Filomat 29:1 (2015), 1–5
DOI 10.2298/FIL1501001M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Maximal Ideals in Some F-Algebras of Holomorphic Functions

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Abstract. For $1 < p < \infty$, the Privalov class N^p consists of all holomorphic functions f on the open unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$ such that

$$\sup_{0\le r<1}\int_0^{2\pi}(\log^+|f(re^{i\theta})|)^p\,\frac{d\theta}{2\pi}<+\infty.$$

M. Stoll [16] showed that the space N^p with the topology given by the metric d_p defined as

$$d_p(f,g)=\Big(\int_0^{2\pi}\Big(\log(1+|f^*(e^{i\theta})-g^*(e^{i\theta})|)\Big)^p\,\frac{d\theta}{2\pi}\Big)^{1/p},\qquad f,g\in N^p,$$

becomes an F-algebra. Since the map $f\mapsto d_p(f,0)$ ($f\in N^p$) is not a norm, N^p is not a Banach algebra.

Here we investigate the structure of maximal ideals of the algebras N^p ($1 < p < \infty$). We also give a complete characterization of multiplicative linear functionals on the spaces N^p. As an application, we show that there exists a maximal ideal of N^p which is not the kernel of a multiplicative continuous linear functional on N^p.

1. Introduction and Preliminaries

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$ and let $\mathbb{T}$ denote the boundary of $\mathbb{D}$. Let $L^q(\mathbb{T})$ ($0 < q \le \infty$) be the familiar *Lebesgue space* on the unit circle $\mathbb{T}$. For $1 < p < \infty$, the *Privalov class* N^p consists of all holomorphic functions f on the disk $\mathbb{D}$ for which

$$\sup_{0\le r<1}\int_0^{2\pi}(\log^+|f(re^{i\theta})|)^p\,\frac{d\theta}{2\pi}<+\infty,\tag{1}$$

where $\log^+|a|=\max\{0,\log|a|\}$. These classes were firstly considered in 1941 by I. I. Privalov [15, p. 93] in the first edition of his monograph, where N^p is denoted as A_p.

Notice that the condition (1) with $p = 1$ defines the *Nevanlinna class* N of holomorphic functions in $\mathbb{D}$ (see, e.g., [1]). Recall that the *Smirnov class* N^+ (see, e.g., [1, p. 26]) consists of those functions $f\in N$ such that

$$\lim_{r\to1}\int_0^{2\pi}\log^+|f(re^{i\theta})|\,\frac{d\theta}{2\pi}=\int_0^{2\pi}\log^+|f^*(e^{i\theta})|\,\frac{d\theta}{2\pi}<+\infty,$$

2010 *Mathematics Subject Classification*. Primary 30H50, 46J20; Secondary 30H10, 30H15, 46J15, 46E10.
Keywords. Privalov space N^p; F-algebra; maximal ideal; multiplicative linear functional.
Received: 1 November 2014; Accepted: 21 November 2014
Communicated by Dragan S. Djordjević
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where f^* is the boundary function of f on $\mathbb{T}$, i.e.,

$$f^*(e^{i\theta}) = \lim_{r\to 1-} f(re^{i\theta})$$

is the *radial limit* of f which exists for almost every $e^{i\theta}$.

Furthermore, the *Hardy space* H^q $(0 < q \le \infty)$ consists of all functions f, holomorphic in $\mathbb{D}$, which satisfy

$$\sup_{0\le r<1} \int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^q \frac{d\theta}{2\pi} < \infty$$

if $0 < q < \infty$, and which are bounded when $q = \infty$:

$$\sup_{z\in\mathbb{D}} |f(z)| < \infty.$$

It is known (see [9] and [14]) that

$$N^r \subset N^p \ (r > p), \quad \bigcup_{q>0} H^q \subset \cap_{p>1} N^p, \quad \text{and} \quad \bigcup_{p>1} N^p \subset N^+ \subset N,$$

where the above containment relations are proper.

