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On Weak and Strong Convergence of an Explicit Iteration Process for a Total Asymptotically Quasi-I-Nonexpansive Mapping in Banach Space

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Abstract. In this paper, we introduce a new class of Lipschitzian maps and prove some weak and strong convergence results for explicit iterative process using a more satisfactory definition of self mappings. Our results approximate common fixed point of a total asymptotically quasi-I-nonexpansive mapping T and a total asymptotically quasi-nonexpansive mapping I, defined on a nonempty closed convex subset of a Banach space.

1. Introduction

Let E be a real normed linear space, K a nonempty subset of E and $T : K \to K$ a mapping. Denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in K : Tx = x\}$ and we denote by $D(T)$ the domain of a mapping T. Throughout this paper, we always assume that E is a real Banach space and $F(T) \neq \emptyset$. Now, we recall the well-known concept and results. A mapping $T : K \to K$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that

$$\left\|T^n x - T^n y\right\| \leq k_n \left\|x - y\right\|$$

for all $x, y \in K$ and $n \geq 1$. A mapping $T : K \to K$ is said asymptotically quasi-nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\left\|T^n x - p\right\| \leq k_n \left\|x - p\right\|$$

for all $x \in K$, $p \in F(T)$ and $n \geq 1$. Let $T : K \to K$, $I : K \to K$ be two mappings of nonempty subset K of a real normed linear space E. Then T is said asymptotically I-nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$

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with $\lim_{n\to\infty} k_n = 1$ such that

$$\left\|T^n x - T^n y\right\| \le k_n \left\|I^n x - I^n y\right\|$$

for all $x, y \in K$ and $n \ge 1$. Let $T : K \to K$, $I : K \to K$ be two mappings of nonempty subset K of a real normed linear space E. Then T is said asymptotically quasi I-nonexpansive (see [11]) if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\left\|T^n x - p\right\| \le k_n \left\|I^n x - p\right\|$$

for all $x \in K$, $p \in F(T) \cap F(I)$ and $n \ge 1$.

Remark 1.1. *If $F(T) \cap F(I) \neq \emptyset$ then an asymptotically I-nonexpansive mapping is asymptotically quasi I-nonexpansive. But, there exists a nonlinear continuous asymptotically quasi I-nonexpansive mappings which is not asymptotically I-nonexpansive.*

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [3]. They proved that if K is a nonempty closed convex bounded subset of a real uniformly convex Banach space and $T : K \to K$ is an asymptotically nonexpansive mappings, then T has a fixed point. Liu [5] studied iterative sequences for asymptotically quasi-nonexpansive mappings. The weak and strong convergence of implicit iteration process to a common fixed point of a finite family of I-asymptotically nonexpansive mappings were studied by Temir [10]. Temir and Gul [11] defined I-asymptotically quasi-nonexpansive mapping in Hilbert space and they proved convergence theorem for I-asymptotically quasi-nonexpansive mapping defined in Hilbert space.

A mapping $T : K \to K$ is called a total asymptotically nonexpansive mapping (see [1]) if there exist nonnegative real sequences $\{\mu_n\}$, $\{l_n\}$ with $\mu_n, l_n \to 0$ as $n \to \infty$ and strictly increasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that for all $x, y \in K$,

$$\left\|T^n x - T^n y\right\| \le \left\|x - y\right\| + \mu_n \phi(\left\|x - y\right\|) + l_n, \quad n \ge 1. \tag{1}$$

Let $T : K \to K$, $I : K \to K$ be two mappings of a nonempty subset K of a real normed space E. T is said to be total asymptotically I-nonexpansive mapping (see [6]) if there exist nonnegative real sequences $\{\mu_n\}$, $\{l_n\}$ with $\mu_n, l_n \to 0$ as $n \to \infty$ and strictly increasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that for all $x, y \in K$,

$$\left\|T^n x - T^n y\right\| \le \left\|I^n x - I^n y\right\| + \mu_n \phi(\left\|I^n x - I^n y\right\|) + l_n, \quad n \ge 1. \tag{2}$$

Note that if $I = Id$ (Id is the indentity mapping), then (2) reduces to (1). One can see that if $\phi(\xi) = \xi$, then (1) reduces to $\left\|T^n x - T^n y\right\| \le (1 + \mu_n)\left\|x - y\right\| + l_n$, $n \ge 1$. In addition, if $l_n = 0$ for all $n \ge 1$, then total asymptotically nonexpansive mappings coincide with asymptotically nonexpansive mappings.

Let K be a nonempty closed subset of a real Banach space E. Then a mapping $T : K \to K$ is called a uniformly L-Lipschitzian mapping if there exists a constant $L > 0$ such that

$$\left\|T^n x - T^n y\right\| \leq L \left\|x - y\right\| \tag{3}$$

for all $x, y \in K$ and $n \geq 1$.

The class of a total asymptotically nonexpansive mappings was introduced by Alber et al. [1] to unify various definitions of asymptotically nonexpansive mappings. They constructed a scheme which convergences strongly to a fixed point of a total asymptotically nonexpansive mappings. Mukhamedov and Saburov [6] studied strong convergence of an explicit iteration process for a totally asymptotically I-nonexpansive mapping in Banach spaces.

