Filomat 29:1 (2015), 111–120
DOI 10.2298/FIL1501111D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# Localic Remote Points Revisited

**Themba Dube[a], Martin M. Mugochi[b]**

[a]*Department of Mathematical Sciences, University of South Africa, P. O. Box 392, 0003 Unisa, SOUTH AFRICA.*
[b]*Department of Mathematics, University of Namibia, P.O. Box 13301, Windhoek, NAMIBIA.*

**Abstract.** We consider remote points in general extensions of frames, with an emphasis on perfect extensions. For a strict extension $\tau_{\mathfrak{X}}L \to L$ determined by a set $\mathfrak{X}$ of filters in $L$, we show that if there is an ultrafilter in $\mathfrak{X}$ then the extension has a remote point. In particular, if a completely regular frame $L$ has a maximal completely regular filter which is an ultrafilter, then $\beta L \to L$ has a remote point, where $\beta L$ is the Stone-Čech compactification of $L$. We prove that in certain extensions associated with radical ideals and $\ell$-ideals of reduced $f$-rings, remote points induced by algebraic data are exactly non-essential prime ideals or non-essential irreducible $\ell$-ideals. Concerning coproducts, we show that if $M_1 \to L_1$ and $M_2 \to L_2$ are extensions of $T_1$-frames, then each of these extensions has a remote point if the extension $M_1 \oplus M_2 \to L_1 \oplus L_2$ has a remote point.

## 1. Introduction

Remote points in pointfree topology have hitherto been considered only in the case of completely regular frames [12], and even then for the extension $\beta L \to L$. In this note we extend the notion of remote point to any extension $M \xrightarrow{h} L$ of an arbitrary $L$. In such a case we shall speak of a point $p$ of $M$ being remote from $L$. Our motivation is that many extensions (and, in fact, all strict extensions [5]) of a frame $L$ are equivalent to extensions constructed by starting with a collection $\mathfrak{X}$ of filters of the frame. Thus, to determine if an extension has a remote point we need only check if certain types of filters are present in the collection $\mathfrak{X}$. Indeed, if $\mathfrak{X}$ contains an ultrafilter, then the strict extension $\tau_{\mathfrak{X}}L \to L$ determined by $\mathfrak{X}$ has a remote point.

The paper consists of five sections. We start with preliminaries where we recall some few pertinent results from frames and $f$-rings, and then proceed to Section 3 where we define remote points and show that the definition is "conservative" if we restrict to sober spaces. Examples are then given in frames that need not be regular. Following that we generate points from algebraic data in the following way. Given a reduced commutative $f$-ring $A$ with identity, let $\mathrm{Rad}(A)$ denote the frame of radical ideals of $A$, and $A^*$ the subring of $A$ consisting of bounded elements. The map $\varepsilon\colon \mathrm{Rad}(A^*) \to \mathrm{Rad}(A)$, given by extension of ideals, is a dense onto frame homomorphism; so that we have an extension in the frame sense. We show that remoteness of points in this extension is intertwined with non-essentiality of prime ideals. Indeed, if $A$ has $C(X)$-like features in a manner we will make precise, then remote points of the extension $\varepsilon\colon \mathrm{Rad}(A^*) \to \mathrm{Rad}(A)$ are exactly the non-essential prime ideals of $A$. Applied to the rings $C(\mathrm{X})$, we have that remote points of the

2010 *Mathematics Subject Classification*. Primary 06D22; Secondary 54D20, 54D60.
*Keywords*. frame; remote point; points in binary coproducts; extensions of frames; $f$-ring; frame of radical ideals; frame of $\ell$-ideals.
Received: 25 September 2014; Accepted: 23 December 2014
Communicated by Dragan Djurčić
The first author was supported by the National Research Foundation of South Africa
*Email addresses:* dubeta@unisa.ac.za (Themba Dube), mmugochi@unam.na (Martin M. Mugochi)

extension $\varepsilon\colon \mathrm{Rad}(C^*(X)) \to \mathrm{Rad}(C(X))$ are, in the notation of Gilman and Jerison [17], exactly the maximal ideals $\boldsymbol{M}^{*p}$ of $C^*(X)$ for $p$ an isolated point of $\beta X$.

Another result with an algebraic flavour deals with extensions associated with $\ell$-ideals of a reduced $f$-ring with bounded inversion. In this case there is an extension $\mathfrak{L}(A^*) \to \mathfrak{L}(A)$ for which we show that the remote points are precisely the non-essential irreducible $\ell$- ideals of $A$, again if $A$ is $C(X)$-like.

In Section 4 we prove the result about coproducts mentioned in the abstract. We precede that by showing how some points in a binary coproduct can be constructed from points in the summands. It turns out that for $T_1$-frames $L$ and $M$, the points of $L \oplus M$ are precisely those formed from the points of $L$ and $M$ in this way.

Section 5 deals with remote points in perfect extensions. We describe their presence in terms of saturated filters. These are filters which contain every dense element. In particular, we establish that if a completely regular frame $L$ has a saturated maximal completely regular filter, then $\beta L \to L$ has a remote point.

## 2. Preliminaries

### 2.1. *Frames*

Our references for frames are [20] and [22]. We follow, to a large extent, the notation of these texts, with minor deviations such as, for instance, denoting the frame of open sets of a topological space $X$ by $\mathfrak{O}X$. By a *point* of $L$ we mean a prime element, that is, an element $p$ such that $p \neq 1$ and $x \wedge y \leq p$ implies $x \leq p$ or $y \leq p$. Following [23] we shall say a frame $L$ is a $T_1$*-frame* if its points are precisely the maximal elements, where the term "maximal" is understood to mean maximal strictly below the top. Every regular frame is a $T_1$-frame. We denote the set of all points of $L$ by $\mathrm{Pt}(L)$. The frame of ideals of $L$ is denoted by $\mathfrak{J}L$. By a *quotient map* we mean a surjective frame homomorphism.

A filter $F$ (throughout assumed to be proper) in a frame $L$ is *completely prime* if, for any $S \subseteq L$, $\bigvee S \in F$ implies $S \cap F \neq \emptyset$. If $p \in \mathrm{Pt}(L)$, then the set

$$F_p = \{x \in L \mid x \nleq p\}$$

is a completely prime filter. On the other hand, if $F$ is a completely prime filter in $L$, then the element

$$p_F = \bigvee (L \smallsetminus F)$$

is a point in $M$. Furthermore, $p_{F_p} = p$ and $F_{p_F} = F$.