In 1977 M. Stoll [16] (with the notation $(\log^+ H)^\alpha$ for N^p) proved the following result.

Theorem A (M. Stoll [16, Theorem 4.2]). *The space N^p with the topology given by the metric d_p defined by*

$$d_p(f,g) = \Big(\int_0^{2\pi} \left(\log(1 + |f^*(e^{i\theta}) - g^*(e^{i\theta})|)\right)^p \frac{d\theta}{2\pi} \Big)^{1/p}, \quad f, g \in N^p, \tag{2}$$

becomes an F-algebra, that is, an F-space (*a complete metrizable topological vector space with the invariant metric*) *in which multiplication is continuous.*

Since the function $f \mapsto d_p(f,0)$ defined for $f \in N^p$ is not a norm, the Privalov space N^p is not a *Banach algebra*.

The function $d_1 = d$ defined on the Smirnov class N^+ by (2) with $p = 1$ induces the metric topology on N^+. In 1973 N. Yanagihara [17] showed that under this topology, N^+ is an F-space.

The study of the spaces N^p $(1 < p < \infty)$ was continued in 1977 by M. Stoll [16] (with the notation $(\log^+ H)^\alpha$ in [16]). Further, the topological and functional properties of these spaces were studied by C. M. Eoff ([2] and [3]), N. Mochizuki [14], Y. Iida and N. Mochizuki [5], Y. Matsugu [6] and in author's works [7]–[13]; typically, the notation varied and Privalov was mentioned in [6], [11], [12] and [13]. In particular, the functional, topological and algebraic properties of the spaces N^p and their Fréchet envelopes were recently investigated in [8], [11] and [13].

It is well known (see, e.g., [1, p. 26]) that every function $f \in N^+$ admits a unique factorization of the form

$$f(z) = B(z)S_\mu(z)F(z), \quad z \in \mathbb{D}, \tag{3}$$

where $B(z)$ is the *Blaschke product* with respect to zeros $\{z_n\} \subset \mathbb{D}$ of f (the set $\{z_n\}$ may be finite), S_μ is a *singular inner function*, and F is an *outer function* for N^+, i.e.,

$$B(z) = z^m \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \bar{z}_n z}, \quad z \in \mathbb{D}, \tag{4}$$

with $\sum_{n=1}^{\infty} \left(1 - |z_n|\right) < \infty$, m a nonnegative integer,

$$S_\mu(z) = \exp\left(- \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \, d\mu(t) \right) \tag{5}$$

with positive singular measure $d\mu$, and

$$F(z) = \lambda \exp\left(\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log\left|f^*\left(e^{it}\right)\right| \frac{dt}{2\pi}\right), \tag{6}$$

where λ is a complex constant such that $|\lambda| = 1$ and $\log|f^*(e^{it})| \in L^1(\mathbb{T})$.

Recall that a function I of the form

$$I(z) = B(z)S_\mu(z), \quad z \in \mathbb{D},$$

is called an *inner function*, and I is a bounded holomorphic function on $\mathbb{D}$ whose boundary values $I^*(e^{i\theta})$ have modulus 1 for almost every $e^{i\theta} \in \mathbb{T}$.

The *inner-outer factorization theorem* for the classes N^p was given and proved by Privalov [15] as follows.

Theorem B (I. I. Privalov [15, pp. 98-100]; also see C. M. Eoff [3]). *A function $f \in N^+$ uniquely factorized by* (3) *belongs to the class N^p if and only if $\log^+|F^*(e^{i\theta})| \in L^p(\mathbb{T})$.*

In 1999 R. Meštrović and A. V. Subbotin [12] characterized the *topological dual space* of N^p (the set of all linear functionals that are continuous with respect to the metric topology d_p) as follows.