Definition 1.2. *[2] Let K be a nonempty closed subset of a real normed linear space E. A mapping $T : K \to K$ is said to be total asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exist nonnegative real sequences $\{\mu_n\}$, $\{l_n\}$ with $\mu_n, l_n \to 0$ as $n \to \infty$ and strictly increasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that for all $x \in K, p \in F(T)$,*

$$\left\|T^n x - p\right\| \leq \left\|x - p\right\| + \mu_n \phi(\left\|x - p\right\|) + l_n, \quad n \geq 1. \tag{4}$$

Definition 1.3. *Let $T : K \to K$, $I : K \to K$ be two mappings of a nonempty closed subset K of a real normed space E. T is said to be total asymptotically quasi-I-nonexpansive if $F(T) \neq \emptyset$ and there exist nonnegative real sequences $\{\mu_n\}$, $\{l_n\}$ with $\mu_n, l_n \to 0$ as $n \to \infty$ and strictly increasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that for all $x \in K, p \in F(T)$,*

$$\left\|T^n x - p\right\| \leq \left\|I^n x - p\right\| + \mu_n \phi(\left\|I^n x - p\right\|) + l_n, \quad n \geq 1. \tag{5}$$

Note that if $I = Id$ (Id is the indentity mapping), then (5) reduces to (4). One can see that if $\phi(\xi) = \xi$, then (4) reduces to $\left\|T^n x - p\right\| \leq (1 + \mu_n) \left\|x - p\right\| + l_n$, $n \geq 1$. In addition, if $l_n = 0$ for all $n \geq 1$, then total asymptotically quasi-nonexpansive mappings coincide with asymptotically quasi-nonexpansive mappings.

Definition 1.4. *Let K be a nonempty closed subset of a real normed linear space E. A mapping $T : K \to K$ is said to be total uniformly L-Lipschitzian if there exist $L > 0$, noninegative real sequences $\{\mu_n\}$, $\{l_n\}$ with $\mu_n, l_n \to 0$ as $n \to \infty$ and strictly increasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that for all $x, y \in K$,*

$$\left\|T^n x - T^n y\right\| \leq L \left[\left\|x - y\right\| + \mu_n \phi(\left\|x - y\right\|) + l_n\right], \quad n \geq 1. \tag{6}$$

One can see that if $\mu_n = 0$ and $l_n = 0$ for all $n \geq 1$, then (6) reduces to (3).

Example 1.5. *Let us consider that $\mathbb{R}$, the set of real numbers, endowed with the usual topology. Let $K = [0, 1] \subset \mathbb{R}$. The mapping $T : K \to K$ is defined by*

$$Tx = \begin{cases} \frac{1}{2}, & x \in \left[0, \frac{1}{2}\right] \\ \frac{\sqrt{1-x^2}}{\sqrt{3}}, & x \in \left[\frac{1}{2}, 1\right] \end{cases}$$

for all $x \in K$*. Let* ϕ *be a strictly increasing continuous function such that* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *with* $\phi(0) = 0$*. Let* $\{\mu_n\}_{n\geq 1}$ *and* $\{l_n\}_{n\geq 1}$ *in* $\mathbb{R}$ *be two sequences defined by* $\mu_n = \frac{1}{n}$ *and* $l_n = \frac{1}{n+1}$*, for all* $n \geq 1$ *(*$\lim_{n\to\infty} \mu_n = \lim_{n\to\infty} \frac{1}{n} = 0$*,* $\lim_{n\to\infty} l_n = \frac{1}{n+1} = 0$*). Note that* $T^n x = \frac{1}{2}$ *for all* $x \in K$ *and* $n \geq 2$ *and* $F(T) = \left\{\frac{1}{2}\right\}$*. Clearly, T is both uniformly continuous and total asymptotically nonexpansive mapping on K. Also, for all* $x, y \in K$ *and* $L > 0$*, we obtain*

$$\left|T^n x - T^n y\right| \leq L\left|x - y\right|. \tag{7}$$

for all $n \geq 1$*.*

In fact, if $x \in \left[0, \frac{1}{2}\right]$*, then* $\left|x - \frac{1}{2}\right| = |x - Tx|$*. Similarly, if* $x \in \left[\frac{1}{2}, 1\right]$*, then* $\left|x - \frac{1}{2}\right| = x - \frac{1}{2} \leq x - \frac{\sqrt{1-x^2}}{\sqrt{3}} = |x - Tx|$*. Hence, we get* $d(x, F(T)) = \left|x - \frac{1}{2}\right| \leq |x - Tx|$*. But, T is not Lipschitzian. Indeed, suppose not, i.e., there exists* $L > 0$ *such that*

$$\left|Tx - Ty\right| \leq L\left|x - y\right|$$

for all $x, y \in K$*. If we take* $x = 1 - \frac{1}{2(1+L)^2} > \frac{1}{2}$ *and* $y = 1$*, then* $\frac{\sqrt{1-x^2}}{\sqrt{3}} \leq L|1 - x| \Longleftrightarrow \frac{1}{3L^2} \leq \frac{1-x}{1+x} = \frac{1}{4L^2+8L+3}$*. This is a contradiction.*