### 2.2. *Extensions determined by sets of filters*

A frame homomorphism is *dense* if it maps only the bottom element to the bottom element. If $h\colon M \to L$ is dense onto, then, for any $a \in M$ and any $b \in L$, we have (i) $h(a^*) = h(a)^*$, (ii) $h_*(b^*) = (h_*(b))^*$, and (iii) $h_*h(a^*) = a^*$. The third identity holds because $h(h_*h(a^*) \wedge a) = 0$, so that, by density of $h$, $h_*h(a^*) \wedge a = 0$, whence $h_*h(a^*) \leq a^*$, which is the nontrivial inequality in the claimed equality.

By an extension of $L$ we mean a pair $(M, h)$ where $h\colon M \to L$ is a dense onto frame homomorphism. We shall frequently write $M \xrightarrow{h} L$ for an extension of $L$. An extension $M \xrightarrow{h} L$ is called *strict* if $h_*[L]$ generates $M$. In [19], Hong defines a *simple extension* of a frame $L$ determined by the set $\mathfrak{X}$ of filters of $L$ as follows. For each $a \in L$, let $\mathfrak{X}_a = \{F \in \mathfrak{X} \mid a \in F\}$ and let $s_{\mathfrak{X}}L$ be the subframe of $L \times \mathcal{P}(\mathfrak{X})$ given by

$$s_{\mathfrak{X}}L = \{(a, \mathfrak{F}) \mid \mathfrak{F} \subseteq \mathfrak{X}_a\}.$$

The map $s\colon s_{\mathfrak{X}}L \to L$ defined by $s(a, \mathfrak{F}) = a$ is a dense onto frame homomorphism whose right adjoint is given by $s_*(a) = (a, \mathfrak{X}_a)$. The strict extension of $L$ determined by $\mathfrak{X}$ is the subframe of $s_{\mathfrak{X}}L$ generated by $s_*[L]$. See [5] for details. The Katětov extension of a frame $L$ is the simple extension $\kappa\colon \kappa L \to L$ determined by the set of all *free* ultrafilters, meaning the ultrafilters $F$ such that $\bigvee\{x^* \mid x \in F\} = 1$. Properties of the Katětov extension can be found in [21] and [24].

*2.3. $f$-Rings*

All rings considered in this paper are commutative with identity 1. A ring is said to be *reduced* if it has no nonzero nilpotent elements. An $f$-ring $A$ is said to have *bounded inversion* if every $a \geq 1$ is invertible. The *bounded part* of $A$ is denoted by $A^*$. The *contraction* of an ideal $I$ of $A$ is the ideal $I^c = A^* \cap I$ of $A^*$. The *extension* $J^e$ of an ideal $J$ of $A^*$ is the ideal of $A$ generated by $J$. An ideal of a ring is a *radical ideal* if whenever it contains a power of an element, then it contains the element. The lattice $\mathrm{Rad}(A)$ of radical ideals of $A$ is a frame, and the map $\varepsilon\colon \mathrm{Rad}(A^*) \to \mathrm{Rad}(A)$ given by $J \mapsto J^e$ is a dense onto frame homomorphism [16, Proposition 3.6], so that we have an extension of the frame $\mathrm{Rad}(A)$. We recall that an ideal of a ring is called *essential* if it meets every nonzero ideal nontrivially. We refer to [4] for information regarding the $f$-ring $\mathcal{R}L$ of real-valued continuous functions on a frame $L$. Let $I \in \beta L$ and $r_L$ denote the right adjoint of the join map $\beta L \to L$. The ideals $\boldsymbol{M}^I$ and $\boldsymbol{O}^I$ are defined by

$$\boldsymbol{M}^I = \{\alpha \in \mathcal{R}L \mid r_L(\operatorname{coz}\alpha) \subseteq I\} \qquad \text{and} \qquad \boldsymbol{O}^I = \{\alpha \in \mathcal{R}L \mid \operatorname{coz}\alpha \in I\}.$$

Maximal ideals of $\mathcal{R}L$ are precisely the ideals $\boldsymbol{M}^I$, for $I \in \mathrm{Pt}(\beta L)$ [13].

## 3. Remote points generally

General remote points in spaces are defined as follows. Let $X$ be a topological space and $Y \supseteq X$ be an extension of $X$. A point $p \in Y \smallsetminus X$ is said to be remote from $X$ (or is called a remote point if there is no danger of confusion) if for any nowhere dense set $D$ in $X$, $p \notin \mathrm{cl}_Y D$. Now recall from [12] that a quotient map $\eta\colon L \to N$ is said to be *nowhere dense* if for every nonzero $x \in L$ there exists a nonzero $y \leq x$ in $L$ such that $\eta(y) = 0$. The terminology is justified by the fact that a subspace $N$ of a topological space $X$ is nowhere dense if and only if the homomorphism $\mathfrak{O}X \to \mathfrak{O}N$, given by $U \mapsto U \cap N$, is nowhere dense. It is shown in [12, Lemma 3.2] that

*a quotient $L \xrightarrow{\eta} N$ of $L$ is nowhere dense iff $h_*(0)$ is dense.*

**Definition 3.1.** *Let $M \xrightarrow{h} L$ be an extension of $L$. A point $p \in \mathrm{Pt}(M)$ is* remote from L *if, for every nowhere dense quotient $L \xrightarrow{\eta} N$ of $L$, $h_*(\eta_*(0)) \nleq p$. We denote the set of points of $M$ that are remote from $L$ by $\mathrm{Pt}(M \ltimes L)$.*

If there is no danger of confusion, we shall simply say $p$ is a remote point. Applied to $\beta L \to L$, this definition is precisely that of remote point employed in [12] because in a $T_1$-frame $L$, $a \nleq p$ if and only if $a \vee p = 1$, for any $a \in L$ and $p \in \mathrm{Pt}(L)$. The following characterisations of remote points are easy to prove (and hence the proofs are omitted) if one takes into account that a closed quotient map $M \to {\uparrow}a$ is nowhere dense if and only if $a$ is a dense element in $M$. Given an extension $M \xrightarrow{h} L$ of $L$ and $p \in \mathrm{Pt}(M)$, we set

$$U^p = \{a \in L \mid h_*(a) \nleq p\}.$$

It is routine to check that $U^p$ is a (proper) filter in $L$. Following [8], we say a filter in a frame is *saturated* if it contains all dense elements of the frame.

