Filomat 28:9 (2014), 1899–1906
DOI 10.2298/FIL1409899S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A New Variation of Weyl Type Theorems and Perturbations

Junli Shen[a], **Alatancang Chen**[*a,b]

[a] *School of Mathematical Sciences, Inner Mongolia University, Hohhot, 010021, China.*
[b] *Department of Mathematics, Hohhot Minzu College, Hohhot, 010051, China.*

Abstract. In this paper, we introduce the new property (aR), which extends property (R) introduced by Aiena and his collaborators. We investigate the property (aR) in connection with Weyl type theorems, and establish sufficient and necessary conditions for which property (aR) holds. We also study the stability of property (aR) under perturbations by finite rank operators, by nilpotent operators, by quasi-nilpotent operators and by algebraic operators commuting with T.

1. Introduction

Throughout this paper, we denote X an infinite dimensional complex Banach space and $L(X)$ the algebra of all bounded linear operators on X. For $T \in L(X)$, we denote the null space, the range, the spectrum, the approximate point spectrum, the surjective spectrum, the isolated points of spectrum and the isolated points of approximate point spectrum by $N(T)$, $R(T)$, $\sigma(T)$, $\sigma_a(T)$, $\sigma_s(T)$, $\text{iso}\sigma(T)$ and $\text{iso}\sigma_a(T)$, respectively. If $R(T)$ is closed and $\alpha(T) = \dim N(T) < \infty$ (resp. $\beta(T) = \dim X/R(T) < \infty$), then T is called an upper (resp. a lower) semi-Fredholm operator. In the sequel $\Phi_+(X)$ (resp. $\Phi_-(X)$) is written for the set of all upper (resp. lower) semi-Fredholm operators. The class of all semi-Fredholm operators is defined by $\Phi_\pm(X) = \Phi_+(X) \cup \Phi_-(X)$, and the index of T is given by $i(T) = \alpha(T) - \beta(T)$. Denote $\Phi(X) = \Phi_+(X) \cap \Phi_-(X)$ the set of all Fredholm operators. Define $W_+(X) = \{T \in \Phi_+(X) : i(T) \leq 0\}$, $W_-(X) = \{T \in \Phi_-(X) : i(T) \geq 0\}$. The set of all Weyl operators is defined by $W(X) = W_+(X) \cap W_-(X) = \{T \in \Phi(X) : i(T) = 0\}$. The classes of operators defined above generate the following spectrums: the Weyl spectrum of T is defined by $\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin W(X)\}$, while the upper semi-Weyl spectrum of T is defined by $\sigma_{uw}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin W_+(X)\}$ and the lower semi-Weyl spectrum of T is defined by $\sigma_{lw}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin W_-(X)\}$. For $T \in L(X)$, let $\Delta(T) = \sigma(T) \backslash \sigma_w(T)$ and $\Delta_a(T) = \sigma_a(T) \backslash \sigma_{uw}(T)$. Following Coburn [9], Weyl's theorem is said to hold for T if $\Delta(T) = \pi_{00}(T)$, where $\pi_{00}(T) = \{\lambda \in \text{iso}\sigma(T) : 0 < \alpha(T - \lambda I) < \infty\}$. According to Rakočević [14], a-Weyl's theorem is said to hold for T if $\Delta_a(T) = \pi^a_{00}(T)$, where $\pi^a_{00}(T) = \{\lambda \in \text{iso}\sigma_a(T) : 0 < \alpha(T - \lambda I) < \infty\}$. It's known that an operator satisfying a-Weyl's theorem satisfies Weyl's theorem, but the converse doesn't hold in general.

Recall that the ascent $p(T)$ of an operator T is defined by $p(T) = \inf\{n \in \mathbb{N} : N(T^n) = N(T^{n+1})\}$ and the descent $q(T)$ of an operator T is defined by $q(T) = \inf\{n \in \mathbb{N} : R(T^n) = R(T^{n+1})\}$. It is well-known that if $p(T)$

2010 *Mathematics Subject Classification*. Primary 47B20; Secondary 47A10
Keywords. property (aR); perturbation; SVEP; Weyl type theorems; polaroid type operator
Received: 16 August 2013; Accepted: 14 December 2014
Communicated by Dragan S. Djordjević
Research supported by the National Natural Science Foundation of China(11371185) and Major subject of Natural Science Foundation of Inner Mongolia of China (2013ZD01)
Email addresses: zuoyawen1215@126.com (Junli Shen), alatanca@imu.edu.cn (Alatancang Chen*)