Also, since ϕ *is strictly increasing continuous function* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *with* $\phi(0) = 0$ *and* $\mu_n = \frac{1}{n}$*,* $l_n = \frac{1}{n+1}$*, for all* $n \geq 1$ *and* $L > 0$*, it follows that we have*

$$L\left(\frac{1}{n}\phi\left(\left|x - y\right|\right) + \frac{1}{n+1}\right) \geq 0 \tag{8}$$

for all $x, y \in K$*. Due to* (7) *and* (8)*, there exists* $L > 0$ *such that for all* $x, y \in K$*,*

$$\left|T^n x - T^n y\right| \leq L\left[\left|x - y\right| + \frac{1}{n}\phi\left(\left|x - y\right|\right) + \frac{1}{n+1}\right], \quad n \geq 1.$$

Then, T is a total uniformly L-Lipschitzian mapping on K.

Mukhamedov and Saburov [6] studied strong convergence of an explicit iteration process for a totally asymptotically I-nonexpansive mapping in Banach spaces. This iteration scheme is defined as follows.

Let K be a nonemty closed convex subset of a real Banach space E. Consider $T : K \to K$ is a total asymptotically quasi I-nonexpansive mapping, where $I : K \to K$ is a total asymptotically quasi-nonexpansive mapping. Then for two given sequences $\{\alpha_n\}$, $\{\beta_n\}$ in $[0, 1]$ we shall consider the following iteration scheme:

$$\begin{cases} x_0 \in K, \\ x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T^n y_n, \ n \geq 0, \\ y_n = (1 - \beta_n) x_n + \beta_n I^n x_n. \end{cases} \tag{9}$$

Inspired and motivated by this facts, we study the convergence theorems of the explicit iterative scheme involving a total asymptotically quasi-I-nonexpansive mapping in a nonempty closed convex subset of uniformly convex Banach spaces.

In this paper, we will prove the weak and strong convergences of the explicit iterative process (9) to a common fixed point of T and I.

2. Preliminaries

Recall that a Banach space E is said to satisfy *Opial condition* [7] if, for each sequence $\{x_n\}$ in E such that $\{x_n\}$ converges weakly to x implies that

$$\lim_{n\to\infty}\inf\|x_n - x\| < \lim_{n\to\infty}\inf\left\|x_n - y\right\| \tag{10}$$

for all $y \in E$ with $y \neq x$. It is weel known that (see [4]) inequality (10) is equivalent to

$$\lim_{n\to\infty}\sup\|x_n - x\| < \lim_{n\to\infty}\sup\left\|x_n - y\right\|.$$

Definition 2.1. *Let K be a closed subset of a real Banach space E and let $T : K \to K$ be a mapping. T is said to be semiclosed (demiclosed) at zero, if for each bounded sequence $\{x_n\}$ in K, the conditions x_n converges weakly to $x \in K$ and Tx_n converges strongly to 0 imply $Tx = 0$.*

Definition 2.2. *Let K be a closed subset of a real Banach space X and let $T : K \to K$ be a mapping. T is said to be semicompact, if for any bounded sequence $\{x_n\}$ in K such that $\|x_n - Tx_n\| \to 0,\ n \to \infty$, then there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to x^* \in K$ strongly.*

Lemma 2.3. *[8] Let X be a uniformly convex Banach space and let $b,\ c$ be two constant with $0 < b < c < 1$. Suppose that $\{t_n\}$ is a sequence in $[b, c]$ and $\{x_n\}$, $\{y_n\}$ are two sequence in X such that*

$$\lim_{n\to\infty}\left\|t_n x_n + (1-t_n)\, y_n\right\| = d, \quad \lim_{n\to\infty}\sup\|x_n\| \le d, \quad \lim\sup\left\|y_n\right\| \le d,$$

holds some $d \ge 0$. Then $\lim_{n\to\infty}\left\|x_n - y_n\right\| = 0$.

Lemma 2.4. *[9] Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be three sequences of nonnegative real numbers with $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} c_n < \infty$. If the following conditions is satisfied:*

$$a_{n+1} \le (1 + b_n)\, a_n + c_n, \quad n \ge 1,$$

then the limit $\lim_{n\to\infty} a_n$ exists.

3. Main Results

In this section, we prove the convergence theorems of an explicit iterative scheme (9) for a total asymptotically quasi-I-nonexpansive mapping in Banach spaces. In order to prove our main results, the following lemmas are needed.

