Filomat 29:1 (2015), 89–97
DOI 10.2298/FIL1501089S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Characterization of Topologically Transitive Attractors for Analytic Plane Flows

Martin Shoptrajanov[a], Nikita Shekutkovski[a]

[a] *University of St. Cyril and Methodius, Faculty of Natural Sciences and Mathematics, Institute of Mathematics, R. Macedonia*

Abstract. We give necessary conditions for a set to be topologically transitive attractor of an analytic plane flow using topological characterization of ω-limit sets and the concept of upper semi-continuity of multi valued maps.

1. Introduction to topological transitivity

Topological dynamics traditionally employees techniques which explore quality properties up to homeomorphisms neglecting the use of smooth structure of the dynamical systems under investigations. The concept of topological transitivity goes back to Birkoff and is one of the basic features when chaotical systems are involved (in the sense of Devaney). Although the local structure of topologically transitive dynamical system fulfills certain conditions, there is variety of such systems. Some of them have dense periodic points while some of them may be minimal and without any periodic points. As a motivation for the notion of topological transitivity one may think of a real physical system, where a state is never given or measured exactly, but always up to certain error. So instead of points one should study small open subsets of the phase space and describe how they move in that space. Intuitively, a topologically transitive map f has points which eventually move under iteration from one arbitrary small neighborhood to any other(discrete version). On the other hand some authors working with the notion of topological transitivity are using definitions of this notion which are different and in general not equivalent to this one. For instance a flow φ is called topologically transitive if there is a point $x \in X$ whose trajectory is dense in X. Günther and Segal for example in the paper [1] using this concept raised the problems involving characterization of topologically transitive attractors on arbitrary manifolds for continuous as well as for discrete dynamical systems. We shall give partial answer to this problem when the phase space is actually the plane.

Let (X, d) be a given metric space. By a flow we mean a continuous group action by the reals on X. For a given flow $\varphi : X \times \mathbb{R} \to X$ we say that a set $M \subseteq X$ is *positively invariant* if $\varphi(M, [0, \infty)) \subseteq M$. If we replace the set of positive reals with the set of reals we obtain the notion of *invariant set*. A set M is *minimal* if it is nonempty, closed, invariant and none of his true subsets have this property. For $x \in X$, $\gamma(x)$, $\gamma^+(x)$, $\gamma^-(x)$ denote respectively the trajectory, the positive semi-trajectory and the negative semi-trajectory through x. For $x \in X$, $T(x, \varepsilon)$ denotes open ε- ball centered at x. A point x is said to be *periodic* if $T \neq 0$ exists such that

2010 *Mathematics Subject Classification*. Primary 54H20, 54C56; Secondary 37B20, 37B25.
Keywords. Flow; Topologically Transitive set; Attractor; Shape; Index Pair.
Received: 8 September 2014; 12 November 2014
Communicated by Dragan Djurčić

Email addresses: martin@pmf.ukim.mk (Martin Shoptrajanov), nikita@pmf.ukim.mk (Nikita Shekutkovski)

$\varphi(x, T) = x$. If $\varphi(x, t) = x$, for all $t \in \mathbb{R}$, the point x is called *critical point*. If a point x is periodic but not critical, then the smallest positive period of x is called its *fundamental period*. Trajectory of periodic points are called *closed trajectories*. Trajectory of critical points are called *degenerate periodic trajectories*. The following characterization of minimal sets from [2] will be useful in the sequel:

Theorem 1.1. *A set M is minimal iff $\overline{\gamma(x)} = M$, for every $x \in M$.*

One of the important problems in dynamical systems concerns the asymptotic behavior of trajectories as time goes to plus or minus infinity. Limit sets are fundamental tools for this problem. Recall that positive limit set for arbitrary subset $N \subseteq X$ is the following set: $\omega(N) = \{x \in X | \exists x_n \in N, t_n \to \infty, \varphi(x_n, t_n) \to x\}$. Analogously, negative limit set for arbitrary subset $N \subseteq X$ is the set: $\alpha(N) = \{x \in X | \exists x_n \in N, t_n \to -\infty, \varphi(x_n, t_n) \to x\}$. Let us note that for $N = \{x\}$, $\omega(N)$, $\alpha(N)$ are just the usual $\omega(x)$, $\alpha(x)$ limit sets from [2].