and $q(T)$ are both finite, then $p(T) = q(T)$ [12, Proposition 38.3]. Moreover, $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ precisely when λ is a pole of the resolvent of T, see Proposition 50.2 of Heuser [12]. The class of all upper semi-Browder operators is defined by $B_+(X) = \{T \in \Phi_+(X) : p(T) < \infty\}$ and the class of all Browder operators is defined by $B(X) = \{T \in \Phi(X) : p(T) = q(T) < \infty\}$. The Browder spectrum of T is defined by $\sigma_b(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin B(X)\}$ and the upper semi-Browder spectrum is defined by $\sigma_{ub}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin B_+(X)\}$, clearly $\sigma_w(T) \subseteq \sigma_b(T)$ and $\sigma_{uw}(T) \subseteq \sigma_{ub}(T)$. For $T \in L(X)$, set $p_{00}(T) = \sigma(T)\backslash\sigma_b(T)$ and $p_{00}^a(T) = \sigma_a(T)\backslash\sigma_{ub}(T)$. Obviously, $p_{00}(T) \subseteq \pi_{00}(T)$. In [11], Browder's theorem is said to hold for T if $\Delta(T) = p_{00}(T)$, or equivalently $\sigma_w(T) = \sigma_b(T)$; a-Browder's theorem is said to hold for T if $\Delta_a(T) = p_{00}^a(T)$, or equivalently $\sigma_{uw}(T) = \sigma_{ub}(T)$. Note that Weyl's theorem for T entails Browder's theorem for T. Moreover, a-Browder's theorem for T entails Browder's theorem for T and the converse doesn't hold in general.

Recall [6, 8] that property (aw) is said to hold for T if $\Delta(T) = \pi_{00}^a(T)$, and property (R) holds for T if $p_{00}^a(T) = \pi_{00}(T)$.

The single valued extension property plays an important role in local spectral theory, see the recent monograph of Laursen and Neumamn [13] and Aiena [2]. In this article we shall consider the following local version of this property.

Let X be a complex Banach space and $T \in L(X)$. The operator T is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (abbrev. SVEP at λ_0), the only analytic function $f : D \to X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in D$ is the function $f \equiv 0$. An operator T is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$.

It is known that both Browder's theorem and a-Browder's theorem hold for T if T or T^* has SVEP. Precisely, we have that a-Browder's theorem holds for T if and only if T has SVEP at every $\lambda \notin \sigma_{uw}(T)$, and dually, a-Browder's theorem holds for T^* if and only if T^* has SVEP at every $\lambda \notin \sigma_{lw}(T)$, see [5, Theorem 2.3].

From the identity theorem for analytic function it easily follows that T, as well as its dual T^*, has SVEP at every point of the boundary of the spectrum $\sigma(T) = \sigma(T^*)$, so both T and T^* have SVEP at every isolated point of the spectrum.

Theorem[5, Theorem 1.2] *If $T \in L(X)$ and suppose that $\lambda_0 I - T \in \Phi_\pm(X)$. Then the following statements are equivalent:*
(i) *T has SVEP at λ_0;* (ii) *$p(T - \lambda_0 I) < \infty$;* (iii) *$\sigma_a(T)$ doesn't cluster at λ_0.*
Dually, if $\lambda_0 I - T \in \Phi_\pm(X)$, then the following statements are equivalent:
(iv) *T^* has SVEP at λ_0;* (v) *$q(T - \lambda_0 I) < \infty$;* (vi) *$\sigma_s(T)$ doesn't cluster at λ_0.*

A bounded operator T is said to be polaroid if every isolated point of $\sigma(T)$ is a pole of the resolvent of T. A bounded operator T is said to be hereditarily polaroid if every part of T is polaroid. T is said to be a-polaroid if every isolated point of $\sigma_a(T)$ is a pole of the resolvent of T. T is said to be a-isoloid if every isolated point of $\sigma_a(T)$ is an eigenvalue of T. T is said to be finite-isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of finite multiplicity.

In section 2, we introduce and study the new property (aR) in connection with Weyl type theorems. We prove that an operator T possessing property (aR) possesses property (R), but the converse is not true in general as shown by Example 2.4. We prove also that if T^* has SVEP at every $\lambda \notin \sigma_{uw}(T)$, then property (aR), property (aw), Weyl's theorem and a-Weyl's theorem are equivalent. In section 3, in Theorem 3.5 we prove that if $T \in L(X)$ and E is a nilpotent operator commuting with T, then T possesses property (aR) if and only if $T + E$ possesses property (aR). And we provide a condition under which the new property (aR) is preserved under commuting finite dimensional operator, we prove in Theorem 3.3 that if $\text{iso}\sigma_a(T) = \phi$ and K is a finite dimensional operator commuting with T, then $T + K$ satisfies property (aR).