Lemma 3.1. *Let E be real Banach space and K be a nonempty closed convex subset of E. Let $T : K \to K$ be a total asymptotically quasi-I-nonexpansive mapping with sequences $\{\mu_n\}$, $\{l_n\}$ and $I : K \to K$ be a total asymptotically quasi-nonexpansive mapping with sequences $\{\widetilde{\mu}_n\}$, $\{\widetilde{l}_n\}$ such that $F = F(T) \cap F(I) \neq \emptyset$. Suppose that there exist M_i, $N_i > 0$, $i = 1, 2$, such that $\phi(\xi) \le M_2\xi$ for all $\xi \ge M_1$ and $\varphi(\zeta) \le N_2\zeta$ for all $\zeta \ge N_1$. Then for any $x, y \in K$ we have*

$$\left\|I^n x - p\right\| \le (1 + N_2\widetilde{\mu}_n)\left\|x - p\right\| + \varphi(N_1)\widetilde{\mu}_n + \widetilde{l}_n \tag{11}$$

$$\begin{aligned}\left\|T^n x-p\right\| &\leq (1+M_2\mu_n)(1+N_2\tilde{\mu}_n)\left\|x-p\right\| \\ &+(1+M_2\mu_n)(\varphi(N_1)\tilde{\mu}_n+\tilde{l}_n)+\phi(M_1)\mu_n+l_n. \end{aligned} \tag{12}$$

Proof. Since $\phi,\varphi:\mathbb{R}^+\to\mathbb{R}^+$ are strictly increasing continuous functions, it follows that $\phi(\xi)\leq\phi(M_1)$, $\varphi(\zeta)\leq\varphi(N_1)$ whenever $\xi\leq M_1$, $\zeta\leq N_1$, respectively. By the hypothesis of lemma we get

$$\phi(\xi)\leq\phi(M_1)+M_2\xi,\quad \varphi(\zeta)\leq\varphi(N_1)+N_2\zeta, \tag{13}$$

for all $\xi,\zeta\geq 0$. Since $T:K\to K$, $I:K\to K$ are a total asymptotically quasi-I-nonexpansive mapping and a total asymptotically quasi-nonexpansive mapping, respectively, then from (13) we obtain

$$\left\|I^n x-p\right\|\leq(1+N_2\tilde{\mu}_n)\left\|x-p\right\|+\varphi(N_1)\tilde{\mu}_n+\tilde{l}_n.$$

Similarly, from (11) and (13) we obtain

$$\begin{aligned}\left\|T^n x-p\right\| &\leq (1+M_2\mu_n)\left\|I^n x-p\right\|+\phi(M_1)\mu_n+l_n \\ &\leq (1+M_2\mu_n)(1+N_2\tilde{\mu}_n)\left\|x-p\right\| \\ &+(1+M_2\mu_n)(\varphi(N_1)\tilde{\mu}_n+\tilde{l}_n)+\phi(M_1)\mu_n+l_n.\end{aligned}$$

This completes the proof. □

Lemma 3.2. *Let E be real Banach space and K be a nonempty closed convex subset of E. Let $T:K\to K$ be a total asymptotically quasi-I-nonexpansive mapping with sequences $\{\mu_n\}$, $\{l_n\}$ and $I:K\to K$ be a total asymptotically quasi-nonexpansive mapping with sequences $\{\tilde{\mu}_n\}$, $\{\tilde{l}_n\}$ such that $F=F(T)\cap F(I)\neq\emptyset$. Also, let $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$. Suppose that $\sum_{n=1}^{\infty}\mu_n<\infty$, $\sum_{n=1}^{\infty}l_n<\infty$, $\sum_{n=1}^{\infty}\tilde{\mu}_n<\infty$, $\sum_{n=1}^{\infty}\tilde{l}_n<\infty$ and there exist $M_i, N_i>0$, $i=1,2$, such that $\phi(\xi)\leq M_2\xi$ for all $\xi\geq M_1$ and $\varphi(\zeta)\leq N_2\zeta$ for all $\zeta\geq N_1$. Then sequence $\{x_n\}$ by (9) is bounded and for each $p\in F=F(T)\cap F(I)$ the limit $\lim_{n\to\infty}\left\|x_n-p\right\|$ exists.*

Proof. Since $F=F(T)\cap F(I)\neq\emptyset$, for any given $p\in F$, it follows from (9) and (12) that

$$\left\|y_n-p\right\|\leq\left(1+N_2\beta_n\tilde{\mu}_n\right)\left\|x_n-p\right\|+\beta_n\left(\varphi(N_1)\tilde{\mu}_n+\tilde{l}_n\right). \tag{14}$$

Using a similar method, from (9), (11) and (14), we have

$$\begin{aligned}\left\|x_{n+1}-p\right\| &\leq (1-\alpha_n)\left\|x_n-p\right\|+\alpha_n\left\|T^n y_n-p\right\| \\ &\leq (1-\alpha_n)\left\|x_n-p\right\|+\alpha_n(1+M_2\mu_n)(1+N_2\tilde{\mu}_n)\left\|y_n-p\right\| \\ &+\alpha_n(1+M_2\mu_n)(\varphi(N_1)\tilde{\mu}_n+\tilde{l}_n)+\alpha_n\left(\phi(M_1)\mu_n+l_n\right) \\ &\leq \left\{1+\alpha_n\left[(1+M_2\mu_n)\left(1+N_2\tilde{\mu}_n\right)\left(1+N_2\beta_n\tilde{\mu}_n\right)-1\right]\right\}\left\|x_n-p\right\| \\ &+\alpha_n\Big[(1+M_2\mu_n)\left(\varphi(N_1)\tilde{\mu}_n+\tilde{l}_n\right)\left(\beta_n\left(1+N_2\tilde{\mu}_n\right)+1\right) \\ &+\phi(M_1)\mu_n+l_n\Big].\end{aligned} \tag{15}$$