**Proposition 3.2.** *Let $M \xrightarrow{h} L$ be an extension of $L$. The following statements about a point $p \in \mathrm{Pt}(M)$ are equivalent.*

1. *$p$ is remote from $L$.*
2. *For any dense $a \in L$, $h_*(a) \nleq p$.*
3. *$U^p$ is a saturated filter.*

Let us show that the definition of remote point is "conservative" in the usual sense of usage of this term in pointfree topology. If $X$ is a sober space and $p \in X$, we write $\tilde{p} = X \smallsetminus \mathrm{cl}_X\{p\}$, so that, by sobriety,

$$\mathrm{Pt}(\mathfrak{O}X) = \{\tilde{p} \mid p \in X\}.$$

For an extension $Y \supseteq X$ of sober spaces we denote by $\mathrm{Rem}(Y \smallsetminus X)$ the set of points of $Y$ that are remote from $X$. Recall that for any continuous map $f\colon X \to Y$ and $U \in \mathfrak{O}X$,

$$(\mathfrak{O}f)_*(U) = Y \smallsetminus \mathrm{cl}_Y(X \smallsetminus U).$$

**Lemma 3.3.** *Let $Y \supseteq X$ be an extension of sober spaces, and denote by $h\colon \mathfrak{O}Y \to \mathfrak{O}X$ the frame homomorphism $U \mapsto X \cap U$. For any $p \in Y \smallsetminus X$,*

$$p \in \mathrm{Rem}(Y \smallsetminus X) \iff \tilde{p} \in \mathrm{Pt}(\mathfrak{O}Y \ltimes \mathfrak{O}X).$$

*Proof.* ($\Rightarrow$) Let $p \in \mathrm{Rem}(Y \smallsetminus X)$ and $U \in \mathfrak{O}X$ be dense. Then $X \smallsetminus U$ is a nowhere dense set in $X$, so that $p \notin \mathrm{cl}_Y(X \smallsetminus U)$, which certainly implies $Y \smallsetminus \mathrm{cl}_Y(X \smallsetminus U) \nsubseteq Y \smallsetminus \mathrm{cl}_Y\{p\}$. That is, $h_*(U) \nleq \tilde{p}$, and hence $\tilde{p} \in \mathrm{Pt}(\mathfrak{O}Y \ltimes \mathfrak{O}X)$.

($\Leftarrow$) Let $N \subseteq X$ be nowhere dense in $X$, and consider $\mathfrak{O}Y \xrightarrow{h} \mathfrak{O}X \xrightarrow{\eta} \mathfrak{O}N$, where $\eta$ is the nowhere dense quotient map given by $V \mapsto N \cap V$. Therefore, by the present hypothesis, $h_*\eta_*(0_N) \nsubseteq \tilde{p}$. Now

$$\eta_*(0_N) = \eta_*(\emptyset) = X \smallsetminus \mathrm{cl}_X(N \smallsetminus \emptyset) = X \smallsetminus \mathrm{cl}_X N,$$

and hence

$$h_*\eta_*(0_N) = Y \smallsetminus \mathrm{cl}_Y\big(X \smallsetminus (X \smallsetminus \mathrm{cl}_X N)\big) = Y \smallsetminus \mathrm{cl}_Y(\mathrm{cl}_X N).$$

Consequently, the relation

$$h_*\eta_*(0_N) \nsubseteq \tilde{p} = Y \smallsetminus \mathrm{cl}_Y\{p\}$$

implies $p \notin \mathrm{cl}_Y(\mathrm{cl}_X N)$, whence $p \notin \mathrm{cl}_Y N$. Therefore $p \in \mathrm{Rem}(Y \smallsetminus X)$. □

Here are examples of remote points in frames which are not necessarily completely regular.

**Example 3.4.** *Call a downset $U \in \mathfrak{D}L$* prime *if, for any $a, b \in L$, $a \wedge b \in U$ implies $a \in U$ or $b \in U$. It is not hard to show that $U \in \mathrm{Pt}(\mathfrak{D}L)$ if and only if $U$ is a prime downset. The right adjoint of the homomorphism $\bigvee\colon \mathfrak{D}L \to L$ is the map $\downarrow\colon L \to \mathfrak{D}L$. Now, for any $a \in L$ and $U \in \mathfrak{D}L$, ${\downarrow}a \nleq U$ if and only if $a \notin U$; so it follows from the proposition that $U \in \mathfrak{D}L$ is a remote point if and only if it is a prime downset containing no dense element.*

**Example 3.5.** *In the case of the extension $\bigvee\colon \mathfrak{J}L \to L$, more can be said. Recall from [20, Lemma II 3.4] that the points of $\mathfrak{J}L$ are precisely the prime ideals of $L$. So remote points in this extension are exactly the minimal prime ideals of $L$ because a prime ideal of $L$ is minimal prime if and only if it contains no dense element.*

**Example 3.6.** *For any frame $L$, the set of points of $\kappa L$, the Katětov extension of $L$, is*

$$\mathrm{Pt}(\kappa L) = \{(1, \mathfrak{X} \smallsetminus \{F\}) \mid F \in \mathfrak{X}\} \cup \{(p, \mathfrak{X}) \mid p \in \mathrm{Pt}(L)\}.$$

*This is proved in [21, Proposition 3.9]. Since every filter in $\mathfrak{X}$ is an ultrafilter, so that it contains every dense element, we have that, for any dense $d \in L$ and any $F \in \mathfrak{X}$,*

$$\kappa_*(d) = (d, \mathfrak{X}_d) = (d, \mathfrak{X}) \nleq (1, \mathfrak{X} \smallsetminus \{F\}).$$

*Therefore remote points of $\kappa L \to L$ are precisely the points $(1, \mathfrak{X} \smallsetminus \{F\})$, for $F \in \mathfrak{X}$. This agrees with the spatial result that if $X$ is any topological space which is not almost compact, then every point of $\kappa X \smallsetminus X$ is remote from $X$.*

Now we aim to determine when, for a reduced $f$-ring $A$ with bounded inversion, the extension $\varepsilon\colon \mathrm{Rad}(A^*) \to \mathrm{Rad}(A)$ has remote points. We first observe that, generally, every non-essential prime ideal of $\mathrm{Rad}(A^*)$ is a remote point. In the case where $A$ resembles $C(X)$ as explained below, we show that these are precisely the remote points. We denote the annihilator of an ideal $I$ in $A$ by $\mathrm{Ann}(I)$, and the annihilator of an ideal $J$ in $A^*$ by $\mathrm{Ann}_*(J)$. Recall that in a reduced ring, an ideal is essential if and only if its annihilator is the zero ideal.