2. Property (aR)

Definition 2.1. *An operator T is said to satisfy property (aR) if $\pi_{00}^a(T) = p_{00}(T)$.*

Lemma 2.2. [1] *Suppose that $T \in L(X)$. Then we have*
(i) *T satisfies Weyl's theorem if and only if Browder's theorem holds for T and $p_{00}(T) = \pi_{00}(T)$.*
(ii) *T satisfies a-Weyl's theorem if and only if a-Browder's theorem holds for T and $p_{00}^a(T) = \pi_{00}^a(T)$.*

Theorem 2.3. *Suppose that T satisfies property (aR). Then property (R) holds for T.*

Proof. Let $\lambda \in \pi_{00}(T)$. Then $\lambda \in \pi_{00}^a(T)$, since T satisfies property (aR), $\pi_{00}^a(T) = p_{00}(T)$, hence $\lambda \in p_{00}(T) \subseteq p_{00}^a(T)$, i.e., $\pi_{00}(T) \subseteq p_{00}^a(T)$. Conversely, let $\lambda \in p_{00}^a(T)$. Then $\lambda \in \pi_{00}^a(T)$, since T satisfies property (aR), $\pi_{00}^a(T) = p_{00}(T)$, hence $\lambda \in p_{00}(T) \subseteq \pi_{00}(T)$, i.e., $p_{00}^a(T) \subseteq \pi_{00}(T)$. Therefore, $p_{00}^a(T) = \pi_{00}(T)$, we have T satisfies property (R).

The following example shows that property (R) is weaker than property (aR).

Example 2.4. Let $R : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be the unilateral right shift operator defined by $R(x_1, x_2, \cdots) = (0, x_1, x_2, \cdots)$ for all $x = (x_1, x_2, \cdots) \in l^2(\mathbb{N})$ and $Q(x_1, x_2, \cdots) = (\frac{x_2}{2^2}, \frac{x_3}{2^3}, \cdots)$ for all $x = (x_1, x_2, \cdots) \in l^2(\mathbb{N})$. Define $T = R \oplus Q$. Then $\sigma(T) = D$, $\sigma_a(T) = \partial D \cup \{0\}$, where D denotes the closed unit disc and ∂D denotes the unit circle, and hence $p_{00}(T) = \pi_{00}(T) = \phi$, but $\pi_{00}^a(T) = \{0\}$, i.e., T doesn't satisfy property (aR). While T satisfies property (R) since $p_{00}^a(T) = \pi_{00}(T) = \phi$.

In the following theorem we give a condition for the equivalence of property (aR) and property (aw).

Theorem 2.5. *T satisfies property (aw) if and only if Browder's theorem holds for T and T has property (aR).*

Proof. If Browder's theorem holds for T and T has property (aR), then $\Delta(T) = p_{00}(T)$ and $\pi_{00}^a(T) = p_{00}(T)$, hence $\Delta(T) = \pi_{00}^a(T)$, i.e., T satisfies property (aw). Conversely, it is easy to prove property (aw) implies Browder's theorem by [8, Theorem 2.4, Theorem 3.5], i.e., $\Delta(T) = p_{00}(T)$, since T satisfies property (aw), $\pi_{00}^a(T) = \Delta(T)$, hence $\pi_{00}^a(T) = p_{00}(T)$, i.e., T has property (aR).

The following example shows that property (aR) is weaker than property (aw).

Example 2.6. Let $R : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be the unilateral right shift operator defined by $R(x_1, x_2, \cdots) = (0, x_1, x_2, \cdots)$ for all $x = (x_1, x_2, \cdots) \in l^2(\mathbb{N})$ and $L : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be the unilateral left shift operator defined by $L(x_1, x_2, \cdots) = (x_2, x_3, \cdots)$ for all $x = (x_1, x_2, \cdots) \in l^2(\mathbb{N})$. Define $T := R \oplus L$. Then $\sigma(T) = \sigma_a(T) = D$. It follows that $p_{00}(T) = \pi_{00}^a(T) = \phi$, then T satisfies property (aR). While T doesn't satisfy property (aw), since $0 \in \sigma(T) \backslash \sigma_w(T) \neq \phi = \pi_{00}^a(T)$.

The following example shows property (aR) for an operator is not transmitted to the dual T^*.

Example 2.7. Let $L : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be the unilateral left shift operator defined by $L(x_1, x_2, \cdots) = (x_2, x_3, \cdots)$ for all $x = (x_1, x_2, \cdots) \in l^2(\mathbb{N})$ and $Q(x_1, x_2, \cdots) = (0, x_2, x_3, \cdots)$ for all $x = (x_1, x_2, \cdots) \in l^2(\mathbb{N})$. Define $T := L \oplus Q$. Then $\sigma(T) = \sigma(T^*) = \sigma_a(T) = D$ and $\sigma_a(T^*) = \partial D \cup \{0\}$. It follows that $p_{00}(T) = \pi_{00}^a(T) = \phi$, then T satisfies property (aR). While T^* doesn't satisfy property (aR), since $0 \in \pi_{00}^a(T^*) \neq \phi = p_{00}(T^*)$.