Definition 1.2. *An ω- limit point of a point x is an element from $\omega(x)$.*

The following results are from [3] and [2]

Theorem 1.3. *The limit sets are closed and invariant.*

Theorem 1.4. *For arbitrary $x \in X$, $\overline{\gamma(x)} = \alpha(x) \cup \gamma(x) \cup \omega(x)$.*

We are ready to introduce the notion of attractor according to Conley which will be used in the sequel:

Definition 1.5. *A compact invariant set $M \subseteq X$ is called an attractor if it admits a neighborhood U such that $\omega(U) = M$. Analogous, a compact invariant set $M \subseteq X$ is called an repeller if it admits a neighborhood U such that $\alpha(U) = M$.*

Definition 1.6. *A limit cycle is a closed trajectory γ, such that $\gamma \subseteq \omega(x)$ or $\gamma \subseteq \alpha(x)$, for some $x \notin \gamma$.*

Definition 1.7. *We say that a set M is topologically transitive if it admits a point $x \in X$ such that $\overline{\gamma(x)} = M$.*

2. Topologically transitive attractors and multi valued maps

The aim of this paper is to characterize topologically transitive attractors(in the sense of Conley) for analytic flows when the phase space X is actually the Euclidean plane. Of course, in this matter limit sets characterisation for planar flows will be needed. The geometry of ω-limit sets we are interested in is best understood by using "pieces", as stated in [13]. In the plane analytic setting, these pieces are so-called cactuses, half-planes and chains. Their precise definitions are given below.

As usual, $intB$, $\overline{B}$, BdB denote, respectively, the interior, the closure and the boundary of a subset B of a topological space A. Recall that a topological space A is called an arc (respectively, an open arc, a circle, a disk) if it is homeomorphic to $[0, 1]$(respectively, to $\mathbb{R}$, to $\{ (x, y) \in \mathbb{R}^2 \mid x^2+y^2 = 1 \}$, to $\{ (x, y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1 \}$. We say that a continuum T is a *cactus* if it is the simple connected union of finitely many disks. Both points and cactuses in $\mathbb{R}^2$ are also called *dots*(the term degenerate dots refers to points).

We say that $A \subset \mathbb{R}^2$ is a *half-plane* if both A and $\mathbb{R}^2 \setminus intA$ are homeomorphic to $\{ (x, y) \in \mathbb{R}^2 \mid x \geq 0 \}$. We say that $A \subset \mathbb{R}^2$ is a *chain* if there are disks $\{D_i\}_{i=1}^{\infty}$ such that:

i) $A = \bigcup_{i=1}^{\infty} D_i$

ii) If $|i - j| = 1$ then $D_i \cap D_j$ consists of exactly one point; otherwise $D_i \cap D_j = \emptyset$.

iii) The disks D_i tend to ∞ as $i \to \infty$; in other words every bounded set of $\mathbb{R}^2$ intersects finitely many disks D_i.

The following theorem from [13] will be used in the sequel:

Theorem 2.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be analytic, let φ be the local flow of $\frac{dz}{dt} = f(z)$ and let $\Omega = \omega_\varphi(u)$ for some $u \in \mathbb{R}^2$. Then $\Omega = BdA$, with A being*

a) the empty set;

b) a single point;

c) a cactus;

d) the union of a circle C and finitely many pairwise disjoint cactuses, each of them contained in the disk enclosed by C and intersecting C at exactly one point;

e) a union of countably many cactuses, half- planes and chains, which are pairwise disjoint except that each cactus intersects either one of the half planes or one of the chains at exactly one point; moreover, every bounded set of $\mathbb{R}^2$ intersects finitely many of these sets.

We"ll need also the following lemma from [4]:

Lemma 2.2. *If $\gamma(p)$ and $\omega(p)$ have a point in common, then $\gamma(p)$ is a periodic trajectory (degenerate or non-degenerate).*

Proposition 2.3. *Every non-critical periodic point has trajectory which is homeomorphic to S^1.*

Proof. First let's write the trajectory in the form $\gamma(x) = \{\varphi(x,t) \mid 0 \le t \le T\}$, where $T \neq 0$ is the fundamental period of x. We consider the sphere S^1 as a quotient space $[0,T]/\{0,T\}$ and define a natural map $n : \gamma(x) \to [0,T]/\{0,T\}$ by:

$$n(\varphi(x,t)) = \begin{cases} t, & \text{if } \ 0 < t < T; \\ \{0,T\}, & \text{if } \ \ t \in \{0,T\} \end{cases}$$

The map is a continuous bijection and hence a homeomorphism. □

Having in mind theorem1.1 we easily conclude that every minimal set is topologically transitive. Now using the above lemma let us describe topologically transitive attractors for plane flows which are also minimal sets.