Theorem C (R. Meštrović and A. V. Subbotin [12, Theorem 2]). *If Φ is a continuous linear functional on N^p, then there exists a sequence $(b_n)_{n=0}^\infty$ of complex numbers with $b_n = O\left(\exp\left(-cn^{1/(p+1)}\right)\right)$ for some $c > 0$, such that*

$$\Phi(f) = \sum_{n=0}^\infty a_n b_n, \tag{7}$$

where $f(z) = \sum_{n=0}^\infty a_n z^n \in N^p$, with convergence being absolute. Conversely, if $(b_n)_{n=0}^\infty$ is a sequence of complex numbers for which

$$b_n = O\left(\exp\left(-cn^{1/(p+1)}\right)\right), \tag{8}$$

then (7) *defines a continuous linear functional on N^p.*

As every space N^p $(1 < p < \infty)$ becomes an F-algebra, and in particular, a topological algebra, it can be of interest to investigate the ideal structure of N^p. Related problems are closely related to those on Banach algebras. For example, the general theory of Banach algebras gives the following information: every maximal ideal of a function algebra A over $\mathbb{C}$ is the kernel of an element of the space M of all non-zero homomorphisms of A into $\mathbb{C}$, and conversely. It is also well known (see, e.g., [4]) that in a Banach algebra every nontrivial multiplicative linear functional is continuous and that every maximal ideal is the kernel of a multiplicative linear functional.

In the next section we show that for any fixed $\lambda \in \mathbb{D}$, the point evaluation $\gamma_\lambda(f) := f(\lambda)$, $f \in N^p$, is a multiplicative continuous linear functional on the space N^p (Proposition 2.1). Moreover, we prove that the set $\mathcal{M}_\lambda = \{f \in N^p : f(\lambda) = 0\}$ is a closed maximal ideal of N^p for every $\lambda \in \mathbb{D}$ (Proposition 2.2). Furthermore, we give a complete characterization of multiplicative linear functionals on the space N^p (Theorem 2.3). In contrast to the Banach algebras we show that there exists a maximal ideal $\mathcal{M}$ of N^p such that $\mathcal{M} \neq \gamma_\lambda$ for all $\lambda \in \mathbb{D}$ (Theorem 2.4).

2. Maximal ideals in the algebras N^p $(1 < p < \infty)$

For an arbitrary point $\lambda \in \mathbb{D}$, the *point evaluation* at λ is the functional γ_λ on the space N^p defined as

$$\gamma_\lambda(f) = f(\lambda), \quad f \in N^p. \tag{9}$$

Proposition 2.1. *For each $\lambda \in \mathbb{D}$ the point evaluation γ_λ defined by* (9) *is a continuous multiplicative linear functional on N^p.*

Proof. Obviously, γ_λ is a linear and multiplicative functional on N^p. It remains to show that γ_λ is continuous. Notice that the sequence $(b_n)_{n=0}^{\infty}$ with $b_n = \lambda^n$ for all $n = 0, 1, 2, \dots$ obviously satisfies the condition (8) from Theorem C. Hence, by Theorem C, the linear functional Φ defined on N^p as

$$\Phi(f) := \sum_{n=0}^{\infty} a_n\lambda^n = f(\lambda) := \gamma_\lambda(f), \quad f \in N^p,$$

is continuous on N^p with respect to the metric topogy d_p given by (2). □

For $\lambda \in \mathbb{D}$, we define

$$\mathcal{M}_\lambda = \{f \in N^p : f(\lambda) = 0\}. \tag{10}$$

Proposition 2.2. *The set $\mathcal{M}_\lambda$ defined by* (10) *is a closed maximal ideal of N^p for all $\lambda \in \mathbb{D}$.*

Proof. By Proposition 2.1, γ_λ is a continuous linear functional on N^p. From this and the fact that $\mathcal{M}_\lambda$ is the kernel of a continuous linear functional on N^p it follows that $\mathcal{M}_\lambda$ is a closed maximal ideal of N^p. □

The following result characterizes multiplicative linear functionals on the space N^p.