Defining

$$\begin{aligned} a_n &= \left\|x_n - p\right\| \\ b_n &= \alpha_n \left[(1 + M_2\mu_n)\left(1 + N_2\tilde{\mu}_n\right)\left(1 + N_2\beta_n\tilde{\mu}_n\right) - 1\right] \\ c_n &= \alpha_n \left[(1 + M_2\mu_n)\left(\varphi(N_1)\tilde{\mu}_n + \tilde{l}_n\right)\left(\beta_n\left(1 + N_2\tilde{\mu}_n\right) + 1\right) + \phi(M_1)\mu_n + l_n\right] \end{aligned}$$

in (15) we have $a_{n+1} \leq (1 + b_n) a_n + c_n$. Since $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} c_n < \infty$, Lemma 2.4 implies the existence of the limit $\lim_{n\to\infty} a_n$. This completes the proof. □

Theorem 3.3. *Let E be real Banach space and K be a nonempty closed convex subset of E. Let $T : K \to K$ be a total uniformly L_1-Lipschitzian asymptotically quasi-I-nonexpansive mapping with sequences $\{\mu_n\}$, $\{l_n\}$ and $I : K \to K$ be a total uniformly L_2-Lipschitzian asymptotically quasi-nonexpansive mapping with sequences $\{\tilde{\mu}_n\}$, $\{\tilde{l}_n\}$ such that $F = F(T) \cap F(I) \neq \emptyset$. Suppose that $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} l_n < \infty$, $\sum_{n=1}^{\infty} \tilde{\mu}_n < \infty$, $\sum_{n=1}^{\infty} \tilde{l}_n < \infty$ and there exist M_i, $N_i > 0$, $i = 1, 2$, such that $\phi(\xi) \leq M_2\xi$ for all $\xi \geq M_1$ and $\varphi(\zeta) \leq N_2\zeta$ for all $\zeta \geq N_1$. Then the sequence $\{x_n\}$ by (9), converges strongly to a common fixed point in $F = F(T) \cap F(I)$ if and only if*

$$\liminf_{n\to\infty} d(x_n, F) = 0. \tag{16}$$

Proof. For any given $p \in F$, we have (see (15))

$$\left\|x_{n+1} - p\right\| \leq (1 + b_n)\left\|x_n - p\right\| + c_n, \quad n \geq 1. \tag{17}$$

It suffices to show that $\lim_{n\to\infty} \inf d(x_n, F) = 0$ implies that $\{x_n\}$ converges to a common fixed point of T and I.

Necessity. Since (17) holds for all $p \in F$, we obtain from it that

$$d(x_{n+1}, F) \leq (1 + b_n)\, d(x_n, F) + c_n, \quad n \geq 1.$$

Lemma 2.4 implies that $\lim_{n\to\infty} d(x_n, F)$ exists. But, $\lim_{n\to\infty} \inf d(x_n, F) = 0$. Hence, $\lim_{n\to\infty} d(x_n, F) = 0$.

Sufficiency. Let us prove that the sequence $\{x_n\}$ converges to a common fixed point of T and I. Firstly, we show that $\{x_n\}$ is a Cauchy sequence in E. In fact, as $1 + t \leq \exp(t)$ for all $t > 0$. For all integer $m \geq 1$, we obtain from inequality (17) that

$$\left\|x_{n+m} - p\right\| \leq \exp\left(\sum_{i=n}^{n+m-1} b_i\right)\left\|x_n - p\right\| + \left(\sum_{i=n}^{n+m-1} c_i\right)\exp\left(\sum_{i=n}^{n+m-1} b_i\right),$$

so that for all integers $m \geq 1$ and all $p \in F$,

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \left\|x_{n+m} - p\right\| + \left\|x_n - p\right\| \\ &\leq \left(1 + \exp\left(\sum_{i=n}^{\infty} b_i\right)\right)\left\|x_n - p\right\| + \exp\left(\sum_{i=n}^{\infty} b_i\right)\sum_{i=n}^{\infty} c_i \\ &\leq A\left(\left\|x_n - p\right\| + \sum_{i=n}^{\infty} c_i\right), \end{aligned} \tag{18}$$

for all $p \in F$, where $0 < A - 1 = \exp\left(\sum_{i=n}^{\infty} b_i\right) < \infty$. Taking the infimum over $p \in F$ in (18) gives

$$\|x_{n+m} - x_n\| \leq A\left(d(x_n, F) + \sum_{i=n}^{\infty} c_i\right). \tag{19}$$

Now, since $\lim_{n\to\infty} d(x_n, F) = 0$ and $\sum_{i=1}^{\infty} c_i < \infty$, given $\varepsilon > 0$, there exists an integer $n_0 > 0$ such that for all $n > n_0$ we have $d(x_n, F) < \frac{\varepsilon}{2A}$ and $\sum_{i=n}^{\infty} c_i < \frac{\varepsilon}{2A}$. So, for all integers $n > n_0$ and $m \geq 1$, we obtain (19) that

$$\|x_{n+m} - x_n\| \leq \varepsilon$$

which means that $\{x_n\}$ is a Cauchy sequence in E, and completeness of E yields the existence of $x^* \in E$ such that $x_n \to x^*$ strongly.