**Lemma 3.7.** *Let $A$ be a reduced $f$-ring with bounded inversion. Every non-essential prime ideal of $A^*$ is a remote point of the extension $\varepsilon\colon \mathrm{Rad}(A^*) \to \mathrm{Rad}(A)$.*

*Proof.* It is easy to verify that, for any ring $B$, the points of the frame $\mathrm{Rad}(B)$ are exactly the prime ideals of $B$. In [18, Remarks 4.2] the authors observe that the pseudocomplements in $\mathrm{Rad}(B)$ are exactly the annihilator ideals if $B$ is reduced. Thus, dense elements of $\mathrm{Rad}(B)$ are precisely the essential radical ideals because in a reduced ring an ideal is essential if and only if its annihilator is the zero ideal.

Now let $P$ be a non-essential prime ideal of $A^*$. Consider any essential radical ideal $I$ in $A$. This implies $\mathrm{Ann}(I) = 0$. Let $x \in \mathrm{Ann}_*(I^c)$ and $u \in I$. Then $\frac{u}{1+|u|} \in I^c$ and so $\frac{xu}{1+|u|} = 0$, so that $x \in \mathrm{Ann}(I)$ and hence $x = 0$. Thus $I^c$ is an essential ideal in $A^*$. Consequently, $I^c \nsubseteq P$, which then shows that $P$ is a remote point. □

Recall that for a prime ideal $P$ in a ring $A$, the ideal $O_P$ of $A$ is defined by

$$O_P = \{a \in A \mid ab = 0 \text{ for some } b \in A \setminus P\}.$$

If $M$ is a maximal ideal, then $O_M$ is exactly the pure part, $mM$, of $M$; that is, the ideal

$$mM = \{a \in M \mid a = ab \text{ for some } b \in M\}.$$

Let us also recall from [13, Lemma 4.3] that

*an ideal $Q$ of $\mathcal{R}L$ is essential if and only if $\bigvee\{\mathrm{coz}\,\alpha \mid \alpha \in Q\}$ is a dense element in $L$.*

Let $M$ be an essential maximal ideal of $\mathcal{R}L$. By [14, Lemma 4.4] and the fact (observed in the proof of [15, Proposition 3.4]) that $m\boldsymbol{M}^I = \boldsymbol{O}^I$, it follows that $O_M$ is essential. In view of the fact that, for any completely regular frame $L$, $\mathcal{R}^*L$ is isomorphic to $\mathcal{R}(\beta L)$, it follows from what we have just observed that for every maximal ideal $M$ of $\mathcal{R}^*L$, the ideal $O_M$ of $\mathcal{R}^*L$ is essential in this subring. This motivates the following definition.

**Definition 3.8.** *An $f$-ring $A$ is* essentially good *if, for every essential maximal ideal $M$ of $A^*$, the ideal $O_M$ is essential in $A^*$.*

Every $C(X)$ is essentially good because $C(X)$ is isomorphic to $\mathcal{R}(\mathfrak{O}X)$.

**Proposition 3.9.** *Let $A$ be a reduced essentially good $f$-ring with bounded inversion. Then the remote points of the extension $\varepsilon\colon \mathrm{Rad}(A^*) \to \mathrm{Rad}(A)$ are exactly the non-essential prime ideals of $A^*$.*

*Proof.* In view of the preceding lemma, we need only show that every remote point of this extension is a non-essential prime ideal. Let $P$ be an essential prime ideal in $A^*$. We aim to show that $P$ is not a remote point, which will then prove the result. Let $M$ be a maximal ideal of $A^*$ with $P \subseteq M$. Since $A$ is essentially good, $O_M$ is an essential ideal in $A^*$. Let $S \subseteq A^*$ be the multiplicatively closed set

$$S = \{a \in A^* \mid a \text{ is a unit in } A\}.$$

By [16, Lemma 3.4], $A = A^*[S^{-1}]$; the ring of fractions of $A^*$ with respect to $S$. Since $O_M \cap S = \emptyset$, $O_M^e$ is a proper ideal in $A$. We claim that it is essential. Indeed, if $x \in \mathrm{Ann}(O_M^e)$, then, for any $c \in O_M$, $\frac{x}{1+|x|}c = 0$, which implies $\frac{x}{1+|x|} \in \mathrm{Ann}_*(O_M) = \{0\}$, and hence $x = 0$. Next, we show that $O_M^{ec} \subseteq P$. Let $a \in O_M^{ec}$. Then there is a $d \in O_M$ and $u \in S$ such that $a = du^{-1}$. Since $d \in O_M$, there is a $b \in A^* \setminus M$ such that $bd = 0$. This implies $ab = 0 \in P$, whence $a \in P$ because $P$ is prime and $b \notin P$. Now, $O_M$ is a radical ideal because if $w^2 \in O_M$, then $w^2y = 0$ for some $y \in A^* \setminus M$, which implies $wy = 0$ because $A^*$ is a reduced ring. Thus, by [16, Lemma 3.5], $O^e \in \mathrm{Rad}(A)$, and is therefore a dense element in the frame $\mathrm{Rad}(A)$ for which $\varepsilon_*(O_M^e) \le P$. It follows from Proposition 3.2 that $P$ is not a remote point. This completes the proof. □

In [1], Azarpanah shows that the non-essential prime ideals of $C(X)$ are exactly the maximal ideals $\boldsymbol{M}^p$, for $p$ an isolated point in $\beta X$. Consequently, we have the following corollary.