Theorem 2.8. *Suppose that T satisfies property (aR). Then $p_{00}^a(T) = \pi_{00}^a(T) = p_{00}(T) = \pi_{00}(T)$.*

Proof. Observe that $p_{00}(T) \subseteq \pi_{00}(T) \subseteq \pi_{00}^a(T)$ holds for every operator T. As T satisfies property (aR), $\pi_{00}^a(T) = p_{00}(T)$, hence $p_{00}(T) = \pi_{00}(T) = \pi_{00}^a(T)$. As $p_{00}(T) \subseteq p_{00}^a(T) \subseteq \pi_{00}^a(T)$ holds for every operator T and $\pi_{00}^a(T) = p_{00}(T)$, then $p_{00}(T) = p_{00}^a(T) = \pi_{00}^a(T)$, i.e., $p_{00}(T) = p_{00}^a(T) = \pi_{00}^a(T) = \pi_{00}(T)$.

The following example shows neither of the two equalities $p_{00}^a(T) = \pi_{00}^a(T)$, $p_{00}(T) = \pi_{00}(T)$ can imply $p_{00}(T) = \pi_{00}^a(T)$.

Example 2.9. Let $R : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be the unilateral right shift operator defined by $R(x_1, x_2, \cdots) = (0, x_1, x_2, \cdots)$ for all $x = (x_1, x_2, \cdots) \in l^2(\mathbb{N})$ and $Q(x_1, x_2, \cdots) = (\frac{1}{2}x_1, x_2, x_3, \cdots)$ for all $x = (x_1, x_2, \cdots) \in l^2(\mathbb{N})$. Define $T = R \oplus Q$. Then $\sigma(T) = \sigma(T^*) = D$, $\sigma_a(T) = \partial D \cup \{\frac{1}{2}\}$ and $\sigma_{uw}(T) = \partial D$, and hence $p_{00}(T) = \pi_{00}(T) = \phi$. We show that T does not satisfy property (aR). Since T has SVEP at the points of ∂D, these points belong to the boundary of the spectrum, and T has SVEP at $\frac{1}{2}$, since this point is an isolated point of $\sigma_a(T)$. Hence, T has SVEP and a-Browder's theorem holds for T, i.e., $\sigma_{uw}(T) = \sigma_{ub}(T) = \partial D$. Observe that the operator T satisfies the equality $p_{00}^a(T) = \pi_{00}^a(T)$. Indeed, $\frac{1}{2}$ is an isolated point of $\sigma_a(T)$, and hence $\pi_{00}^a(T) = \{\frac{1}{2}\} = \sigma_a(T)\backslash\sigma_{ub}(T) = p_{00}^a(T)$. While T does not satisfy property (aR) since $\pi_{00}^a(T) = \{\frac{1}{2}\} \neq p_{00}(T)$.

As noted in Example 2.9 the condition $p_{00}^a(T) = \pi_{00}^a(T)$ is strictly weaker than property (aR). However, we have:

Theorem 2.10. *T satisfies property (aR) if and only if the following two conditions hold:*
(i) $\pi_{00}^a(T) \subseteq \text{iso}\sigma(T)$.
(ii) $p_{00}^a(T) = \pi_{00}^a(T)$.

Proof. If T satisfies property (aR), then $\pi_{00}^a(T) = p_{00}(T) \subseteq \text{iso}\sigma(T)$, and by Theorem 2.8 $p_{00}^a(T) = \pi_{00}^a(T)$. Conversely, since $p_{00}(T) \subseteq \pi_{00}^a(T)$ holds for every operator T, it suffices to show that $\pi_{00}^a(T) \subseteq p_{00}(T)$, suppose that both (i) and (ii) hold and let $\lambda \in \pi_{00}^a(T)$. Then $\lambda \in p_{00}^a(T)$ and $\lambda \in \text{iso}\sigma(T)$, hence $p(\lambda I - T) = q(\lambda I - T) < \infty$, and so $\lambda \in p_{00}(T)$.

The following example shows that a-Weyl's theorem does not entail property (aR).

Example 2.11. Let T be defined as in Example 2.9. As already observed, T does not satisfy property (aR). While T has SVEP and hence a-Browder's theorem holds for T, since $p_{00}^a(T) = \pi_{00}^a(T)$. By part (ii) of Lemma 2.2, then a-Weyl's theorem holds for T.

The following example shows that property (aR) does not entail a-Weyl's theorem.

Example 2.12. Let T be defined as in Example 2.6. We have $\alpha(T) = \beta(T) = 1$ and $p(T) = \infty$. Therefore, $0 \notin \sigma_w(T)$, while $0 \in \sigma_b(T)$, so Browder's theorem (and hence a-Weyl's theorem) does not hold for T. On the other hand, since $\sigma(T) = \sigma_a(T) = D$, we have $p_{00}(T) = \pi_{00}^a(T) = \phi$, and hence property (aR) holds for T.