Theorem 2.4. *Every minimal attractor M of a plane flow is a periodic trajectory.*

Proof. From the assumption M is a minimal set and hence according to theorem 1.1 $\overline{\gamma(x)} = M$, for every $x \in M$. From the definition of attractor, M is also a compact set. Hence $\omega(x) \neq \varnothing$, for every $x \in M$. We pick arbitrary point $q \in M$ and a point $p \in \omega(q)$. From the invariance of the set $\omega(q)$ we conclude that $\gamma(p) \subseteq \omega(q)$. Using theorem 1.1 we obtain that $M = \overline{\gamma(p)} \subseteq \omega(q) \subseteq M$. Hence $\gamma(q) \cap \omega(q) \neq \emptyset$. So to finish up, using lemma 2.2 we conclude that q is a degenerate or non-degenerate periodic point. Either way $M = \overline{\gamma(q)} = \gamma(q)$ is a periodic trajectory. □

Corollary 2.5. *Minimal attractor of a plane flow is either a point or homeomorphic to S^1.*

The following proposition proved in [5] will be used latter on.

Proposition 2.6. *Topologically transitive set of a plane flow has empty interior.*

The first question that naturally rises from theorem 2.4 is the following: Is there a non minimal topologically transitive attractor for a plane flow?

Proposition 2.7. *There exists a non minimal topologically transitive attractor for a plane flow.*

Proof. We consider the flow defined by the following differential equations:

$$\frac{dr}{dt} = r(1-r) \tag{1}$$
$$\frac{d\theta}{dt} = sin^2\left(\frac{\theta}{2}\right) \tag{2}$$

From the phase portrait it is easily seen that the attractor for this system is a homoclinic trajectory, namely the unit sphere S^1 with $(1,0)$ as a critical point. The sphere S^1 is topologically transitive set, but however not minimal. □

Example 2.8. *Let's consider the plane flow defined by the following two-parameter family of vector fields:*

$$\frac{dx}{dt} = \mu_1 + \mu_2 x - x^3 \tag{3}$$
$$\frac{dy}{dt} = -y \tag{4}$$

For $\mu_2 > 0$ the cubic equation $x^3 - \mu_2 x - \mu_1 = 0$ has three roots iff $\mu_1^2 < 4\frac{\mu_2^3}{27}$, two roots (at $x = \pm\sqrt{\frac{\mu_2}{3}}$) iff $\mu_1^2 = 4\frac{\mu_2^3}{27}$ and one root if $\mu_1^2 > 4\frac{\mu_2^3}{27}$. If for example $\mu_2 = 3$ and $\mu_1 = -2$ the points A$(-2,0)$ and B$(1,0)$ are critical points. The line segment $\overline{AB}$ joining this two points is topologically transitive attractor.

This example encourage us to impose a question. Is it possible to describe topologically transitive attractors for analytic plane flows? For answering this question the notion of upper-semi-continuity of multi valued maps turns out to be quite useful:

Definition 2.9. *A multi valued map $\beta : \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ is upper semi-continuous in the Cauchy sense at the point $x \in \mathbb{R}^2$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that the following holds: $y \in \mathbb{R}^2, \|x-y\| < \delta \Rightarrow \beta(y) \subseteq T(\beta(x), \varepsilon)$.*

We"ll need the following lemma from [2]:

Lemma 2.10. *In a plane flow $\alpha(x)$ is nonempty and compact iff $\overline{\gamma^-(x)}$ is compact.*

Remark 2.11. *The previous lemma holds true for arbitrary locally compact metric space.*

Remark 2.12. *We can consider the multi valued map $\alpha : \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ associating the limit set $\alpha(x)$ to a point $x \in \mathbb{R}^2$.*

The following lemma is the key lemma used in the proof of the main theorem. Intuitively the lemma states: If a point y is close to the point x of an attractor $\overline{\gamma(x)}$ then the α - limit set of y is not close to the attractor.

Lemma 2.13. *If $\overline{\gamma(x)}$ is an attractor for a plane flow and for arbitrary $\delta > 0$, there exists $y \in T(x,\delta) \setminus \overline{\gamma(x)}$ such that $\overline{\gamma^-(y)}$ is compact then α is not upper semi-continuous in the Cauchy sense at x.*

Proof. Let us suppose the opposite, namely let α is upper semi-continuous at x. From the assumption $\overline{\gamma(x)}$ is an attractor hence a neighborhood $\mathcal{U}$ exists such that $\omega(\mathcal{U}) = \overline{\gamma(x)}$. Let's note that $\varepsilon > 0$ exists such that $T(\alpha(x),\varepsilon) \subseteq \mathcal{U}$, because $\alpha(x) \subseteq \overline{\gamma(x)}$. Now from Cauchy upper semi-continuity at x, there exists $\delta > 0$ such that the following holds: $p \in \mathbb{R}^2, \|x-p\| < \delta \Rightarrow \alpha(p) \subseteq T(\alpha(x),\varepsilon)$. We choose $y \in T(x,\delta) \setminus \overline{\gamma(x)}$(such a point exists by assumption). From the inclusion $\alpha(y) \subseteq T(\alpha(x),\varepsilon)$ and from lemma 2.10 having in mind that $\overline{\gamma^-(y)}$, $\overline{\gamma^-(x)}$ are compact sets we can conclude that a sequence $t_n \to \infty$ exists such that $\varphi(y,-t_n) \in T(\alpha(x),\varepsilon) \subseteq \mathcal{U}$, for sufficiently large n, which would imply that the following holds: $y = \varphi(\varphi(y,-t_n),t_n) \to y$ and hence $y \in \omega(\mathcal{U}) = \overline{\gamma(x)}$ surely a contradiction. □

