Filomat 28:8 (2014), 1575–1584
DOI 10.2298/FIL1408575G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Iteration by Cesàro Means for Quasi-contractive Mappings

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Abstract. Let C be a nonempty closed convex subset of a Banach space E and T be a quasi-contractive mapping on C. We prove, the sequence $\{x_n\}$, iteratively defined by,

$$\begin{cases} x_1 \in C \\ y_n = s_n x_n + (1-s_n)T^n x_n \\ x_{n+1} = t_n x_n + (1-t_n)\frac{1}{n+1}\sum_{j=0}^{n} T^j y_n, \end{cases}$$

is weakly convergent to a point of $F(T)$. Moreover, by a numerical example (using Matlab software), the main result and the rate of convergence are illustrated.

1. Introduction

Let C be a nonempty subset of a Banach space E, $T : C \to C$ and

$$F(T) = \{x \in C, Tx = x\},$$

denotes the set of fixed points of T. A mapping T is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\}$ of positive numbers with $\lim_{n\to\infty} k_n = 1$ such that for all $x, y \in C$ and $n \geq 1$,

$$\|T^n x - T^n y\| \leq k_n \|x - y\|.$$

The convergence of various iteration sequence was studied broadly by many authors over the last twenty years for construction of fixed points of nonlinear mappings (see [4–7, 10]). In 1975, Baillon [1] proved the first nonlinear ergodic theorem as follows:

Theorem 1.1. *Suppose C is a nonempty, closed and convex subset of Hilbert space H and $T : C \to C$ is a nonexpansive mapping such that $F(T) \neq \emptyset$, then for every $x \in C$, the Cesàro means*

$$T_n x = \frac{1}{n+1}\sum_{j=0}^{n} T^j x,$$

2010 *Mathematics Subject Classification*. 47H10
Keywords. Cesáro means, Fixed point, Quasi-contractive mapping.
Received: 17 July 2013; Accepted: 01 December 2013
Communicated by Vladimir Rakočević
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converges weakly to a fixed point of T.

Then, Shimizu-Takahashi [8] proved for an asymptotically nonexpansive mapping T in a Hilbert space, the approximating sequence

$$x_n = t_n u + (1 - t_n)\frac{1}{n+1}\sum_{j=0}^{n} T^j x_n,$$

converges strongly to the element of $F(T)$. In addition, Several authors surveyed iterative approximation of Cesàro means for asymptotically nonexpansive mappings (see[2, 3]).

The property of Cesàro means for nonexpansive mappings in uniformly convex Banach spaces was studied by Bruck for the first time. Bruck proved the nonlinear ergodic theorem for nonexpansive mapping in uniformly convex Banach space with Fréchet differentiable norms. In 1999, Shioji-Takahashi [9] surveyed the strong convergence of sequence

$$x_{n+1} = t_n u + (1 - t_n)\frac{1}{n+1}\sum_{j=0}^{n} T^j x_n,$$

in uniformly convex Banach spaces with uniformly Gateaux differentiable norms.

2. Preliminaries

Within this section, we recall some preliminary definitions and lemmas which are needed in the next section.

Definition 2.1. *Suppose E is a Banach space. E is said to satisfy Opial's condition, if for each sequence $\{x_n\}$ in E the condition $x_n \rightharpoonup x$ implies*

$$\overline{\lim}_{n\longrightarrow\infty}\|x_n - x\| < \overline{\lim}_{n\longrightarrow\infty}\|x_n - y\|,$$

for all $y \in E$ and $y \neq x$.

Definition 2.2. *E is said to have a Fréchet differentiable norm if, for each $x \in S(E)$, the unit sphere of E, $\lim_{t\longrightarrow 0}\frac{\|x+ty\|-\|x\|}{t}$ exists and is attained uniformly in $y \in S(E)$.*

Definition 2.3. *Let E be an arbitrary real Banach space with norm $\|.\|$ and E^* be the dual space of E. The duality mapping $J : E \to E^*$ is defined by*

$$Jx = \{f \in E^* :< x, f >= \|x\|^2, \|f\| = \|x\|\},$$

where $< x, f >$ denotes the value of the continuous linear function $f \in E^*$ at $x \in E$.