Theorem 2.3. *Let γ be a nontrivial multiplicative linear functional on N^p. Then there exists $\lambda \in \mathbb{D}$ such that $\gamma(f) = f(\lambda)$ for every $f \in N^p$. Consequently, γ is a continuous map.*

Proof. Take $\lambda = \gamma(z)$. Then $\gamma(z - \lambda) = 0$. If we suppose that $\lambda \notin \mathbb{D}$, then $z \mapsto 1/(z - \lambda)$ $(z \in \mathbb{D})$ is a bounded function on the closed unit disk $\overline{\mathbb{D}} : |z| \leq 1$, and hence $z \mapsto z - \lambda$ $(z \in \mathbb{D})$ is an invertible element of the algebra N^p. However, for each invertible element $f \in N^p$ we have $1 = \gamma(1) = \gamma(f)\gamma(f^{-1})$, which implies that $\gamma(f) \neq 0$. In particular, it follows that $\gamma(z - \lambda) \neq 0$. This contradiction shows that must be $\lambda \in \mathbb{D}$.

Now consider the set $(z - \lambda)N^p = \{(z - \lambda)f(z) : f \in N^p\}$. If we suppose that $f(\lambda) = 0$ for some $f \in N^p$, then by Theorem B, the function g defined by $g(z) = f(z)/(z - \lambda)$ $(z \in \mathbb{D})$ belongs to the class N^p. Therefore, we have

$$\mathcal{M}_\lambda = (z - \lambda)N^p \subset \ker\gamma, \tag{11}$$

where $\ker\gamma$ is the kernel of the functional γ. By Proposition 2.2, $\mathcal{M}_\lambda$ is a closed maximal ideal of N^p. Hence, by (11) we conclude that $\mathcal{M}_\lambda = \ker\gamma$. Moreover, γ is continuous and $\gamma(f) = f(\lambda)$ for all $f \in N^p$. This completes the proof of the theorem. □

In contrast to the Banach algebras in which every maximal ideal is the kernel of a multiplicative linear functional (see e.g., [4]), the following assertion shows that this is not true for the F-algebras N^p $(1 < p < \infty)$.

Theorem 2.4. *There exists a maximal ideal $\mathcal{M}$ of N^p which is not the kernel of a multiplicative linear functional on N^p.*

Proof. Let

$$S_\mu(z) = \exp\left(-\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z}\, d\mu(t)\right), \quad z \in \mathbb{D},$$

be a singular inner function. By Theorem B, S_μ is not an invertible element of the algebra N^p. Therefore, $1 \notin S_\mu N^p := \{S_\mu f : f \in N^p\}$, whence it follows that $S_\mu N^p$ is a proper ideal of N^p. By Zorn's lemma, there exists a maximal ideal $\mathcal{M}$ which contains the ideal $S_\mu N^p$. If we suppose that $\mathcal{M}$ is the kernel of a multiplicative linear functional on N^p, then by Theorem 2.3, $\mathcal{M} = \mathcal{M}_\lambda$ for some $\lambda \in \mathbb{D}$. Therefore, $(S_\mu f)(\lambda) = 0$ for each $f \in N^p$. The previous equality with $f(z) = 1$ $(z \in \mathbb{D})$ yields $S_\mu(\lambda) = 0$. Hovewer, $S_\mu(\lambda) \neq 0$ for each $\lambda \in \mathbb{D}$. This contradiction shows that must be $\mathcal{M} \neq \mathcal{M}_\lambda$ for each $\lambda \in \mathbb{D}$. This completes the proof of the theorem. □

Corollary 2.5. *Does not exist a norm defined on the space N^p which induces the same topolgy on N^p as the metric topology d_p, such that N^p is a Banach algebra with respect to this norm.*

Proof. The assertion immediately follows from Theorem 2.4 and the well known fact that in Banach algebras every maximal ideal is the kernel of a multiplicative linear functional (see, e.g., [4]). □

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