We are ready to state the following:

Theorem 2.14. *Let $\alpha(x)$ be a limit cycle for a non periodic point x of a plane flow. Then $\overline{\gamma(x)}$ is not an attractor.*

Proof. First let us note that x has a neighborhood $\mathcal{V}$ such that $\alpha(x) = \alpha(y)$, for every $y \in \mathcal{V}$. Namely, we shall prove that the set $\mathcal{V} = \{\, y \mid \alpha(x) = \alpha(y) \,\} \setminus \alpha(x)$ is open. For sufficiently large $t > 0$, $\varphi(x, -t) \in int\mathcal{V}$. Hence $\varphi(y, -t) \in \mathcal{V}$ for y sufficiently close to x, which implies that $\mathcal{V}$ is open. Now for arbitrary $\delta > 0$ from proposition 2.6 there exists $y \in T(x, \delta) \setminus \overline{\gamma(x)}$ such that the set $\overline{\gamma^-(y)}$ is compact since $\overline{\gamma^-(y)} = \gamma^-(y) \cup \alpha(y) = \gamma^-(y) \cup \alpha(x)$. Also let us note that $\alpha(x)$ is upper semi-continuous in the Cauchy sense at x, so the possibility that $\overline{\gamma(x)}$ is an attractor is excluded by the previous lemma2.13. $\square$

Example 2.15. *Let us discuss the limit set α of the previously defined plane flows in proposition 2.7 and example 2.8. In the example of proposition 2.7 with S^1 as topologically transitive attractor the limit set α is:*

$$\alpha(x) = \begin{cases} \emptyset, & if \ \ \|x\| > 1; \\ (1,0), & if \ \ \ x \in S^1; \\ (0,0), & if \ \ \|x\| < 1; \end{cases}$$

Obviously α is not upper semi-continuous in the Cauchy sense at $x \in S^1$. On the other hand in example 2.8 with the line segment $\overline{AB}$ as topologically transitive attractor, α is upper semi-continuous in the Cauchy sense at $x \in \overline{AB}, x \neq A, B$. Let us observe that $\alpha(y) = \emptyset$, for arbitrary close point to x not on the line segment $\overline{AB}$ and $\alpha(y) = B$, if $y \in \overline{AB}, y \neq A$. So the Cauchy upper semi-continuity appears because the fact that the topologically transitive attractor $\overline{AB}$ does not have arbitrary close point y not on the line segment $\overline{AB}$ such that $\overline{\gamma^-(y)}$ is compact.

Corollary 2.16. *Let $\overline{\gamma(x)}$ be a non minimal attractor for an analytic plane flow. Then $\alpha(x)$ is a single point.*

Proof. If we suppose the opposite, having in mind that $\alpha(x) \neq \emptyset$ by compactness of $\overline{\gamma(x)}$, using theorem 2.1 for analytic systems we deduce that $\alpha(x) = BdA$ where A is either a cactus, union of a circle C and finitely many pairwise disjoint cactuses, or a union of a countably many cactuses, half-planes and chains. Of course the possibility which involves half-planes or chains can not even appear because the limit set $\alpha(x)$ is bounded . In the remaning cases using the well known flow-box theorem [14] we obtain that α is upper semi-continuous in the Cauchy sense at x. But from lemma 2.13 having in mind that $\overline{\gamma(x)}$ is an attractor we conclude that α is not upper semi-continuous in the Cauchy sense at x, surely a contradiction. $\square$

3. Conley index and shape of attractors

In the realm of dynamical systems the study of the existence and structure of invariant sets is of great importance. There are three types of difficulty:

- Invariant sets can be extremely complicated.
- The structure of invariant sets can change dramatically with respect to perturbations(bifurcation theory).
- The bifurcation points need not be isolated.

Conley's approach is an attempt to overcome this issues by avoiding the direct study of invariant sets. In his own words:

"...many significant properties of the flow are reflected in the existence of isolating neighborhoods, or perhaps more accurately, in the companion isolated invariant set... This is true in some generality of those properties which are stable under perturbation."

The Conley index of an isolated invariant set is the fundamental concept that allows the study of a flow inside an isolating neighborhood by topological methods which is central to the index approach. Using Conley's index pairs we shall present new proof of the well known theorem from [1](here theorem 3.11) which claims that every attractor of a flow on a manifold has the shape of a finite polyhedron. As a consequence we state corollary 3.12 in the special case when the manifold is actually the Euclidean plane.