Definition 2.4. *Let C be a nonempty subset of Banach space E. A mapping $T : C \to C$ is said to be quasi-contraction if there exists $k \in [0, 1)$ such that for all $x, y \in C$ and $n \geq 1$*

$$\|T^n x - T^n y\| \leq k \max\{\|x - y\|, \|T^n x - x\|, \|y - T^n y\|, \|x - T^n y\|, \|T^n x - y\|\}. \tag{1}$$

Definition 2.5. *Suppose E is a uniformly convex Banach space with a Fréchet differentiable norm. If $\{x_n\} \subseteq E$, $w_w(\{x_n\})$ is as follows:*

$$w_w(\{x_n\}) = \{x \in E | \ \exists \{x_{n_j}\} \ s.t. \ x_{n_j} \rightharpoonup x\}. \tag{2}$$

In order to prove the main result, we need the following lemmas.

Lemma 2.6. *(Osilike-Aniagbosor [6, Lemma 1]). Let a_n, b_n and δ_n be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n.$$

If $\sum_{n=0}^{\infty} \delta_n < \infty$ and $\sum_{n=0}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. In particular, $\lim_{n\to\infty} a_n = 0$ whenever there exists a subsequence $\{a_{n_k}\}$ in $\{a_n\}$ which converges strongly to zero.

Lemma 2.7. *Let $\{x_n\}$ be a bounded sequence on a reflexive Banach space X. If $w_w(\{x_n\}) = \{x\}$, then $x_n \rightharpoonup x$.*

Lemma 2.8. *Let X be a normed space, C a convex subset of E and $T : C \to C$ a quasi-contractive mapping with $k \in (0, \frac{1}{2})$. If $\lim_{n\to\infty} \|x_n - T^n x_n\| = 0$ and $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$. Then $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.*

Proof. Set $r_n = \|x_n - T^n x_n\|$, then

$$\begin{aligned} \|x_{n+1} - Tx_{n+1}\| \leq & \ \|x_{n+1} - T^{n+1}x_{n+1}\| + \|Tx_{n+1} - T^{n+1}x_{n+1}\| \\ \leq & \ r_{n+1} + k \max\{\|x_{n+1} - T^n x_{n+1}\|, \|x_{n+1} - Tx_{n+1}\|, \|T^{n+1}x_{n+1} - T^n x_{n+1}\|, \|Tx_{n+1} - T^n x_{n+1}\| \\ & , \|T^{n+1}x_{n+1} - x_{n+1}\|\}. \end{aligned} \tag{3}$$

Let $\|T^{n+1}x_{n+1} - x_{n+1}\|$ be the maximum, then

$$\|x_{n+1} - Tx_{n+1}\| \leq r_{n+1} + kr_{n+1},$$

and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Let $\|x_{n+1} - Tx_{n+1}\|$ be the maximum, so

$$(1 - k)\|x_{n+1} - Tx_{n+1}\| \leq r_{n+1},$$

therefore $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Let $\|x_{n+1} - T^n x_{n+1}\|$ be the maximum. We have

$$\|x_{n+1} - T^n x_{n+1}\| \leq \|x_{n+1} - x_n\| + \|x_n - T^n x_n\| + \|T^n x_n - T^n x_{n+1}\|, \tag{4}$$

and also

$$\|T^n x_n - T^n x_{n+1}\| \leq k \max\{\|x_n - x_{n+1}\|, \|T^n x_n - x_n\|, \|T^n x_{n+1} - x_{n+1}\|, \|T^n x_n - x_{n+1}\|, \|T^n x_{n+1} - x_n\|\}.$$

According to the above, the only case that has to be verified is, when $\|T^n x_{n+1} - x_{n+1}\|$ is the maximum. From (4), we have

$$\|T^n x_{n+1} - x_{n+1}\| \leq \|x_{n+1} - x_n\| + \|x_n - T^n x_n\| + k\|x_{n+1} - T^n x_{n+1}\|.$$