Now, we show that x^* is a common fixed point of T and I. Suppose that $x^* \notin F$. Since F is closed subset of E, one has $d(x^*, F) > 0$. However, for all $p \in F$, we have

$$\left\|x^* - p\right\| \leq \|x_n - x^*\| + \left\|x_n - p\right\|.$$

This implies that

$$d(x^*, F) \leq \|x_n - x^*\| + d(x_n, F),$$

so, we obtain $d(x^*, F) = 0$ as $n \to \infty$, which contradicts $d(x^*, F) > 0$. Hence, x^* is a common fixed point of T and I. This comletes the proof. □

Lemma 3.4. *Let E be a real uniformly Banach space and K be a nonempty closed convex subset of E. Let $T : K \to K$ be a total uniformly L_1-Lipschitzian asymptotically quasi-I-nonexpansive mapping with sequences $\{\mu_n\}$, $\{l_n\}$ and $I : K \to K$ be a total uniformly L_2-Lipschitzian asymptotically quasi-nonexpansive mapping with sequences $\{\widetilde{\mu}_n\}$, $\{\widetilde{l}_n\}$ such that $F = F(T) \cap F(I) \neq \emptyset$. Suppose that $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} l_n < \infty$, $\sum_{n=1}^{\infty} \widetilde{\mu}_n < \infty$, $\sum_{n=1}^{\infty} \widetilde{l}_n < \infty$ and there exist $M_i, N_i > 0$, $i = 1, 2$, such that $\phi(\xi) \leq M_2\xi$ for all $\xi \geq M_1$ and $\varphi(\zeta) \leq N_2\zeta$ for all $\zeta \geq N_1$. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[t, 1-t]$, where $0 < t < 1$. Then the sequence $\{x_n\}$ by (9) satisfies the following:*

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0, \tag{20}$$

$$\lim_{n\to\infty} \|x_n - Ix_n\| = 0. \tag{21}$$

Proof. By Lemma 3.2, $\lim_{n\to\infty} \left\|x_n - p\right\|$ exists. Assume that, for any p$\in F = F(T) \cap F(I)$, $\lim_{n\to\infty} \left\|x_n - p\right\| = r$. If $r = 0$, the conclusion is obvious. Suppose $r > 0$.

First, we will prove that

$$\lim_{n\to\infty} \|x_n - T^n x_n\| = 0, \quad \lim_{n\to\infty} \|x_n - I^n x_n\| = 0. \tag{22}$$

It follows from (9) that

$$\left\|x_{n+1} - p\right\| = \left\|(1-\alpha_n)(x_n - p) + \alpha_n(T^n y_n - p)\right\| \to r, \tag{23}$$

as $n \to \infty$. By means of $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} l_n < \infty$, $\sum_{n=1}^{\infty} \tilde{\mu}_n < \infty$, $\sum_{n=1}^{\infty} \tilde{l}_n < \infty$, from (12) and (14) we get

$$\lim_{n\to\infty} \sup \left\| T^n y_n - p \right\| \le \lim_{n\to\infty} \sup \left\| y_n - p \right\| \le \lim_{n\to\infty} \sup \left\| x_n - p \right\| = r. \tag{24}$$

Hence, using (23), (24) and Lemma 2.3, we obtain

$$\lim_{n\to\infty} \left\| x_n - T^n y_n \right\| = 0. \tag{25}$$

From (9) and (25) we have

$$\lim_{n\to\infty} \| x_{n+1} - x_n \| = 0. \tag{26}$$

From (25) and (26) we get

$$\lim_{n\to\infty} \left\| x_{n+1} - T^n y_n \right\| \le \lim_{n\to\infty} \| x_{n+1} - x_n \| + \lim_{n\to\infty} \left\| x_n - T^n y_n \right\| = 0. \tag{27}$$

On the other hand, from (12) and (14) we have

$$\begin{aligned} \left\| x_n - p \right\| &\le \left\| x_n - T^n y_n \right\| + (1 + M_2 \mu_n) \left(1 + N_2 \tilde{\mu}_n\right) \left\| y_n - p \right\| \\ &\quad + (1 + M_2 \mu_n) \left(\varphi(N_1) \tilde{\mu}_n + \tilde{l}_n\right) + \phi(M_1) \mu_n + l_n \\ &\le \left\| x_n - T^n y_n \right\| + (1 + M_2 \mu_n) \left(1 + N_2 \tilde{\mu}_n\right) \left(1 + N_2 \beta_n \tilde{\mu}_n\right) \left\| x_n - p \right\| \\ &\quad + (1 + M_2 \mu_n) \left(\varphi(N_1) \tilde{\mu}_n + \tilde{l}_n\right) \left(\beta_n \left(1 + N_2 \tilde{\mu}_n\right) + 1\right) + \phi(M_1) \mu_n \\ &\quad + l_n. \end{aligned} \tag{28}$$