Definition 3.1. *A subset $A \subseteq X$ is called isolated invariant if it is the maximal invariant set in some compact neighborhood of itself. Such a neighborhood is called isolating neighborhood.*

Definition 3.2. *A compact pair (N, L) is called an index pair for an isolated invariant set S if:*

- *$\overline{N \setminus L}$ is an isolating neighborhood for S.*
- *L is positively invariant relative to N, i.e if $x \in L, t \geq 0, \varphi(x, [0, T]) \subseteq N$ then $\varphi(x, [0, T]) \subseteq L$.*
- *L is the exit set of N, i.e if $x \in N, t \in \mathbb{R}^+$ and $\varphi(x, t) \notin N$ then there exists $t' \in [0, t]$ such that $\varphi(x, t') \in L$.*

Remark 3.3. *If the exit set is immediate we call the index pair proper.*

Good reference paper for index theory is [6] from which we use the following:

Theorem 3.4. *Let V be an isolating neighborhood for (A, φ) so that $A = I(V, \varphi)$ is the maximal invariant set in V for a flow φ defined on a manifold M. Let $v \in L(M)$ be the vector field induced by the flow φ, i.e $v(x) = \frac{d}{dt}\varphi^t(x)|_{t=0} \in T_xM$. Then there exists a neighborhood $\mathcal{F}$ of v in $L(M)$ and a compact pair (N, L) which is an index pair for $(I(V, \psi), \psi)$ for any flow ψ whose induced vector field w lies in $\mathcal{F}$. One can choose the pair (N, L) so that N/L is a homotopy polyhedron.*

Lemma 3.5. *Every attractor A is an isolated invariant set.*

Proof. Let $A = \omega(V)$, for some compact neighborhood V of A. Let $A \subseteq Q \subseteq V$ and Q is invariant. For arbitrary $x \in Q$ we have that $x = \varphi(\varphi(x, -n), n) \in A$ which means that A is isolated invariant set. □

Lemma 3.6. *Compact and isolated invariant set A is an attractor iff there exists an index pair (N, L) for A such that $L = \emptyset$.*

Proof. First let us assume that the compact isolated set A is an attractor. This means that a compact neighborhood V exists such that $\omega(V) = A$. From the definition of index pairs we need a positively invariant isolated neighborhood of A. Let us note that we can choose V as positively invariant set. Namely if V is not positively invariant then we can work with $V^+ = \{x \in V \mid \varphi(x, [0, \infty)) \subseteq V\}$. We shall prove that V^+ is again compact neighborhood of A such that $\omega(V^+) = A$. Compactness is provided for V^+ by being closed subset of V. The relation $\omega(V^+) = A$ is obvious, so the only thing remaining is to prove that V^+ is still a neighborhood of A. Let us prove that $A \subseteq intV^+$. We choose arbitrary point $a \in A$. If we suppose that $a \notin intV^+$ then a sequence $a_n \to a$ will exists such that $a_n \notin V^+, a_n \in V$. This means that $t_n \in \mathbb{R}^+$ exists such that $\varphi(a_n, t_n) \notin V$. There are two possibilities for this sequence. First let us assume that t_n is bounded and hence convergent. But this implies that $\varphi(a_n, t_n) \to \varphi(a, t) \in A$ which is a contradiction with $\varphi(a_n, t_n) \notin V$, for arbitrary n. So let us discuss the second case, namely let a subsequence (t_{n_k}) exists such that $(t_{n_k}) \to \infty$. But then a sequence $\tau_{n_k} \in [0, t_{n_k}]$ exists such that $\varphi(a_{n_k}, \tau_{n_k}) \in \partial V$. Let us assume that the sequence τ_{n_k} is bounded and hence convergent(passing to subsequence if necessary). But then $\varphi(a_{n_k}, \tau_{n_k}) \to \varphi(a, \tau) \in A$ which is again contradiction with $\varphi(a_{n_k}, \tau_{n_k}) \in \partial V$. So the remaining case is the existence of a subsequence $\tau_{n_{k_p}} \to \infty$. But the from $\varphi(a_{n_{k_p}}, \tau_{n_{k_p}}) \in \partial V$ and from the compactness of ∂V we have that there exists convergent subsequence of $\varphi(a_{n_{k_p}}, \tau_{n_{k_p}})$ with limit $a_0 \in \partial V$. But from $\omega(V) = A$ we conclude that $a_0 \in A$ which is again a contradiction. Hence $A = \omega(V^+)$, for a positively invariant neighborhood V^+ of A. We shall prove that V^+ is an isolating neighborhood of A. Namely if we suppose that $K \subseteq V^+$ is an arbitrary invariant subset of V^+ then from $x = \varphi(\varphi(x, -n), n)$, for arbitrary $x \in K$ and $A = \omega(V^+)$ we conclude that $K \subseteq A$ which means that $(V, \emptyset)$ is indeed an index pair for A. Now for the opposite direction, let the isolated invariant set A has an index pair of the form $(N, \emptyset)$. From the definition of index pairs it follows that N is positively invariant set. Hence $\omega(N) \subseteq N$ since N is closed. But the limit set is invariant and hence $\omega(N) \subseteq A$. Now again from the relation $x = \varphi(\varphi(x, -n), n)$, for arbitrary $x \in A$ we conclude that $\omega(N) = A$. □