**Corollary 3.10.** *The remote points of the extension $\varepsilon\colon \mathrm{Rad}(C^*(X)) \to \mathrm{Rad}(C(X))$ are precisely the maximal ideals $\boldsymbol{M}^{*p}$ of $C^*(X)$, for $p$ an isolated point of $\beta X$.*

The following result is in the same vein as the preceding one. Recall that an *ℓ-ideal* of an $f$-ring (or, more generally, an ℓ-ring) is a ring ideal $I$ such that

$$|a| \leq |b| \text{ and } b \in I \quad \Longrightarrow \quad a \in I.$$

Let $A$ be a reduced $f$-ring and $\mathfrak{L}(A)$ be the frame of its ℓ-ideals (see [3] for details). If $\tau\colon \mathfrak{L}(A^*) \to \mathfrak{L}(A)$ is the map $I \mapsto \bigcup\{[a] \mid a \in I\}$, where $[a]$ denotes the ℓ-ideal generated by $a$, then $\tau$ is a frame homomorphism whose right adjoint is the contraction map (see [3, p. 141]). This homomorphism can be shown to be dense onto by essentially the same argument as in the proof of [16, Proposition 3.6]. An ℓ-ideal $I$ of $A$ is called *irreducible* if $A/I$ is totally ordered. For reduced $f$-rings, this is equivalent to saying whenever $ab = 0$, for $a, b \in A$, then $a \in I$ or $b \in I$. As mentioned in [3, Remark 2.3],

$$\mathrm{Pt}(\mathfrak{L}(A)) = \{I \in \mathfrak{L}(A) \mid I \text{ is irreducible}\}.$$

It is easy to verify that every annihilator ideal in a reduced $f$-ring $A$ is an ℓ-ideal; and, in fact, for any $I \in \mathfrak{L}(A)$, $\mathrm{Ann}(I)$ is the pseudocomplement of $I$ in the frame $\mathfrak{L}(A)$. Thus, $I$ is a dense element in this frame if and only if $I$ is an essential ideal in $A$.

**Proposition 3.11.** *Let $A$ be a reduced essentially good $f$-ring with bounded inversion. Then the remote points of the extension $\tau\colon \mathfrak{L}(A^*) \to \mathfrak{L}(A)$ are exactly the non-essential irreducible ℓ-ideals of $A^*$.*

*Proof.* An argument similar to the case of $\varepsilon\colon \mathrm{Rad}(A^*) \to \mathrm{Rad}(A)$ shows that non-essential irreducible ℓ-ideals of $\mathfrak{L}(A^*)$ are remote points of the extension $\tau\colon \mathfrak{L}(A^*) \to \mathfrak{L}(A)$. We show that there are no others. The argument mimics the one employed in Proposition 3.9, with some minor changes. Let $P$ be an essential irreducible ℓ-ideal in $A^*$. Let $M$ be a maximal ideal of $A^*$ (and hence an ℓ-ideal) containing $P$. It is easy to check that $O_M$ is an ℓ-ideal in$A^*$. We show that its extension is an ℓ-ideal in $A$. In fact, for any ℓ-ideal $I$ of $A^*$, $I^e$ is an ℓ-ideal in $A$. Indeed, suppose $|a| \leq |b|$ for some $a \in A$ and $b \in I^e$. Pick $u \in I$ and $s \in S$ (the set $S$ as above) such that $b = us^{-1}$, which is possible because $A = A^*[S{-}1$. This implies $|as| \leq |u|$, whence $|\frac{a}{1+|a|} \cdot \frac{s}{1+|s|}| \leq |u|$. Since $\frac{a}{1+|a|}$ and $\frac{s}{1+|s|}$ are in $A^*$ and $I$ is an ℓ-ideal, it follows that $|\frac{a}{1+|a|} \cdot \frac{s}{1+|s|}| \in I$, hence $as \in I$, and thence $a = (as)s^{-1} \in I^e$ because $I^e$ is an ideal of $A$. From here the rest follows as before since $O_M^{ec} \subseteq P$ as $P$ is irreducible. □

## 4. Remote points and coproducts

In this section we show that there are instances where, informally speaking, summands in a binary coproduct inherit remote points from the coproduct. We start by showing how points of a coproduct are constructed from those of the summands. In fact, this is done in [11], but we shall give an alternative proof based on a result of Banaschewski and Vermeulen [7] which we shall also use in another instance.

Recall that if, for $i = 1, 2$, $h_i\colon M_i \to L_i$ are frame homomorphisms, then the induced frame homomorphism $h_1 \oplus h_2\colon M_1 \oplus M_2 \to L_1 \oplus L_2$ is given by

$$(h_1 \oplus h_2)\Big(\bigvee_\alpha (x_\alpha \oplus y_\alpha)\Big) = \bigvee_\alpha \Big(h_1(x_\alpha) \oplus h_2(y_\alpha)\Big).$$

If the $h_i$ are dense (resp. onto), then $h_1 \oplus h_2$ is also dense (resp. onto).

**Lemma 4.1.** *Let $L$ and $M$ be frames, $p \in \mathrm{Pt}(L)$ and $q \in \mathrm{Pt}(M)$. Then*

$$(p \oplus 1_M) \vee (1_L \oplus q) \in \mathrm{Pt}(L \oplus M).$$

*Proof.* Let $\xi\colon L \to \mathbf{2}$ and $\zeta\colon M \to \mathbf{2}$ be the frame homomorphisms determined by $p$ and $q$ respectively. Recall that $\mathbf{2} \oplus \mathbf{2} \cong \mathbf{2}$. Consider the frame homomorphism

$$\xi \oplus \zeta\colon L \oplus M \to \mathbf{2} \oplus \mathbf{2} \quad \text{given by} \quad a \oplus b \mapsto \xi(a) \oplus \zeta(b).$$

By [7, Lemma 2],

$$\begin{aligned}(\xi\oplus\zeta)_*(0) &= \bigvee\{\xi_*(a)\oplus\zeta_*(b)\mid a\oplus b\le 0_{L\oplus M}\}\\ &= \big(\xi_*(0)\oplus\zeta_*(0)\big)\vee\big(\xi_*(0)\oplus\zeta_*(1)\big)\vee\big(\xi_*(1)\oplus\zeta_*(0)\big)\\ &= (p\oplus q)\vee(p\oplus 1_M)\vee(1_L\oplus q)\\ &= (p\oplus 1_M)\vee(1_L\oplus q).\end{aligned}$$

Therefore $(p\oplus 1_M)\vee(1_L\oplus q)$ is a point of $L\oplus M$. □

**Corollary 4.2.** *If $L$ and $M$ are $T_1$-frames, then*

$$\mathrm{Pt}(L\oplus M)=\{(p\oplus 1_M)\vee(1_L\oplus q)\mid p\in\mathrm{Pt}(L) \text{ and } q\in\mathrm{Pt}(M)\}.$$

*Proof.* Let $p\in\mathrm{Pt}(L\oplus M)$ and $L\xrightarrow{i}L\oplus M\xleftarrow{j}M$ be the coproduct injections. Then $i_*(p)$ and $j_*(p)$ are points of $L$ and $M$ respectively such that

$$\big(i_*(p)\oplus 1\big)\vee\big(1\oplus j_*(p)\big)=ii_*(p)\vee jj_*(p)\le p,$$

whence $p=\big(i_*(p)\oplus 1\big)\vee\big(1\oplus j_*(p)\big)$ because coproducts of $T_1$-frames are $T_1$-frames [23]. The result therefore follows from the foregoing lemma. □

In the proof that follows, we write the right adjoint of a homomorphism $h_i\colon M_i\to L_i$ as $h_{i*}$ instead of $(h_i)_*$. By a $T_1$-extension of a frame $L$ we mean an extension $M\to L$ where $M$ is a $T_1$-frame. In this case $L$ is then also a $T_1$-frame.