Then

$$\|x_{n+1} - T^n x_{n+1}\| \leq \frac{1}{1-k}(\|x_{n+1} - x_n\| + \|x_n - T^n x_n\|).$$

Therefore

$$\|x_{n+1} - Tx_{n+1}\| \leq r_{n+1} + \frac{1}{1-k}(\|x_{n+1} - x_n\| + r_n).$$

Thus, $\|x_{n+1} - Tx_{n+1}\| \to 0$ as $n \to \infty$. Let $\|T^{n+1}x_{n+1} - T^n x_{n+1}\|$ be the maximum. We have

$$\|T^{n+1}x_{n+1} - T^n x_{n+1}\| \leq \|T^{n+1}x_{n+1} - x_{n+1}\| + \|x_{n+1} - T^n x_{n+1}\|,$$

thus

$$\|x_{n+1} - Tx_{n+1}\| \leq r_{n+1} + kr_{n+1} + k\|x_{n+1} - T^n x_{n+1}\|.$$

Since

$$\|x_{n+1} - T^n x_{n+1}\| \leq \frac{1}{1-k}(\|x_{n+1} - x_n\| + \|x_n - T^n x_n\|),$$

we have

$$\|x_{n+1} - Tx_{n+1}\| \leq r_{n+1} + kr_{n+1} + \frac{k}{1-k}(\|x_{n+1} - x_n\| + r_n).$$

Therefore, $\|x_{n+1} - Tx_{n+1}\| \to 0$ as $n \to \infty$. Let $\|Tx_{n+1} - T^n x_{n+1}\|$ be the maximum. We have

$$\|Tx_{n+1} - T^n x_{n+1}\| \leq \|Tx_{n+1} - x_{n+1}\| + \|T^n x_{n+1} - x_{n+1}\|,$$

therefore

$$(1-k)\|Tx_{n+1} - x_{n+1}\| \leq r_{n+1} + k\|T^n x_{n+1} - x_{n+1}\|.$$

Since

$$\|x_{n+1} - T^n x_{n+1}\| \leq \frac{1}{1-k}(\|x_{n+1} - x_n\| + \|x_n - T^n x_n\|),$$

we get

$$(1-k)\|Tx_{n+1} - x_{n+1}\| \leq r_{n+1} + \frac{k}{1-k}(\|x_{n+1} - x_n\| + r_n\|).$$

Thus $\|x_{n+1} - Tx_{n+1}\| \to 0$ as $n \to \infty$. □

Proposition 2.9. *Let E be a uniformly convex Banach space and C be a nonempty, closed and convex subset of E. Suppose $T : C \longrightarrow C$ is a quasi-contractive mapping such that $k \in (0, \frac{1}{2})$ and $F(T) \neq \emptyset$. Then $I - T$ is demiclosed at zero in the sense that if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x \in C$ and $\limsup_{m\to\infty} \limsup_{n\to\infty} \|x_n - T^m x_n\| = 0$, then $(I - T)x = 0$.*

Proof. Assume that T is a continuous quasi-contractive mapping such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C such that $x_n \rightharpoonup x \in C$ and $\limsup_{m\to\infty} \limsup_{n\to\infty} \|x_n - T^m x_n\| \to 0$, then $\{x_n\}$ is bounded.
Now, the sets $\{T^m x_n, m, n \geq 1\}$ and $\{T^m x, m \geq 1\}$ are bounded. Indeed, for every $p \in F(T)$,

$$\|T^m x_n - p\| = \|T^m x_n - T^m p\| \leq k \max\{\|x_n - p\|, \|x_n - T^m x_n\|, \|T^m x_n - p\|\}.$$