In the sequel we"ll use few results from shape theory. Good reference book for shape theory is [7] and the papers [10], [8], [9]. We"ll need the following lemma from [11]:

Lemma 3.7. *Let (N, L) be a proper index pair for an isolated invariant set A. If we introduce the notation $N^- = \{ x \in N \mid \varphi(x, (-\infty, 0)) \subseteq N \}$ then the inclusion map $i : N^- \cup L \to N$ is in fact a shape equivalence.*

Lemma 3.8. *Let A be an isolated invariant set with an index pair (N, L). Then A is an attractor iff $A = N^-$.*

Proof. Let A be an attractor. Then a neighborhood V of A exists such that $\omega(V) = A$. The inclusion $A \subseteq N^-$ is obvious. Let us choose a point $x \in N^-$. From the definition of N^- we have that $\alpha(x) \neq \emptyset, \alpha(x) \subseteq N$. Also let us note that $\alpha(x) \cap L = \emptyset$ because the limit set $\alpha(x)$ is invariant. Hence $\alpha(x) \subseteq N \setminus L$. Now because $\overline{N \setminus L}$ is an isolating neighborhood for A we have that $\alpha(x) \subseteq A$. For sufficiently large n, $\varphi(x, -n) \in V$ which implies that $x = \varphi(\varphi(x, -n), n) \in A$. Conversely, let $A = N^-$. First we shall prove that N^+ is a neighborhood of A. If we suppose that $A \nsubseteq intN^+$ then a point $a \in A$ exists such that $a \notin intN^+$. This means that a sequence $x_n \in N \setminus N^+$ exists such that $x_n \to a$. Now from $x_n \notin N^+$, a sequence $t_n \in \mathbb{R}^+$ exists such that $\varphi(x_n, [0, t_n]) \subseteq N, \varphi(x_n, t_n) \in L$. First we shall assume that the sequence $t_n \in \mathbb{R}^+$ is bounded, hence convergent(passing to subsequence if necessary). But then $\varphi(x_n, t_n) \to \varphi(a, t) \in L \cap A = \emptyset$. So let us consider the second possibility, namely let a subsequence (t_{n_k}) exists such that $t_{n_k} \to \infty$. From the compactness of L we can suppose that $\lim_{k\to\infty} \varphi(x_{n_k}, t_{n_k}) = p \in L$. Now for arbitrary $t \in \mathbb{R}^-$ we have that $\varphi(p, t) = \lim_{k\to\infty} \varphi(x_{n_k}, t_{n_k} + t)$, but $0 < t_{n_k} + t < t_{n_k}$, for sufficiently large k which implies that $\varphi(x_{n_k}, t_{n_k} + t) \in N$ and hence $\varphi(p, t) \in N$. So $p \in N^- \cap L = A \cap L = \emptyset$. This completes the proof of the statement: N^+ is a neighborhood of A. Now let us prove that $\omega(N^+) = A$. Of course from the invariance of A, $A \subseteq \omega(N^+)$. Let $p \in \omega(N^+)$ is an arbitrary point. Then $p = \lim_{n\to\infty} \varphi(p_n, t_n)$, $p_n \in N^+, t_n \to \infty$. Now, similarly for arbitrary $t \in \mathbb{R}^-$ we have $\varphi(p, t) = \lim_{n\to\infty} \varphi(p_n, t_n + t)$, but for sufficiently large n, $0 < t_n + t$ and hence $\varphi(p_n, t_n + t) \in N$, since $p_n \in N^+$. This implies that $\varphi(p, t) \in N$ and hence $p \in N^- = A$. □

The following lemma is also from [11]:

Lemma 3.9. *Let A be an isolated invariant set with an proper index pair (N, L). Then A is an attractor iff the inclusion map $i : A \cup \{*\} \to N/L$ is a shape equivalence.*