From (28) we obtain

$$\begin{aligned} \lim_{n\to\infty} \left\| x_n - p \right\| &\le \lim_{n\to\infty} \left\| x_n - T^n y_n \right\| + \lim_{n\to\infty} \left\| y_n - p \right\| \\ &\le \lim_{n\to\infty} \left\| x_n - T^n y_n \right\| + \lim_{n\to\infty} \left\| x_n - p \right\|. \end{aligned} \tag{29}$$

Then (29) with the squeeze theorem, imply that

$$\lim_{n\to\infty} \left\| y_n - p \right\| = r.$$

From (9) we can see that

$$\left\| y_n - p \right\| = \left\| (1 - \beta_n)(x_n - p) + \beta_n (I^n x_n - p) \right\| \to r, \quad n \to \infty. \tag{30}$$

Furthermore, from (11) we get

$$\lim_{n\to\infty} \sup \left\| I^n x_n - p \right\| \le \lim_{n\to\infty} \sup \left\| x_n - p \right\| = r. \tag{31}$$

Now, applying Lemma 2.3 to (30) we obtain

$$\lim_{n\to\infty} \| x_n - I^n x_n \| = 0. \tag{32}$$

From (26) and (32) we have

$$\lim_{n\to\infty} \|x_{n+1} - I^n x_n\| \leq \lim_{n\to\infty} \|x_{n+1} - x_n\| + \lim_{n\to\infty} \|x_n - I^n x_n\| = 0. \quad (33)$$

It follows from (9) that

$$\left\|y_n - x_n\right\| = \beta_n \|x_n - I^n x_n\|. \quad (34)$$

Hence, from (32) and (34) we obtain

$$\lim_{n\to\infty} \left\|y_n - x_n\right\| = 0. \quad (35)$$

Consider

$$\|x_n - T^n x_n\| \leq \left\|x_n - T^n y_n\right\| + L_1 \left\|y_n - x_n\right\| + L_1 \left(\mu_n \phi\left(\left\|y_n - x_n\right\|\right) + l_n\right). \quad (36)$$

Then, from (25), (35) and (36) we obtain

$$\lim_{n\to\infty} \|x_n - T^n x_n\| = 0. \quad (37)$$

From (26) and (35) we have

$$\lim_{n\to\infty} \left\|x_{n+1} - y_n\right\| \leq \lim_{n\to\infty} \|x_{n+1} - x_n\| + \lim_{n\to\infty} \left\|y_n - x_n\right\| = 0. \quad (38)$$

Finally, from

$$\begin{aligned} \|x_n - Tx_n\| &\leq \|x_n - T^n x_n\| + L_1 \left\|x_n - y_{n-1}\right\| + L_1 \left(\mu_n \phi\left(\left\|x_n - y_{n-1}\right\|\right) + l_n\right) \\ &\quad + L_1 \left\|T^{n-1} y_{n-1} - x_n\right\| + L_1 \left(\mu_n \phi\left(\left\|T^{n-1} y_{n-1} - x_n\right\|\right) + l_n\right), \end{aligned} \quad (39)$$

which with (27), (37) and (38) we get

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \quad (40)$$

Similarly, we obtain

$$\begin{aligned} \|x_n - Ix_n\| &\leq \|x_n - I^n x_n\| + L_2 \|x_n - x_{n-1}\| + L_2 \left(\tilde{\mu}_n \varphi\left(\|x_n - x_{n-1}\|\right) + \tilde{l}_n\right) \\ &\quad + L_2 \left\|I^{n-1} x_{n-1} - x_n\right\| + L_2 \left(\tilde{\mu}_n \varphi\left(\left\|I^{n-1} x_{n-1} - x_n\right\|\right) + \tilde{l}_n\right), \end{aligned} \quad (41)$$

which with (26), (32) and (33) implies

$$\lim_{n\to\infty} \|x_n - Ix_n\| = 0. \quad (42)$$

This completes the proof. □

Theorem 3.5. *Let E be a real uniformly Banach space satisfying Opial condition and let K be a nonempty closed convex subset of E. Let $C : E \to E$ be an identity maping. Let $T : K \to K$ be a total uniformly L_1-Lipschitzian asymptotically quasi-I-nonexpansive mapping with sequences $\{\mu_n\}$, $\{l_n\}$ and $I : K \to K$ be a total uniformly L_2-Lipschitzian asymptotically quasi-nonexpansive mapping with sequences $\{\tilde{\mu}_n\}$, $\{\tilde{l}_n\}$ such that $F = F(T) \cap F(I) \neq \emptyset$. Suppose that $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} l_n < \infty$, $\sum_{n=1}^{\infty} \tilde{\mu}_n < \infty$, $\sum_{n=1}^{\infty} \tilde{l}_n < \infty$ and there exist $M_i, N_i > 0$, $i = 1, 2$, such that $\phi(\xi) \leq M_2\xi$ for all $\xi \geq M_1$ and $\varphi(\zeta) \leq N_2\zeta$ for all $\zeta \geq N_1$. Assume that $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in $[t, 1-t]$, where $0 < t < 1$. If the mappings $C - T$ and $C - I$ are semiclosed at zero, then the explicit iterative sequence $\{x_n\}$ defined by (9) converges weakly to a common fixed point of T and I.*