**Proposition 4.3.** *Let $M_i\xrightarrow{h_i}L_i$ (for $i=1,2$) be $T_1$-extensions of the frames $L_i$. If the extension $M_1\oplus M_2\xrightarrow{h_1\oplus h_2}L_1\oplus L_2$ has a remote point, then each $M_i\xrightarrow{h_i}L_i$ has a remote point.*

*Proof.* Pick $p_i\in\mathrm{Pt}(L_i)$ such that $(p_1\oplus 1)\vee(1\oplus p_2)$ is a point of $M_1\oplus M_2$ remote from $L_1\oplus L_2$. Let $a\in L_1$ be dense. Then $a\oplus 1$ is dense in $L_1\oplus L_2$ because, as shown in [6],

$$(a\oplus 1)^{**}=a^{**}\oplus 1^{**}=1\oplus 1=1_{L_1\oplus L_2}.$$

Therefore, by Proposition 3.2, $(h_1\oplus h_2)_*(a\oplus 1)\nleq(p_1\oplus 1)\vee(1\oplus p_2)$. By [7, Lemma 2],

$$(h_1\oplus h_2)_*(a\oplus 1)=\bigvee\{h_{1*}(x)\oplus h_{2*}(y)\mid x\oplus y\le a\oplus 1\},$$

and so there exist $x\in L_1$ and $y\in L_2$ such that $x\oplus y\le a\oplus 1$ and

$$h_{1*}(x)\oplus h_{2*}(y)\nleq(p_1\oplus 1)\vee(1\oplus p_2).$$

This implies $h_{1*}(x)\oplus h_{2*}(y)\nleq p_1\oplus 1$, and hence $h_{1*}(x)\nleq p_1$. Since $h_{1*}(x)\oplus h_{2*}(y)\ne 0$, $h_{1*}(x)\ne 0$ and $h_{2*}(y)\ne 0$, so that, by density of these homomorphisms, $x\ne 0$ and $y\ne 0$. Thus, the inequality $0\ne x\oplus y\le a\oplus 1$ implies $x\le a$, and hence $h_{1*}(a)\nleq p_1$. Therefore $p_1$ is a remote point of the extension $M_1\xrightarrow{h_1}L_1$. The proof for the other extension is similar. □

**Remark 4.4.** *The same proof shows that even if the $L_i$ are not $T_1$-frames, if $p_i\in\mathrm{Pt}(L_i)$ and $(p_1\oplus 1)\vee(1\oplus p_2)$ is a remote point of $M_1\oplus M_2\to L_1\oplus L_2$, then each $p_i$ is a remote point of $M_i\to L_i$.*

## 5. Remote points in perfect extensions

Following [2], we say an extension $M \xrightarrow{h} L$ is *perfect* if $h_*(a \vee a^*) = h_*(a) \vee h_*(a^*)$ for every $a \in L$. This is equivalent to saying $h_*(a \vee b) = h_*(a) \vee h_*(b)$ for all disjoint $a$ and $b$ in $L$. The extensions $\beta L \to L$ and $\kappa L \to L$ are perfect. For perfect extensions there are more equivalent conditions for a point to be remote. As in [8], we say a filter $F$ in a frame $L$ is *disjoint-prime* if, for any $a \in L$, $a \vee a^* \in F$ implies $a \in F$ or $a^* \in F$. Because a filter is an ultrafilter if and only if, for every $a \in L$, either $a \in F$ or $a^* \in F$, it follows easily that a filter is an ultrafilter if and only if it is saturated and disjoint-prime. Observe that if $M \xrightarrow{h} L$ is a perfect extension, then $U^p$ is saturated for every $p \in \mathrm{Pt}(M)$. Call an ideal $I$ in a frame $L$ *balanced* if, for any $a \in L$, $a^{**} \in I$ whenever $a \in I$. Minimal prime ideals are balanced because they do not contain dense elements, so that if one such contains $a^{**}$, then it does not contain $a^*$, and hence it must contain $a$ by primeness. For an extension $M \xrightarrow{h} L$ and $p \in \mathrm{Pt}(M)$, we set

$$I_p = \{a \in L \mid h_*(a) \leq p\},$$

so that $I_p = L \smallsetminus U^p$.

**Proposition 5.1.** *Let $M \xrightarrow{h} L$ be a perfect extension of $L$. The following statements about a point $p \in \mathrm{Pt}(M)$ are equivalent.*

1. *$p$ is a remote point.*
2. *For any dense $a \in L$, $h_*(a) \nleq p$.*
3. *For any $a \in L$, $h_*(a) \leq p$ implies $h_*(a^*) \nleq p$.*
4. *For any $a \in L$, $h_*(a^*) \leq p$ implies $h_*(a) \nleq p$.*
5. *For any $b \in M$, $b^* \leq p$ implies $h_*h(b) \nleq p$.*
6. *$U^p$ is an ultrafilter.*
7. *$I_p$ is a minimal prime ideal of $L$.*
8. *$I_p$ is a balanced ideal of $L$.*

*Proof.* (2) ⇒ (3): Let $a \in L$ be such that $h_*(a) \leq p$. Since $a \vee a^*$ is dense, (2) implies $h_*(a \vee a^*) \nleq p$. Since $h_*(a \vee a^*) = h_*(a) \vee h_*(a^*)$ and $h_*(a) \leq p$, it follows that $h_*(a^*) \nleq p$.

(3) ⇒ (4): Clearly the denial of (4) contradicts (3).

(4) ⇒ (5): For any $b \in M$, $b^* = h_*h(b^*)$, so $b^* \leq p$ implies $h_*h(b^*) \leq p$, that is, $h_*(h(b)^*) \leq p$, so that, $h_*h(b) \nleq p$ by (4).