Proof. Let A be an attractor. From the previous lemma $A = N^-$ and the inclusion map $i : A \cup \{*\} \to (N^- \cup L)/L$ is in fact the identity map. From the previous lemma 3.7 the inclusion map $i : (N^- \cup L)/L \to N/L$ is a shape equivalence and hence the inclusion map $i : A \cup \{*\} \to N/L$ is a shape equivalence. Conversely, let the inclusion map $i : A \cup \{*\} \to N/L$ is a shape equivalence. If A is not an attractor then $A \neq N^-$, and hence a point $x \in N^- \cap L$ exists. This means that $\overline{\gamma^-(x)} \cap L \neq \emptyset$. But $\overline{\gamma^-(x)}$ is connected and $\alpha(x) \subseteq \overline{\gamma^-(x)}$, which implies that $\alpha(x), \{*\}$ are in the same connected component of N/L. But $\alpha(x), \{*\}$ are in different connected components of $A \cup \{*\}$, because $\alpha(x) \cap L = \emptyset$, $\alpha(x)$ is invariant and hence $\alpha(x) \subseteq A$. Since shape equivalence preserves connected components(see [7] and [15]) we obtain a contradiction. □

Lemma 3.10. *(The homotopy Conley index [6]) Let A be an isolated invariant set for the flow φ on manifold M and let (N_α, L_α) and (N_β, L_β) be index pairs for (A, φ). Then the spaces N_α/L_α and N_β/L_β are homotopy equivalent:*

$$[N_\alpha/L_\alpha] = [N_\beta/L_\beta]$$

The common homotopy type is called the homotopy Conley index.

Finally we can state:

Theorem 3.11. *Every attractor A of a flow on a manifold has the shape of a finite polyhedron.*

Proof. First from lemma 3.5 A is an isolated invariant set. Now let us choose index pair for $A, (N, L)$ such that N/L has the homotopy type of a finite polyhedron. This is possible according to theorem 3.4. Now from lemma 3.6 an index pair for A exists of the form $(N_1, \emptyset)$(the index pair is obviously proper). Now from lemma 3.10 $[N/L] = [N_1/\emptyset]$. Now from $N_1/\emptyset = N_1 \cup \{*\}$ and from the fact that $\{*\}, N_1$ are in different connected components we get that N_1 as well has the homotopy type of a finite polyhedron. Now from lemma 3.9 the inclusion map $i : A \cup \{*\} \to N_1 \cup \{*\}$ is a shape equivalence and hence $Sh(A) = Sh(N_1) = Sh(P)$. P is a finite polyhedron.

□

Now if the manifold is actually the Euclidean plane we obtain the following:

Corollary 3.12. *A compact subset of the plane is an attractor of a suitable plane flow iff it has the shape of a finite union of finite wedges of circles.*

4. Main Result

In this section we will give a characterization of topologically transitive attractor A of an analytic plane flow. In the previous section we showed that:

$$Sh(A) = Sh(\cup_{i=1}^{n} S_1^i \wedge S_2^i \wedge \dots S_{k_i}^i) \tag{5}$$

i.e. A has the shape of a finite union of finite wedges of circles. In what follows we will present a finer description of the structure of topologically transitive attractors of analytic plane flows.

Before stating our main claim we will need the following definitions from [12]. We say that the open arc $E \subseteq \mathbb{R}^2$ ends at the point $p \in \mathbb{R}^2$ if $E \cup \{p\}$ is homeomorphic to $[0, 1)$. We say that the open arc $E \subseteq \mathbb{R}^2$ ends at the cactus $T \subseteq \mathbb{R}^2$, if E spirals towards T, that is, there is a homeomorphism $\tau : \mathbb{R} \to E$ such that $d(\tau(t), T) \to 0$ as $t \to \infty$, and there are continuous functions $\rho : \mathbb{R} \to [0, \infty)$, $\theta : \mathbb{R} \to \mathbb{R}$ and a point $p \in T$ such that $\tau(t) = p + \rho(t)e^{i\theta(t)}$, for arbitrary t and $|\theta(t)| \to \infty$, as $t \to \infty$. A disjoint union of an open arc E and two dots P and Q with E ending both at P and Q is called *generalized arc*. A generalised arc with at least one degenerate dot is called d-generalised arc. A disjoint union of a point P, an open arc E and a union of a circle and finitely many pairwise disjoint cactuses each of them contained in the disk enclosed by C and intersecting C at exactly one point such that the open arc ends in P and spirals towards the cactuses and the circle C is called inverted d-generalised arc.

Now we are ready to state:

Theorem 4.1. *Every topologically transitive attractor A of an analytic plane flow is either:*

i) a periodic trajectory;

ii) closure of a homoclinic trajectory;

iii) boundary of a d-generalised arc;

iv) boundary of an inverted d-generalised arc.