Theorem 2.13. *Suppose that T satisfies both a-Browder's theorem and property (aR). Then T satisfies a-Weyl's theorem. Moreover, $\sigma_a(T)\backslash\sigma_{uw}(T) = p_{00}(T)$.*

Proof. Since T satisfies a-Browder's theorem and property (aR), $p_{00}^a(T) = \pi_{00}^a(T)$ by Theorem 2.8. Therefore, a-Weyl's theorem holds for T by part (ii) of Lemma 2.2, i.e. $\sigma_a(T)\backslash\sigma_{uw}(T) = \pi_{00}^a(T)$. Property (aR) implies $\sigma_a(T)\backslash\sigma_{uw}(T) = p_{00}(T)$.

In [7] an operator T is said to have property (b) if $\sigma_a(T)\backslash\sigma_{uw}(T) = p_{00}(T)$.

The following example shows that property (aR) does not entail property (b).

Example 2.14. Let T be defined as in Example 2.6. Then T satisfies property (aR), while property (b) does not hold for T, since $0 \in \sigma_a(T)\backslash\sigma_{uw}(T)$, while $p_{00}(T) = \phi$. This example also shows that without the assumption that T satisfies a-Browder's theorem, the result of Theorem 2.13 does not hold.

The following example shows that property (b) does not entail property (aR).

Example 2.15. Let $Q(x_1, x_2, \cdots) = (\frac{x_2}{2^2}, \frac{x_3}{2^3}, \cdots)$ for all $x = (x_1, x_2, \cdots) \in l^2(\mathbb{N})$. Clearly, Q is quasi-nilpotent and hence $\sigma(Q) = \sigma_a(Q) = \{0\}$ and $\alpha(Q) = 1$, we have $0 \in \pi_{00}^a(Q)$, $p_{00}(Q) = \phi$, it then follows that Q does not satisfy property (aR). On the other hand, Q has property (b) since $\sigma_a(Q)\backslash\sigma_{uw}(Q) = p_{00}(Q) = \phi$.

The next result shows that the equivalence of property (aR), property (aw), Weyl's theorem and a-Weyl's theorem is true whenever we assume that T^* has SVEP at the points $\lambda \notin \sigma_{uw}(T)$.

Theorem 2.16. *Suppose that T^* has SVEP at every $\lambda \notin \sigma_{uw}(T)$. Then the following statements are equivalent:*
(i) $\pi_{00}(T) = p_{00}(T)$;
(ii) $\pi_{00}^a(T) = p_{00}^a(T)$;
(iii) $\pi_{00}^a(T) = p_{00}(T)$.
Consequently, property (aR), property (aw), Weyl's theorem and a-Weyl's theorem are equivalent for T.

Proof. Since T^* has SVEP at every $\lambda \notin \sigma_{uw}(T)$, $\sigma(T) = \sigma_a(T)$, hence $\pi_{00}(T) = \pi_{00}^a(T)$. In the following we would show $p_{00}^a(T) = p_{00}(T)$, observe first that $p_{00}(T) \subseteq p_{00}^a(T)$ holds for every operator T. To show the opposite inclusion, let $\lambda \in p_{00}^a(T) = \sigma_a(T) \backslash \sigma_{ub}(T)$. Then $T - \lambda I \in B_+(X)$, and hence both $\alpha(\lambda I - T)$ and $p(\lambda I - T)$ are finite. But $\sigma_{uw}(T) \subseteq \sigma_{ub}(T)$ holds for every operator T, thus $\lambda \notin \sigma_{uw}(T)$ and the SVEP of T^* at λ implies that $q(\lambda I - T) < \infty$, therefore, by [2, Theorem 3.4], we have $\alpha(\lambda I - T) = \beta(\lambda I - T) < \infty$, so $\lambda \in p_{00}(T)$. Therefore, $p_{00}(T) = p_{00}^a(T)$. From which the equivalence of (i), (ii) and (iii) easily follows. To show the last statement observe that the SVEP of T^* at the points $\lambda \notin \sigma_{uw}(T)$ entails that a-Browder's theorem (and hence Browder's theorem) holds for T, see [5, Theorem 2.3]. By Lemma 2.2 and Theorem 2.5, then property (aR), property (aw), Weyl's theorem and a-Weyl's theorem are equivalent for T.

Dually, we have

Corollary 2.17. *Suppose that T has SVEP at every $\lambda \notin \sigma_{lw}(T)$. Then the following statements are equivalent:*
(i) $\pi_{00}(T^*) = p_{00}(T^*)$;
(ii) $\pi_{00}^a(T^*) = p_{00}^a(T^*)$;
(iii) $\pi_{00}^a(T^*) = p_{00}(T^*)$.
Consequently, property (aR), *property* (aw), *Weyl's theorem and a-Weyl's theorem are equivalent for* T^*.

Proof. The proof is similar to Theorem 2.16.

Theorem 2.18. *Suppose that T is a-polaroid. Then T satisfies property* (aR).