(5) ⇒ (6): Let $a \in L$ be such that $a^* \notin U^p$. Then $h_*(a^*) \leq p$, that is $h_*(a)^* \leq p$. So, by (5), $h_*hh_*(a) \nleq p$, that is, $h_*(a) \nleq p$, so that $a \in U^p$. Therefore $U^p$ is an ultrafilter.

(6) ⇒ (7): Since $I_p = L \smallsetminus U^p$, it follows from [10, Corollary 3], which states that a filter is an ultrafilter if and only if its set-theoretic complement is a minimal prime ideal, that $I_p$ is a minimal prime ideal in $L$.

(7) ⇒ (8): Minimal prime ideals are balanced.

(8) ⇒ (1): If $L \xrightarrow{\eta} N$ is a nowhere dense quotient of $L$, then $\eta_*(0)$ is dense, and is therefore not in $I_p$, otherwise $1 = \eta_*(0)^{**} \in I_p$ because $I_p$ is balanced. Thus, $h_*\eta_*(0) \nleq p$, hence $p$ is remote from $L$. □

We shall now determine, in terms of filters, when a perfect extension has remote points. It is easy to check that the image of a filter under a dense onto homomorphism is a (proper) filter. Recall from Section 2 the notation and the one-one correspondence between points and completely prime filters.

**Proposition 5.2.** *Let $M \xrightarrow{h} L$ be a perfect extension of $L$ and $p \in \mathrm{Pt}(M)$. Then $p$ is a remote point iff $h[F_p]$ is an ultrafilter.*

*Proof.* Assume first that $p$ is a remote point. By Proposition 5.1, the filter $U^p$ is an ultrafilter. If $a \in U^p$, then $h_*(a) \nleq p$, which is to say $h_*(a) \in F_p$, and hence $a \in h[F_p]$ because $a = hh_*(a)$. Therefore $U^p \subseteq h[F_p]$. Since $h[F_p]$ is a (proper) filter, the maximality of $U^p$ implies $h[F_p] = U^p$, and hence $h[F]$ is an ultrafilter.

Conversely, assume $h[F_p]$ is an ultrafilter. We shall show that $U^p$ is an ultrafilter, whence we shall be done. Observe that $U^p \subseteq h[F_p]$ because the verification above for this assertion did not require that $p$ be a remote point. Now suppose, by way of contradiction, that there is an $a \in F_p$ for which $h(a) \notin U^p$. Then $h_*h(a) \leq p$, which implies $a \leq \bigvee(L \setminus F_p)$. Since $a \in F_p$ and $F_p$ is an upset, this implies $\bigvee(L \setminus F_p) \in F_p$, which is false because $F_p$ is completely prime. Thus, $h[F_p] \subseteq U^p$, and hence $U^p = h[F_p]$. Therefore $U^p$ is an ultrafilter, so that $p$ is a remote point by Proposition 5.1. □

**Corollary 5.3.** *A perfect extension $M \xrightarrow{h} L$ of $L$ has a remote point iff it has a completely prime filter $F$ for which $h[F]$ is an ultrafilter. Furthermore, remote points are in bijective correspondence with completely prime filters $G$ for which $h[G]$ is an ultrafilter.*

*Proof.* This follows immediately from the proposition because $p = p_{F_p}$ and $F_{p_F} = F$. □

It is shown in [5] that if $\mathfrak{X}$ is a set of filters in $L$, then, for any $F \in \mathfrak{X}$, the set

$$P_F = \left\{(a, \mathfrak{X}_W) \mid a = \bigvee W \text{ and } F \in \mathfrak{X}_W\right\}$$

is a completely prime filter in $\tau_{\mathfrak{X}}L$ for which $\tau[P_F] = F$. Consequently, we have the following corollary.

**Corollary 5.4.** *If a set $\mathfrak{X}$ of filters of $L$ contains an ultrafilter and the strict extension $\tau_{\mathfrak{X}}L \to L$ is perfect, then it has a remote point.*

We recall from [5] that an extension $M \xrightarrow{h} L$ of $L$ is said to be *spatial over* $L$ if whenever $h(a) = h(b)$ and $a \nleq b$, then there exists $p \in \mathrm{Pt}(M)$ such that $b \leq p$ and $a \nleq p$. In the cited paper this is expressed in terms of completely prime filters. A filter $F \subseteq L$ is called a *trace filter* [5] if it is not completely prime but $F = h[P]$ for some completely prime filter $P$ of $M$. We aim to show that if the perfect extension $M \xrightarrow{h} L$ is spatial over $L$, then it has a remote point precisely if it has a completely prime filter whose image is saturated. We first observe the following result.

**Lemma 5.5.** *The following conditions on an extension $M \xrightarrow{h} L$ of $L$ which is spatial over $L$ are equivalent.*

1. *The extension is perfect.*
2. *For every $p \in \mathrm{Pt}(M)$, $h[F_p]$ is disjoint-prime.*
3. *Every trace filter of the extension is disjoint-prime.*

*Proof.* (1) ⇒ (2): Assume the extension is perfect and let $p \in \mathrm{Pt}(M)$. Let $a \in L$ be such that $a \vee a^* \in h[F_p]$. Pick $u \in F_p$ such that $a \vee a^* = h(u)$. Then $u \leq h_*(a \vee a^*) = h_*(a) \vee h_*(a^*)$. Thus, $h_*(a) \vee h_*(a^*) \in F_p$, which implies $h_*(a) \in F_p$ or $h_*(a^*) \in F_p$. Hence $a \in h[F_p]$ or $a^* \in h[F_p]$.

(2) ⇒ (3): This is trivial.