Let $\|x_n - T^m x_n\|$ be maximum, then

$$\|x_n - T^m x_n\| \leq \|x_n - p\| + \|p - T^m x_n\|,$$

so

$$\|T^m x_n - p\| \leq \frac{k}{1-k}\|x_n - p\|.$$

Then

$$\|T^m x_n - p\| \leq r\|x_n - p\|,$$

where $\frac{k}{1-k} = r$. Also

$$\|T^m x - p\| \leq r\|x - p\|.$$

Thus $\{T^m x_n, m, n \geq 1\}$ and $\{T^m x, m \geq 1\}$ are bounded. Consequently

$$\begin{aligned} \|x - T^m x\| \leq & \quad \|x - x_n\| + \|x_n - T^m x\| \\ \leq & \quad \|x - x_n\| + \|x_n - T^m x_n\| + \|T^m x_n - T^m x\|. \end{aligned}$$

Since T is a quasi-contractive mapping, we have

$$\|T^m x_n - T^m x\| \leq k \max\{\|x_n - x\|, \|x_n - T^m x\|, \|x - T^m x\|, \|x_n - T^m x_n\|, \|x - T^m x_n\|\}.$$

Suppose $\|T^m x - x\|$ is the maximum, then

$$\|x - T^m x\| \leq \frac{1}{1-k}(\|x - x_n\| + \|x_n - T^m x_n\|).$$

Consider $\|x - T^m x_n\|$ is the maximum, so

$$\|x - T^m x_n\| \leq \|x_n - T^m x_n\| + \|x_n - x\|,$$

thus

$$\|x - T^m x\| \leq (1+k)[\|x_n - x\| + \|x_n - T^m x_n\|].$$

Assume $\|x_n - T^m x\|$ is the maximum, then

$$\|x_n - T^m x\| \leq \|T^m x - x\| + \|x - x_n\|,$$

therefore

$$\|x - T^m x\| \leq \frac{1}{1-k}[(1+k)\|x - x_n\| + \|x_n - T^m x_n\|].$$

Since $\limsup_{m\to\infty} \limsup_{n\to\infty} \|x_n - T^m x_n\| = 0$ and $\|x - x_n\| \to 0$ as $n \to \infty$. Then $T^m x \to x$ as $m \to \infty$. Therefore, the continuity of T implies $(I - T)x = 0$. This completes the proof. □

3. Weak Convergence

Theorem 3.1. *Let E be a uniformly convex Banach space which satisfies Opial's condition. Suppose C is a nonempty closed convex subset of E and $T : C \to C$ a uniformly continuous quasi-contractive mapping such that $k \in (0, \frac{1}{2})$ and $F(T) \neq \emptyset$. Assume $\{t_n\}$ and $\{s_n\}$ are real sequences such that $t_n \to 1$ or $\sum t_n < \infty$, $\{s_n\}$ is bounded away from 0 and 1 and $\lim_{n\to\infty} s_n = 1$. Then sequence $\{x_n\}$ defined by*

$$\begin{cases} x_1 \in C \\ y_n = s_n x_n + (1 - s_n)T^n x_n \\ x_{n+1} = t_n x_n + (1 - t_n)\frac{1}{n+1}\sum_{j=0}^{n} T^j y_n, \end{cases} \tag{5}$$

converges weakly to a fixed point of T.

Proof. Let p be an element of $F(T)$. We have

$$\|T^n y_n - p\| \leq k \max\{\|y_n - p\|, \|T^n y_n - y_n\|, \|T^n y_n - p\|\}.$$

Let $\|T^n y_n - p\|$ be the maximum. Since $0 < k < \frac{1}{2}$, this is a contradiction. Suppose $\|T^n y_n - y_n\|$ is the maximum. So

$$\|T^n y_n - y_n\| \leq \|T^n y_n - p\| + \|y_n - p\|,$$

thus

$$\|T^n y_n - p\| \leq \frac{k}{1-k}\|y_n - p\|. \tag{6}$$

Assume $\|y_n - p\|$ is the maximum, then

$$\|T^n y_n - p\| \leq k\|y_n - p\|.$$

Set $r = \frac{k}{1-k}$. Since $r > k$, we have

$$\|T^n y_n - p\| \leq r\|y_n - p\|. \tag{7}$$

According to the equations (6) and (7), we get

$$\begin{aligned} \|x_{n+1} - p\| \leq\ & t_n\|x_n - p\| + (1 - t_n)\tfrac{1}{1+n}\textstyle\sum_{i=0}^{n}\|T^i y_n - p\| \\ \leq\ & t_n\|x_n - p\| + (1 - t_n)r\|y_n - p\|. \end{aligned}$$