Proof. Since $p_{00}(T) \subseteq \pi_{00}^a(T)$ holds for every operator T. To show the opposite inclusion, let $\lambda \in \pi_{00}^a(T)$. Then λ is an isolated point of $\sigma_a(T)$, λ is a pole of the resolvent of T and $\alpha(T - \lambda) < \infty$, hence $\lambda \in p_{00}(T)$, i.e., T satisfies property (aR).

Corollary 2.19. [6] *Suppose that T is a-polaroid. Then T satisfies property* (R).

The next example shows that under a weaker condition of being polaroid the result of Theorem 2.18 does not hold.

Example 2.20. Let $R : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be the unilateral right shift operator defined by $R(x_1, x_2, \cdots) = (0, x_1, x_2, \cdots)$ for all $x = (x_1, x_2, \cdots) \in l^2(\mathbb{N})$ and $Q(x_1, x_2, \cdots) = (\frac{x_2}{2^2}, \frac{x_3}{2^3}, \cdots)$ for all $x = (x_1, x_2, \cdots) \in l^2(\mathbb{N})$. Define $T := R \oplus Q$. Then $\sigma(T) = D$, it follows that iso$\sigma(T) = p_{00}(T) = \phi$. Therefore, T is polaroid. Moreover, $\sigma_a(T) = \partial D \cup \{0\}$ and $\pi_{00}^a(T) = \{0\}$, and hence $\pi_{00}^a(T) \neq p_{00}(T)$, thus T does not satisfy property (aR).

From the proof of Theorem 2.16 we know that if T^* has SVEP, then $\sigma(T) = \sigma_a(T)$. Therefore if T^* has SVEP, then T is a-polaroid $\Leftrightarrow$ T is polaroid.

Corollary 2.21. *Suppose that T is polaroid and T^* has SVEP. Then T satisfies property* (aR).

Note that the result of Corollary 2.21 does not hold if we replace the SVEP for T^* by the SVEP for T.

Example 2.22. Let T be defined as in Example 2.20. Then T has SVEP and is polaroid, while T does not satisfy property (aR).

3. Property (aR) under Perturbations

Theorem 3.1. [10] *Suppose T is a-isoloid and satisfies a-Weyl's theorem. Then T+K satisfies a-Weyl's theorem for every finite-dimensional operator K commuting with T.*

The following example shows that an analogous result of Theorem 3.1 does not hold for property (aR), even with the class of a-isoloid operators.

Example 3.2. Let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be defined by

$$T(x_1, x_2, \cdots) = (2x_1, 2x_2, 0, x_3, x_4, \cdots) \text{ for all } x = (x_1, x_2, \cdots) \in l^2(\mathbb{N})$$

and

$$K(x_1, x_2, \cdots) = (-2x_1, -2x_2, 0, 0, 0, \cdots) \text{ for all } x = (x_1, x_2, \cdots) \in l^2(\mathbb{N}).$$

Then K is a finite-dimensional operator, $KT = TK$, $\sigma(T) = D \cup \{2\}$ and $\sigma_a(T) = \partial D \cup \{2\}$, it follows that $\pi^a_{00}(T) = p_{00}(T) = \{2\}$. Therefore, T is a-isoloid operators, and satisfies property (aR). While $\sigma(T + K) = D$ and $\sigma_a(T + K) = \partial D \cup \{0\}$, it follows that $p_{00}(T + K) = \text{iso}\sigma(T + K) = \phi \neq \{0\} = \pi^a_{00}(T + K)$. Therefore, $T + K$ does not satisfy property (aR).

Theorem 3.3. *Suppose $T \in L(X)$ and $\text{iso}\sigma_a(T) = \phi$. If K is a finite dimensional operator commuting with T, then T+K satisfies property* (aR).

Proof. Since $\text{iso}\sigma_a(T) = \phi$ and K is a finite dimensional operator commuting with T, by the proof of [3, Theorem 2.8], $\sigma_a(T) = \sigma_a(T + K)$, then $\text{iso}\sigma_a(T + K) = \phi$. Since $\text{iso}\sigma(T + K) \subseteq \text{iso}\sigma_a(T + K)$, $\text{iso}\sigma(T + K) = \phi$. It follows that $p_{00}(T + K) = \pi^a_{00}(T + K) = \phi$, i.e., $T + K$ satisfies property (aR).

Corollary 3.4. *Suppose $T \in L(X)$ and $\text{iso}\sigma_a(T) = \phi$. If K is a finite dimensional operator commuting with T, then T+K satisfies property (R).*

The next result shows that property (aR) for T is transmitted to $T + E$ in the case where E is a nilpotent operator which commutes with T. Recall first that the equality $\sigma_a(T) = \sigma_a(T + Q)$ holds for every quasi-nilpotent operator Q which commutes with T.