Proof. From the fact that A is topologically transitive a point $x \in A$ exists such that $\overline{\gamma(x)} = A$. We shall assume that $\overline{\gamma(x)} = A$ is not a periodic trajectory. From corollary 2.16 $\alpha(x)$ is a single point which means that we can choose it as a dot P. Having in mind theorem 2.7 which states that $\overline{\gamma(x)} = \alpha(x) \cup \gamma(x) \cup \omega(x)$ we choose the open arc E to be $E = \gamma(x)$. From theorem 2.1, there are several possibilities for $\omega(x)$. The first case is excluded because of the compactness of $\overline{\gamma(x)}$. Let us note that the last possibility e) which involves chains and half-planes can not appear also because of the compactness of $\overline{\gamma(x)}$. If $\omega(x)$ is a single critical point L which coincides with P then A is actually a clousure of homoclinic trajectory. If $L \neq P$ then we conclude that $A = P \cup E \cup L = Bd(P \cup E \cup L)$ and hence A is a boundary of a d-generalized arc $P \cup E \cup L$(E ends in P and L, which is easy to prove). Now let the ω-limit set $\omega(x)$ coincides with the boundary of some cactus Q.

It remains to prove that the open arc $E = \gamma(x)$ ends both in P and Q which would imply that $W = E \cup P \cup Q$ is in fact a d-generalized arc and $BdW = \overline{\gamma(x)}$. Namely to prove that it ends in P we need to construct a homeomorphism $\psi : [0, 1) \to E \cup P$. We shall define the required map by:

$$\psi(t) = \varphi(x, tg(\frac{\pi}{2}(2t - 1))), 0 < t < 1, \psi(0) = P. \tag{6}$$

It is easily proved that ψ is in fact a homeomorphic map. For the second part we choose the homeomorphism $\tau : \mathbb{R} \to E$ by $\tau(t) = \varphi(x, t)$. Let $p \in intQ$. If we write $\tau(t) = p + \rho(t)e^{i\theta(t)}$, then the possibility that $|\theta(t_n)| \leq M$, for some sequence $t_n \to \infty$ and some positive real number M, would imply that $\omega(x) \neq BdQ$, which surely leads to contradiction. The last case similarly leads to the conclusion that the attractor is in fact a boundary of an inverted d-generalized arc. □

In the Theorem 4.1 it is not shown that all the sets in the list can indeed be realized as (topologically transitive) attractors of analytic plane flows.

Question 1. Is there an analytic plane flow with a topologically transitive attractor which is boundary of a given d-generalised arc?

Question 2. Is there an analytic plane flow with a topologically transitive attractor which is boundary of a given inverted d-generalised arc?

References

[1] B. Günther. J. Segal, Every attractor of a flow on a manifold has the shape of a finite polyhedron, Procedings of the American Mathematical Society 119 (1993), 321-329

[2] N. P. Bhatia, G. P. Szego, Stability theory of dynamical systems, Springer, Berlin, 1970

[3] C. Zuo, X. Wang, Attractors and Quasi-Attractors of a Flow, J.Appl.Math and Computing 23 No.1-2(2007) 411-417

[4] L. Perko, Differential equations and dynamical systems, Springer-Verlag, New York, 2002

[5] M. Shoptrajanov, N. Shekutkovski, Chain recurrence and topological transitivity in the plane, Proceedings of the Fourth International Science Conference, Blagoevgrad 1 (2011), 50-54

[6] J. W. Robbin, D. Salamon, Dynamical systems, shape theory and the Conley index, Ergodic Theor.& and Dynam. Sys.8 (1988), 375-393

[7] K. Borsuk, Theory of shape, Polish Scientific Publishers, Warszawa, 1975

[8] N. Shekutkovski, Intrinsic definition of strong shape for compact metric spaces, Topology Proceedings 39 (2012), 27-39

[9] S. Gabites, Dynamical systems and shape, Revista de la Real Academia de Ciencias Exactas, Fizicas y Naturales. Serie A, Matematicas 102 (1),(2008), 127-159

[10] J. M. Sanjurjo, A non continuous description of the shape category of compacta, Quart.J.Math.Oxford 40 (1989), 351-359

[11] J. M. Sanjurjo, Morse equations and unstable manifolds of isolated invariant sets, Nonlinearity 16 (2003), 1435-1448

[12] V. Jiménez. López, D. Peralta- Salas, Global attractors of analytic plane flows, Ergodic Theor.& Dynam. Sys.29 (2009), 967-981

[13] V. Jiménez. López, J. Libre, A topological characterization of the ω-limit sets for analytic flows on the plane, the sphere and the projective plane, Adv. Math. 216 (2007), 677-710

[14] M. Hirsh, Differential equations, dynamical systems and linear algebra, Academic Press, San Diego, 1974

[15] N. Shekutkovski, T. A. Pachemska, G. Markoski, Maps of quasicomponents induced by a shape morphism, Glasnik Matematicki 47 (2012), 431-438