Also

$$\begin{aligned} \|y_n - p\| =\ & \|s_n x_n + (1 - s_n)T^n x_n - p\| \\ \leq\ & s_n\|x_n - p\| + (1 - s_n)\|T^n x_n - p\| \\ \leq\ & s_n\|x_n - p\| + (1 - s_n)r\|x_n - p\|. \end{aligned}$$

Thus

$$\begin{aligned} \|x_{n+1} - p\| \leq\ & t_n\|x_n - p\| + (1 - t_n)r\|y_n - p\| \\ \leq\ & t_n\|x_n - p\| + (1 - t_n)rs_n\|x_n - p\| + (1 - t_n)(1 - s_n)r^2 s_n\|x_n - p\|, \end{aligned}$$

then

$$\|x_{n+1}-p\| \le (t_n+(1-t_n)rs_n+(1-t_n)(1-s_n)s_nr^2)\|x_n-p\|.$$

By Lemma 2.6, $\lim_{n\to\infty}\|x_n-p\|$ exists. Suppose $\lim_{n\to\infty}\|x_n-p\|=h$ for $h>0$; similar to the above $\lim_{n\to\infty}\|y_n-p\|$ exists. Also

$$\begin{aligned}\|y_n-x_n\| &= \|s_nx_n+(1-t_n)T^nx_n-x_n\|\\ &= (1-s_n)\|x_n-T^nx_n\|\to 0.\end{aligned}$$

Let $A_nx_n=\frac{1}{n+1}\sum_{j=0}^{n}T^jx_n$. Therefore

$$\begin{aligned}\|x_{n+1}-x_n\| &= \|t_nx_n+(1-t_n)A_nx_n-x_n\|\\ &= (1-t_n)\|x_n-A_nx_n\|.\end{aligned}$$

Since $\|x_{n+1}-x_n\|\to 0$; $\|x_n-T^nx_n\|\to 0$ as $n\to 0$, $F(T)\neq\emptyset$, $\{x_n\}$ is a bounded sequence in C and T is uniformly continuous. According to Lemma 2.8, $\|x_n-Tx_n\|\to 0$ as $n\to\infty$. Since $\{x_n\}$ is bounded; there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup x$. Because T is uniformly continuous $\|x_n-Tx_n\|\to 0$. Then we have $\|x_n-T^mx_n\|\to 0$ for all $m\ge 1$; so by Proposition 2.9 $x\in F(T)$. For the rest of the proof it is enough to prove $w_w(x_n)$ contains exactly one point that is called x. In order to prove the uniqueness, consider there is another subsequence $\{x_{n_j}\}$ of $\{x_n\}$ that converges weakly to $z\neq x$. According to above, we have to have $z\in F(T)$. So $\lim_{n\to\infty}\|x_n-x\|$ and $\lim_{n\to\infty}\|x_n-z\|$ exists. Since E satisfies Opial's condition, we have

$$\begin{aligned}\lim_{n\to\infty}\|x_n-x\| &= \lim_{k\to\infty}\|x_{n_k}-x\|\\ &< \lim_{n\to\infty}\|x_n-z\|,\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty}\|x_n-z\| &= \lim_{j\to\infty}\|x_{n_j}-z\|\\ &< \lim_{n\to\infty}\|x_n-x\|,\end{aligned}$$

which leads to a contradiction. Hence $x=z$. This shows that $w_w(\{x_n\})$ is a singleton. Therefore, $\{x_n\}$ converges weakly to x by Lemma 2.7. This completes the proof. □