Proof. Let $p \in F = F(T) \cap F(I)$. By Lemma 3.2, we know that $\lim_{n\to\infty} \|x_n - p\|$ exists and $\{x_n\}$ is bounded. Since E is uniformly convex, then every bounded subset of E is weakly compact. Since $\{x_n\}$ is a bounded sequence in K, then there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $q_1 \in K$. Thus, from (40) and (42) it follows that

$$\lim_{n_k\to\infty} \left\|x_{n_k} - Tx_{n_k}\right\| = 0, \quad \lim_{n_k\to\infty} \left\|x_{n_k} - Ix_{n_k}\right\| = 0. \tag{43}$$

Since the mappings $C - T$ and $C - I$ are semiclosed at zero, we find $Tq_1 = q_1$ and $Iq_1 = q_1$. Namely, $q_1 \in F = F(T) \cap F(I)$.

Finally, let us prove that $\{x_n\}$ converges weakly to q_1. Actually, suppose the contrary, that is, there exists some subsequence $\left\{x_{n_j}\right\} \subset \{x_n\}$ such that $\left\{x_{n_j}\right\}$ converges weakly to $q_2 \in K$ and $q_1 \neq q_2$. Then by the same method as given above, we can also prove that $q_2 \in F = F(T) \cap F(I)$.

Since $q_1, q_2 \in F = F(T) \cap F(I)$, according to Lemma 3.2 $\lim_{n\to\infty} \left\|x_n - q_1\right\|$ and $\lim_{n\to\infty} \left\|x_n - q_2\right\|$ exist, we have

$$\lim_{n\to\infty} \left\|x_n - q_1\right\| = r_1, \quad \lim_{n\to\infty} \left\|x_n - q_2\right\| = r_2, \tag{44}$$

where $d_1, d_2 \geq 0$. Because of the Opial condition of E, we obtain

$$\begin{aligned} r_1 &= \limsup_{n_k\to\infty} \left\|x_{n_j} - q\right\| < \limsup_{n_k\to\infty} \left\|x_{n_k} - q_1\right\| = r_2 \\ &= \limsup_{n_j\to\infty} \left\|x_{n_j} - q_1\right\| < \limsup_{n_j\to\infty} \left\|x_{n_j} - q\right\|. \end{aligned} \tag{45}$$

This is a contradiction. Hence $q_1 = q_2$. This implies that $\{x_n\}$ converges weakly to q. This completes the proof. □

Theorem 3.6. *Let E be a real uniformly Banach space and K be a nonempty closed convex subset of E. Let $T : K \to K$ be a total uniformly L_1-Lipschitzian asymptotically quasi-I-nonexpansive mapping with sequences $\{\mu_n\}$, $\{l_n\}$ and $I : K \to K$ be a total uniformly L_2-Lipschitzian asymptotically quasi-nonexpansive mapping with sequences $\{\tilde{\mu}_n\}$, $\{\tilde{l}_n\}$ such that $F = F(T) \cap F(I) \neq \emptyset$. Suppose that $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} l_n < \infty$, $\sum_{n=1}^{\infty} \tilde{\mu}_n < \infty$, $\sum_{n=1}^{\infty} \tilde{l}_n < \infty$ and there exist $M_i, N_i > 0$, $i = 1, 2$, such that $\phi(\xi) \leq M_2\xi$ for all $\xi \geq M_1$ and $\varphi(\zeta) \leq N_2\zeta$ for all $\zeta \geq N_1$. Assume that $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in $[t, 1-t]$, where $0 < t < 1$. If at least one mapping of the mappings T and I is semicompact, then the explicit iterative sequence $\{x_n\}$ defined by (9) converges strongly to a common fixed point of T and I.*

Proof. Without any loss of generality, we may assume that T is semicompact. This with (40) means that there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to x^*$ strongly and $x^* \in K$. Since $T,\ I$ are continuous, then from (40) and (42) we find

$$\|x^* - Tx^*\| = \lim_{n_k\to\infty} \left\|x_{n_k} - Tx_{n_k}\right\| = 0, \quad \|x^* - Ix^*\| = \lim_{n_k\to\infty} \left\|x_{n_k} - Ix_{n_k}\right\| = 0. \tag{46}$$

This shows that $x^* \in F = F(T) \cap F(I)$. According to Lemma 3.2 the limit $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Then

$$\lim_{n\to\infty} \|x_n - x^*\| = \lim_{n_k\to\infty} \left\|x_{n_k} - x^*\right\| = 0,$$

which means that $\{x_n\}$ converges to $x^* \in F$. This completes the proof. □

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