Theorem 3.5. *Suppose $T \in L(X)$ and let $E \in L(X)$ be a nilpotent operator which commutes with T. Then we have:*
(i) $\pi^a_{00}(T + E) = \pi^a_{00}(T)$.
(ii) *T satisfies property* (aR) *if and only if $T + E$ satisfies property* (aR).
(iii) *If T is a-polaroid, then $T + E$ satisfies property* (aR).

Proof. (i) Let $\lambda \in \pi^a_{00}(T + E)$. We can assume $\lambda = 0$. Clearly, $0 \in \text{iso}\sigma_a(T + E) = \text{iso}\sigma_a(T)$. Let $p \in \mathbb{N}$ be such that $E^p = 0$. If $x \in N(T + E)$, then $T^p x = (-1)^p E^p x = 0$, thus $N(T + E) \subseteq N(T^p)$, since by assumption $\alpha(T + E) > 0$, it then follows that $\alpha(T^p) > 0$ and this obviously implies that $\alpha(T) > 0$. By assumption we also have $\alpha(T + E) < \infty$ and this implies that $\alpha(T + E)^p < \infty$. It is easily seen that if $x \in N(T)$, then $(T + E)^p x = E^p x = 0$, so $N(T) \subseteq N(T + E)^p$ and hence $\alpha(T) < \infty$. Therefore, $0 \in \pi^a_{00}(T)$ and consequently $\pi^a_{00}(T + E) \subseteq \pi^a_{00}(T)$. $\pi^a_{00}(T) \subseteq \pi^a_{00}(T + E)$ follows by symmetry.

(ii) Suppose that T has property (aR). Then $\pi^a_{00}(T + E) = \pi^a_{00}(T) = \sigma(T)\backslash\sigma_b(T) = \sigma(T + E)\backslash\sigma_b(T + E) = p_{00}(T + E)$, therefore $T + E$ has property (aR). The converse follows by symmetry.

(iii) Obviously, by part (ii), since T satisfies property (aR) by Theorem 2.18.

This example shows that the commutativity hypothesis in (ii) of Theorem 3.5 is essential.

Example 3.6. Let $Q : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be defined by

$$Q(x_1, x_2, \cdots) = \left(0, 0, \frac{x_1}{2}, \frac{x_2}{2^2}, \frac{x_3}{2^3}, \cdots\right) \text{ for all } x = (x_1, x_2, \cdots) \in l^2(\mathbb{N})$$

and

$$E(x_1, x_2, \cdots) = \left(0, 0, -\frac{x_1}{2}, 0, 0, \cdots\right) \text{ for all } x = (x_1, x_2, \cdots) \in l^2(\mathbb{N}).$$

Clearly E is a nilpotent operator and $p_{00}(Q) = \pi^a_{00}(Q) = \phi$, i.e., Q satisfies property (aR). While $p_{00}(Q + E) = \phi$ and $\pi^a_{00}(Q + E) = \{0\}$, it follows that $p_{00}(Q + E) \neq \pi^a_{00}(Q + E)$, i.e., $Q + E$ does not satisfy property (aR).

The previous theorem does not extend to commuting quasi-nilpotent operators as shown by the following example.

Example 3.7. Let $Q : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be defined by $Q(x_1, x_2, \cdots) = (\frac{x_2}{2^2}, \frac{x_3}{2^3}, \frac{x_4}{2^4}, \cdots)$ for all $x = (x_1, x_2, \cdots) \in l^2(\mathbb{N})$ and $T = 0$. Clearly T satisfies property (aR). While Q is quasi-nilpotent and $TQ = QT$, so $\sigma(Q) = \sigma_b(Q) = \{0\}$ and hence $\{0\} = \pi^a_{00}(Q) \neq \sigma(Q)\backslash\sigma_b(Q) = \phi$, i.e., $T + Q = Q$ does not satisfy property (aR).

Theorem 3.8. *Suppose T is a-polaroid and finite-isoloid, Q is a quasi-nilpotent operator which commutes with T. Then $T + Q$ has property (aR).*

Proof. Clearly by the proof of [3, Theorem 2.13].

In the case of injective quasi-nilpotent perturbation, we have a very simple situation:

Theorem 3.9. *Suppose that for $T \in L(X)$ there exists an injective quasi-nilpotent operator Q commuting with T. Then both T and $T+Q$ satisfy property (aR).*

Proof. It's evident that $\pi^a_{00}(T)$ is empty by [4, Lemma 3.9], since $p_{00}(T) \subseteq \pi^a_{00}(T)$, $p_{00}(T) = \phi$, it follows that $p_{00}(T) = \pi^a_{00}(T) = \phi$, i.e., T satisfies property (aR). Property (aR) for $T+Q$ is clear, since also $T+Q$ commutes with Q.

In Theorem 3.9, the condition quasi-nilpotent can't be replaced by the condition compact.