Theorem 3.2. *Let X be a Banach space and let C be a nonempty, bounded, closed and convex subset of X; let $T_1,T_2,T_3,\cdots,T_N : C\to C$ be quasi-contractive mapping with constants $k_i\in(0,\frac{1}{2})$ for $i=1,2,\cdots,N$. Suppose $F=\bigcap_{i=1}^{n}F(T_i)\neq\emptyset$ and the sequence $\{x_n\}$ is defined by*

$$\begin{cases} x_1\in C\\ x_n^1 = \alpha_n^1x_n+\beta_n^1T_1x_n+\gamma_n^1u_n^1,\\ x_n^2 = \alpha_n^2x_n+\beta_n^2T_2x_n^1+\gamma_n^2u_n^2,\\ \quad\vdots\\ x_n^{N-1} = \alpha_n^{N-1}x_n+\beta_n^{N-1}T_{N-1}x_n^{N-2}+\gamma_n^{N-1}u_n^{N-1},\\ x_{n+1} = x_n^N=\alpha_n^Nx_n+\beta_n^N\frac{1}{n+1}\sum_{j=1}^{N}T_N^jx_n^{N-1}+\gamma_n^Nu_n^N, \end{cases} \tag{8}$$

where $\{u_n^1\},\dots,\{u_n^N\}$ are non-negative sequences in $\mathbb{R}$ such that $\sum_{n=0}^{\infty}u_n^N<\infty$ and $\{\alpha_n^1\},\dots,\{\alpha_n^N\}$, $\{\beta_n^1\},\dots,\{\beta_n^N\}$ and $\{\gamma_n^1\},\dots,\{\gamma_n^N\}$ are sequences in $[0,1]$ such that $\alpha_n^i+\beta_n^i+\gamma_n^i=1$ for all $i=1,2,\dots,N$ and $\sum_i\alpha_n^i+\frac{k_i}{1-k_i}\beta_n^i<\infty$ and also satisfy the following conditions:

$$\begin{aligned} &i)\lim_{n\to\infty}\beta_n^i=0,\ \ \forall i=1,2,\dots,N\quad \sum_{n=0}^{\infty}\beta_n^N=\infty,\\ &ii)\lim_{n\to\infty}\gamma_n^i=0. \end{aligned}\tag{9}$$

Then $\{x_n\}$ converges strongly to the unique common fixed point of $T_1,\dots,T_N$.

Proof. Let p be an element of F. By using the same technique of Theorem 3.1, we have

$$\begin{aligned} \|x_n - p\| = & \ \|\alpha_n^N x_n + \beta_n^N \tfrac{1}{n+1} \textstyle\sum_{j=1}^N T_N^j x_n^{N-1} + \gamma_n^N u_n^N - p\| \\ \leq & \ \alpha_n^N \|x_n - p\| + \tfrac{\beta_n^N}{n+1} \textstyle\sum_{j=1}^N \|T_N^j x_n^{N-1} - p\| + \gamma_n^N \|u_n^N - p\| \\ \leq & \ \alpha_n^N \|x_n - p\| + \beta_n^N r_N \|x_n^{N-1} - p\| + \gamma_n^N \|u_n - p\|, \end{aligned} \tag{10}$$

where $r_i = \frac{k_i}{1-k_i}$ for $i = 1, 2, \cdots, N$. Let $M = \max\{\sup_{1\leq i\leq N} \|u_n^i - p\|\}$. So

$$\begin{aligned} \|x_n - p\| \leq & \ \alpha_n^N \|x_n - p\| + \tfrac{\beta_n^N}{r}{}_N \|x_n^{N-1} - p\| + \gamma_n^N M \\ \leq & \ (\alpha_n^N + \beta_n^N r_N) \|x_n^{N-1} - p\| + \gamma_n^N M \\ \leq & \ h_N \|x_n^{N-1} - p\| + \gamma_n^N M. \end{aligned}$$