(3) ⇒ (1): Let $\mathfrak{X}$ be the set of trace filters of the extension. By [5, Lemma 3], there is an isomorphism $\hat{h}\colon M \to \tau_{\mathfrak{X}}L$ such that the $\tau\hat{h} = h$. Therefore it suffices to show that the extension $\tau\colon \tau_{\mathfrak{X}}L \to L$ is perfect if every filter in $\mathfrak{X}$ is disjoint-prime. Recall that, for any $b \in L$, $\tau_*(b) = (b, \mathfrak{X}_b)$, where

$$\mathfrak{X}_b = \{F \in \mathfrak{X} \mid b \in F\}.$$

Let $a \in L$. If $F \in \mathfrak{X}_{a\vee a^*}$, then $a \vee a^* \in F$, and hence $a \in F$ or $a^* \in F$ by disjoint-primeness. This implies $F \in \mathfrak{X}_a \cup \mathfrak{X}_{a^*}$, so that $\mathfrak{X}_{a\vee a^*} \subseteq \mathfrak{X}_a \cup \mathfrak{X}_{a^*}$, and hence $\mathfrak{X}_{a\vee a^*} = \mathfrak{X}_a \cup \mathfrak{X}_{a^*}$ because the other inclusion holds anyway. Thus,

$$\begin{aligned} \tau_*(a) \vee \tau_*(a^*) &= (a, \mathfrak{X}_a) \vee (a^*, \mathfrak{X}_{a^*}) \\ &= (a \vee a^*, \mathfrak{X}_a \cup \mathfrak{X}_{a^*}) \\ &= \tau_*(a \vee a^*), \end{aligned}$$

which proves that the extension is perfect. □

**Corollary 5.6.** *A perfect extension* $M \xrightarrow{h} L$ *of* $L$ *which is spatial over* $L$ *has a remote point iff there is a completely prime filter* $F \subseteq M$ *such that* $h[F]$ *is saturated.*

*Proof.* The left-to-right implication follows from Corollary 5.3 because ultrafilters are saturated. Conversely, suppose the extension has a completely prime filter $F$ as stated. For the point $p_F$ of $M$ we have that $h[F_{p_F}]$ is disjoint-prime by the foregoing lemma. Since $F_{p_F} = F$, it follows that $h[F]$ is saturated and disjoint-prime, and hence is an ultrafilter. Therefore $M$ has a remote point. □

We end with a sufficient condition, in terms of completely regular filters, for $\beta L \to L$ to have a remote point. Recall that a filter $F$ is said to be *completely regular* if, for every $a \in F$, there exists $b \in F$ such that $b \ll a$. The following lemma appears in the pointed version as [8, Theorem 2.22].

**Lemma 5.7.** *Every maximal completely regular filter in a completely regular frame is disjoint-prime. Hence it is an ultrafilter iff it is saturated.*

*Proof.* Let $F$ be a maximal completely regular filter in a completely regular frame $L$. We know from [5] that $\beta L \to L$ is (isomorphic to) the strict extension $\tau_{\mathfrak{X}} L \to L$, where $\mathfrak{X}$ is the set of all maximal completely regular filters in $L$. Thus, if $p$ is the point of $\tau_{\mathfrak{X}} L$ corresponding to the completely prime filter

$$P_F = \{(a, \mathfrak{X}_W) \mid a = \bigvee W, F \in \mathfrak{X}_W\},$$

then $\tau[F_p] = F$. Since $\beta L \to L$ is a perfect extension, it follows from Lemma 5.5 that $F$ is disjoint-prime. □

**Corollary 5.8.** *If a completely regular frame* $L$ *has a saturated maximal completely regular filter, then* $\beta L \to L$ *has a remote point.*

## References


[1] F. Azarpanah, Essential ideals in $C(X)$, Periodica Mathematica Hungarica 31 (1995) 105–112.
[2] D. Baboolal, Perfect compactifications of frames, Czechoslovak Mathematical Journal 61 (2011) 845–861.
[3] B. Banaschewski, Pointfree topology and the spectra of $f$-rings, in: Ordered Algebraic Structures, Proceedings of the Curaçao Conference, 1995, Kluwer Academic Publishers, Dodrecht, 1997, pp. 123–148.
[4] B. Banaschewski, The real numbers in pointfree topology, Textos de Matemática Série B, No. 12, Departamento de Matemática da Universidade de Coimbra, (1997).
[5] B. Banaschewski, S.S. Hong, Filters and strict extensions of frames, Kyungpook Mathematical Journal 39 (1999) 215–230.
[6] B. Banaschewski, A. Pultr, Booleanization, Cahiers de Topologie et Géométrie Différentielle Catégoriques 37 (1996) 41–60.
[7] B. Banaschewski, J. Vermeulen, On the completeness of localic groups, Commentationes Mathematicae Universitate Carolinae 40 (1999) 293–307.
[8] N.A. Carlson, J.R. Porter, On open ultrafilters and maximal points, Topology and its Applications 156 (2009) 2317–2325.
[9] E.K. van Douwen, Remote points, Dissertationes Mathematicae (Rozprawy Matematyczne) 188 (1981), 50pp.
[10] T. Dube, Balanced and closed-generated filters in frames, Quaestiones Mathematicae 26 (2003) 73–81.
[11] T. Dube, Irreducibility in pointfree topology, Quaestiones Mathematicae **27** (2004), 231–241.
[12] T. Dube, Remote points and the like in pointfree topology, Acta Mathematica Hungarica 123 (2009) 202–223.
[13] T. Dube, Some ring-theoretic properties of almost $P$-frames, Algebra Universalis 60 (2009) 145–162.
[14] T. Dube, Some algebraic characterizations of $F$-frames, Algebra Universalis, 62 (2009) 273–288.
[15] T. Dube, On the ideal of functions with compact support in pointfree function rings, Acta Mathatica Hungarica 129 (2010) 205–226.
[16] T. Dube, O. Ighedo, Comments regarding $d$-ideals of certain $f$-rings, Journal of Algebra and its Applications 12 (2013) 135008 16pp.
[17] L. Gillman, M. Jerison, Rings of Continuous Functions, Van Nostrand, Princeton, 1960.
[18] A.W. Hager, J. Martínez, Patch-generated frames and projectable hulls, Applied Categorical Structures 17 (2007) 49–80.
[19] S.S. Hong, Simple extensions of frames, Proceedings of Recent Developments in General Topology and its Application, Akademia Verlag, Berlin, (1992) 156–159.
[20] P.T. Johnstone, Stone Spaces, Cambridge University Press, Cambridge, 1982.
[21] J. Paseka, B. Šmarda, $T_2$-frames and almost compact frames, Czechoslovak Mathematical Journal 42 (1992) 385–402.
[22] J. Picado, A. Pultr, Frames and Locales: topology without points, Frontiers in Mathematics, Springer/Basel, 2012.
[23] J. Rosicky, B. Šmarda, $T_1$-locales, Mathematical Proceedings of the Cambridge Philosophical Society 98 (1985) 81–86.
[24] F.L. Zhang, D. Strauss, Katětov-extension of a frame, Proceedings of a Symposium on Categorical Topology, University of Cape Town, (1994) 261–271.