o}}(x)$ for all $x \in \mathcal{X}$. Then

$$\begin{aligned}
\mu_{\mathcal{D}f_{\mathrm{e}}(x,y)}(t) &= \mu_{\left(\frac{1}{2}\mathcal{D}f(x,y)+\frac{1}{2}\mathcal{D}f(-x,-y)\right)}(t) \geq \tau_M\left(\mu_{\frac{1}{2}\mathcal{D}f(x,y)}\left(\frac{t}{2}\right), \mu_{\frac{1}{2}\mathcal{D}f(-x,-y)}\left(\frac{t}{2}\right)\right)\\
&= \tau_M\left(\mu_{\mathcal{D}f(x,y)}(t), \mu_{\mathcal{D}f(-x,-y)}(t)\right)\\
&\geq \tau_M\left(\Lambda_{\psi(x,y)}(t), \Lambda_{\psi(-x,-y)}(t)\right)\\
&= \Lambda_{\psi(x,y)}(t).
\end{aligned}$$

for all $x, y \in \mathcal{X}$ and all $t > 0$. Similarly, one can show that $\mu_{\mathcal{D}f_{\mathrm{e}}(x,y)}(t) \geq \Lambda_{\psi(x,y)}(t)$. By Theorems 3.5 and 3.9, there exists a unique quratic mapping $Q : \mathcal{X} \longrightarrow \mathcal{Y}$ and a unique additive mapping $A : \mathcal{X} \longrightarrow \mathcal{Y}$ such that

$$\mu_{f_{\mathrm{e}}(x)-Q(x)}(t) \geq \Lambda_{\psi(0,x)}\left(\frac{2(16-\alpha)}{7}t\right) \text{ and } \mu_{f_0(x)-A(x)}(t) \geq \Lambda_{\psi(0,x)}\left(\frac{12(2-\alpha)}{7}t\right) \tag{49}$$

for all $x \in \mathcal{X}$ and all $t > 0$. The relations in(49) implies that

$$\mu_{f(x)-A(x)-Q(x)}(t) \geq \tau_M\left(\mu_{f_{\mathrm{e}}(x)-Q(x)}\left(\frac{t}{2}\right), \mu_{f_0(x)-A(x)}\left(\frac{t}{2}\right)\right) \tag{50}$$

$$\geq \tau_M\left(\Lambda_{\psi(0,x)}\left(\frac{16-\alpha}{7}t\right), \Lambda_{\psi(0,x)}\left(\frac{6(2-\alpha)}{7}t\right)\right) \tag{51}$$

$$= \min\left\{\Lambda_{\psi(0,x)}\left(\frac{16-\alpha}{7}t\right), \Lambda_{\psi(0,x)}\left(\frac{6(2-\alpha)}{7}t\right)\right\} \tag{52}$$

for all $x \in \mathcal{X}$ and all $t > 0$. This finishes the proof. □

Corollary 3.13. *Let $\mathcal{X}$ be a linear space, $(\mathcal{Z}, \Lambda, \tau_M)$ be an RN-space and let $(\mathcal{Y}, \mu, \tau_M)$ be a complete RN-space. Let r, s be real numbers such that $r, s \in [0, 1)$ and $z_0 \in \mathcal{Z}$. If $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is an mapping with $f(0) = 0$ such that*

$$\mu_{\mathcal{D}f(x,y)}(t) \geq \Lambda_{(\|x\|^r+\|y\|^s)z_0}(t)$$

for all $x, y \in \mathcal{X}$ and all $t > 0$, then there exists a unique additive mapping $A : \mathcal{X} \longrightarrow \mathcal{Y}$ and a unique quartic mapping $Q : \mathcal{X} \longrightarrow \mathcal{Y}$ satisfying

$$\mu_{f(x)-A(x)-Q(x)}(t) \geq \min\left\{\Lambda_{\|x\|^s z_0}\left(\frac{16-2^s}{7}t\right), \Lambda_{\|x\|^s z_0}\left(\frac{6(2-2^s)}{7}t\right)\right\}$$

for all $x \in \mathcal{X}$ and all $t > 0$.

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References

[1] T. Aoki, On the stability of the linear transformation in Banach spaces, J. Math. Soc. Japan. 2 (1950) 64-66.
[2] E. Baktash, Y. J. Cho, M. Jalili, R. Saadati and S. M. Vaezpour, On the stability of cubic Mappings and quadratic mappings in random normed spaces, J. Inequal. Appl. 2008, Article ID 902187, 11 pages, doi:10.1155/2008/902187.
[3] A. Bodaghi and I. A. Alias, Approximate ternary quadratic derivations on ternary Banach algebras and C^*-ternary rings, Adv. Differene Equa., 2012, Article No. 11, (2012).
[4] A. Bodaghi, I. A. Alias and M. H. Ghahramani, Ulam stability of a quartic functional equation, Abst. Appl. Anal. 2012, Art. ID 232630 (2012).
[5] A. Bodaghi, I. A. Alias and M. H. Ghahramani, Approximately cubic functional equations and cubic multipliers, J. Inequal. Appl. 2011, Article No. 53, (2011).
[6] L. Cădariu and V. Radu, Fixed points and the stability of quadratic functional equations, An. Univ. Timişoara, Ser. Mat. Inform. 41 (2003), 25–48.
[7] J. K. Chung and P. K. Sahoo, On the general solution of a quartic functional equation, Bull. Korean Math. Soc. 40 (2003), 565–576.
[8] S. Czerwik, On the stability of the quadratic mapping in normed spaces, Abh. Math. Sem. Univ. Hamburg. 62 (1992) 59–64.
[9] M. Eshaghi Gordji, Stability of a functional equation deriving from quartic and additive functions, Bull. Korean Math. Soc. 47, No. 3 (2010), 491–502.
[10] M. Eshaghi Gordji and M. Bavand Savadkouhi, Stability of a mixed type additive, quadratic and cubic functional equation in random normed space, Filomat. 25, No. 3 (2011), 43–54.
[11] D. H. Hyers, On the stability of the linear functional equation, Proc. Nat. Acad. Sci. USA. 27 (1941), 222–224.
[12] D. H. Hyers, G. Isac and Th. M. Rassias, Stability of Functional Equations in Several Variables, Birkhäuser, Basel, 1998.
[13] S. Lee, S. Im and I. Hwang, Quartic functional equations, J. Math. Anal. Appl. 307 (2005) 387–394.
[14] H. Lee, S. W. Kim, B. J. Son, D. H. Lee and S. Y. Kang, Additive-quartic functional equation in non-archimedean orthogonality spaces, Korean J. Math. 20, (2012) No. 1, 33-46.
[15] C. Park and A. Bodaghi, On the stability of $*$-derivations on Banach $*$-algebras, Adv. Difference Equ. 2012, Article No. 138 (2012).
[16] J. M. Rassias, Solution of the Ulam stability problem for quartic mappings, Glasnik Matematicki Series III. 34 (2) (1999) 243–252.
[17] Th. M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. 72 (2) (1978) 297-300.
[18] B. Schweizer and A. Sklar, Probabilistic metric spaces, Elsevier, North Holand, New York, 1983.
[19] A. N. Šerstnev, On the motion of a random normed space, Dokl. Akad. Nauk SSSR 149 (1963) 280-283.
[20] S. M. Ulam, Problems in Modern Mathematics, Chapter VI, Science Ed., Wiley, New York, 1940.