Suppose $1 \leq i \leq N-1$, then by using the technique of Theorem 3.1, we get

$$\begin{aligned} \|x_n^i - p\| = & \ \|\alpha_n^i x_n + \beta_n^i T_i x_n^{i-1} + \gamma_n^i u_n^i - p\| \\ \leq & \ \alpha_n^i \|x_n - p\| + \beta_n^i \|T_i x_n^{i-1} - p\| + \gamma_n^i \|u_n^i - p\| \\ \leq & \ \alpha_n^i \|x_n - p\| + \beta_n^i r_i \|x_n^{i-1} - p\| + M\gamma_n^i \\ \leq & \ (\alpha_n^i + r_i \beta_n^i) \|x_n^{i-1} - p\| + M\gamma_n^i. \end{aligned}$$

Set $h_i = \alpha_n^i + r_i \beta_n^i$. Clearly, $h_i < 1$. So

$$\begin{aligned} \|x_{n+1} - p\| \leq & \ h_N (h_{N-1} \|x_n^{N-1} - p\| + M\gamma_n^{N-1}) + \gamma_n^N M \\ \leq & \ h_N h_{N-1} \|x_n^{N-1} - p\| + (h_N \gamma_n^{N-1} + \gamma_n^N) M \\ \leq & \ h_N h_{N-1} h_{N-2} \|x_n^{N-2} - p\| + M(\gamma_n^N + h_N \gamma_n^{N-1} + h_N h_{N-1} \gamma_n^{N-2}) \\ \leq & \ h_N \cdots h_1 \|x_n - p\| + M(\gamma_n^N + h_N \gamma_n^{N-1} + \ldots + h_N h_{N-1} \cdots h_1 \gamma_n^1) \end{aligned}$$

By Lemma 2.6, $\lim_{n\to\infty} \|x_n - p\|$ exists. The rest of the proof is the same as Theorem 3.1. □

4. A Numerical Example

The purpose of our example is to illustrate our main result by a numerical test based on computer programs with Matlab.

Example 4.1. *Suppose $T : \mathbb{R} \to \mathbb{R}$ is defined as follows:*

$$Tx = \frac{x}{16}. \tag{11}$$

It is clear, T is quasi-contraction and $k = \frac{1}{16}$. Set $t_n = \frac{n}{3n+1}$ and $s_n = \frac{1}{3n}$. Then sequence x_n which has been defined by (5), is convergent.

In the next graph, the rate of convergence is shown and the favorite result is obtained after 5 iterations.

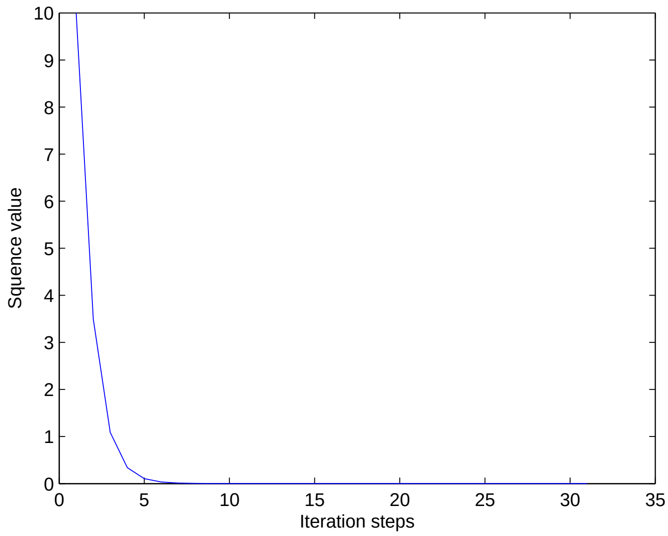

Figure 1: The iteration chart with initial value $x_1 = 10$.

The sequence $\{x_n\}$ for $n = 1, 2, \ldots, 18$ is given in the following table.

n	x_n	n	x_n
1	10.00000	10	0.00012
2	3.49615	11	0.00005
3	0.33985	12	0.00003
4	0.10696	13	0.00005
5	0.03394	14	0.00001
6	0.01085	15	0.00000
7	0.00352	16	0.00000
8	0.00113	17	0.00000
9	0.00041	18	0.00000

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