Example 3.10. Let $U : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be defined by $U(x_1, x_2, \cdots) = (0, \frac{x_2}{2^2}, \frac{x_3}{2^3}, \cdots)$ for all $x = (x_1, x_2, \cdots) \in l^2(\mathbb{N})$ and $V(x_1, x_2, \cdots) = (x_1, -\frac{x_2}{2^2}, -\frac{x_3}{2^3}, \cdots)$ for all $x = (x_1, x_2, \cdots) \in l^2(\mathbb{N})$. Define $T = U \oplus I$ and $K = V \oplus Q$, where Q is an injective compact quasi-nilpotent operator. Clearly $\sigma(T) = \sigma_a(T) = \{\frac{1}{2^n} : n = 2, 3, \cdots\} \cup \{0, 1\}$ and $\sigma_b(T) = \{0, 1\}$, it follows that $p_{00}(T) = \sigma(T)\backslash\sigma_b(T) = \{\frac{1}{2^n} : n = 2, 3, \cdots\} = \pi^a_{00}(T)$, thus property (aR) holds for T. Note that K is an injective compact operator, $KT = TK$ and $\sigma(T + K) = \sigma_b(T + K) = \{0, 1\}$, so $p_{00}(T + K) = \phi$, while $\pi^a_{00}(T + K) = \{1\}$, it follows that $T + K$ does not satisfy property (aR).

Recall that a bounded operator T is said to be algebraic if there exists a non-constant polynomial h such that $h(T) = 0$. Trivially, every nilpotent operator is algebraic. If for some $n \in \mathbb{N}$, K^n is a finite dimensional operator, then K is an algebraic operator. And every algebraic operator has a finite spectrum.

Theorem 3.11. *Suppose $T \in L(X)$ and $K \in L(X)$ is an algebraic operator which commutes with T.*
(i) *If T is hereditarily polaroid and has SVEP, then $T^* + K^*$ satisfies property (aR).*
(ii) *If T^* is hereditarily polaroid and has SVEP, then $T+K$ satisfies property (aR).*

Proof. (i) Since $T^* + K^*$ is a-polaroid by the proof of [3, Theorem 2.15], Property (aR) for $T^* + K^*$ follows from Theorem 2.18.

(ii) The proof is similar to (i).

In the following theorem, recall that $H(\sigma(T))$ is the space of functions analytic in an open neighborhood of $\sigma(T)$.

Theorem 3.12. *Suppose $T \in L(X)$ and $K \in L(X)$ is an algebraic operator which commutes with T.*
(i) *If T is hereditarily polaroid and has SVEP, then $f(T^* + K^*)$ satisfies property (aR) for all $f \in H(\sigma(T))$.*
(ii) *If T^* is hereditarily polaroid and has SVEP, then $f(T + K)$ satisfies property (aR) for all $f \in H(\sigma(T))$.*

Proof. (i) Since $f(T^* + K^*)$ is a-polaroid by the proof of [3, Theorem 2.17], Property (aR) for $f(T^* + K^*)$ follows from Theorem 2.18.

(ii) The proof of (ii) is analogous.

References

[1] P. Aiena, Classes of operators satisfying a-Weyl's theorem, Studia Math. 169(2005), 105–122.
[2] P. Aiena, Fredholm and Local Spectral Theory, with Application to Multipliers, Kluwer Academic Publishers, Dordrecht, 2004.
[3] P. Aiena, E. Aponte, J.R. Guillén, P. Peña, Property (R) under perturbations, Mediterr. J. Math. 10(2013), 367–382.
[4] P. Aiena, M.T. Biondi, F. Villafañe, Property (w) and perturbations III, J. Math. Anal. Appl. 353(2009), 205–214.
[5] P. Aiena, C. Carpintero, E. Rosas, Some characterization of operators satisfying a-Browder theorem, J. Math. Anal. Appl. 311(2005), 530–544.
[6] P. Aiena, J.R. Guillén, P. Peña, Property (R) for bounded linear operators, Mediterr. J. Math. 5(2011), 491–508.
[7] M. Berkani, H. Zariouh, Extended Weyl type theorems, Math. Bohemica 134(2009), 369–378.
[8] M. Berkani, H. Zariouh, New extended Weyl type theorems, Mat. Vesnik 62(2010), 145–154.
[9] L.A. Coburn, Weyl's theorem for nonnormal operators, Michigan Math. J. 13(1966), 285–288.
[10] D.S. Djordjević, Operators obeying a-Weyl's theorem, Publ. Math. Debrecen 55(3-4)(1999), 283–298.
[11] R. Harte, W.Y. Lee, Another note on Weyl's theorem, Trans. Amer. Math. Soc. 349(1997), 2115–2124.
[12] H.G. Heuser, Functional Analysis, John Wiley & Sons, New York, 1982.
[13] K.B. Laursen, M.M. Neumann, Introduction to Local Spectral Theory, Clarendon Press, Oxford, 2000.
[14] V. Rakočević, Operators obeying a-Weyl's theorem, Rev. Roumaine Math. Pures Appl. 34(1989), 915–919.