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The Non-Equivalence of τ-Ultracompactness and τ-Boundedness

Buras Boljiev

Insitute of mathematics and CS, Univerity of Latvia, Raina Bulv. 29, Riga LV-1459, Latvia

Abstract. The main result presented here is a solution to the following problem of V. Saks: Does there exist $\mathfrak{M} > \aleph_0$ and a Hausdorff $\mathfrak{M}$-ultracompact space which is not $\mathfrak{M}$-bounded? The main result is given in a stronger form than the problem suggests itself: For each infinite cardinal τ there is a Hausdorff τ-ultracompact not τ-bounded space of density τ.

In [1] A. Bernstein introduced the following definitions: let $p \in \beta\omega \setminus \omega$ be a free ultrafilter on ω, the (discrete space) of positive integers. Now let $(x_n : n \in \omega)$(for short (x_n)) be a sequence of points in a topological space X and $x \in X$. Then x is a p-limit point of (x_n) provided that for each neighborhood U of x the set $\{n \in \omega : x_n \in U\}$ belongs to p, in this case we write $x = p - lim\ x_n$. If every sequence in X has a p-limit point then X is called p-compact. Each infinite cardinal is identified with the initial ordinal of the same cardinality.

V. Saks [2] generalizes the notion of a p-limit point to transfinite sequences in the following way: let τ be an infinite cardinal; if $p \in \beta\tau \setminus \tau$ is a free ultrafilter on τ (with the discrete topology) and $(x_\alpha : \alpha \in \tau)$ (for short (x_α)) is a τ-sequence in a space X, then $x \in X$ is a p-limit point of (x_α), denoted by $x = p - lim\ x_\alpha$, if for each neighborhood U of x, $\{\alpha : x_\alpha \in U\} \in p$ and we can say, in this case, that (x_α) p-converges to x. Saks further extends p-compactness for any ultrafilter $p \in \beta\tau \setminus \tau$ where a space X is p-compact if any τ-sequence in X has a p-limit point. He proves there that in the class of regular spaces the notions of τ-boundedness and τ-ultracompactness are equivalent for any infinite cardinal τ, where τ-boundedness means that the closure of any subset of cardinality not exceeding τ is compact and τ-ultracompactness means that X is p-compact for any $p \in \beta\tau \setminus \tau$. In case of $\tau = \aleph_0$ we obtain the notions of ultracompactness and $\aleph_0$-boundedness which are not equivalent in the class of Hausdorff spaces as demonstrates an example in [2] but the space in this example is not separable so V. Saks asks there: Does there exist a separable Hausdorff ultracompact space which is not compact? The positive answer to the problem is in [4] and the theorem 3 in the present article covers not only this result but also give a positive answer in a stronger form to another question of V. Saks [2]: Does there exist $\mathfrak{M} > \aleph_0$ and a Hausdorff $\mathfrak{M}$-ultracompact space which is not $\mathfrak{M}$-bounded?

A. P. Kombarov introduced in [3] the notion of a p-sequential space for $p \in \beta\omega \setminus \omega$ and this notion was extended for any $p \in \beta\tau \setminus \tau$ by L. Kočinac [5] in the context of chain-net spaces but for our goals we prefer here to use the name which offered A. P. Kombarov: a space X is p-sequential if for any nonclosed $A \subset X$ there are some τ-sequence $(x_\alpha) \subset A$ and a point $x \notin A$ such that $x = p - lim\ x_\alpha$. In this case we can say that (x_α) p-converges to x.

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Email address: boljievb@mail.ru, buras.boljiev@lumii.lv (Buras Boljiev)

Let (X, γ) be a topological space, $O \subset X$ and $p \in \beta\tau \setminus \tau$, then O is said to be p-sequentially open if $x = p - lim\ x_\alpha$ for some $x \in O$ and some τ-sequence (x_α) imply $\{\alpha : x_\alpha \in O\} \in p$.

Let γ_p be the set of all p-sequentially open sets in (X, γ). It is clear that the union of any number of p-sequentially open sets is again p-sequentially open and the intersection of a finite number of p-sequentially open sets is p-sequentially open. Obviously, each open set is p-sequentially open so we get the following statement.

Proposition 1. *Let (X, γ) be a topological space, then the family γ_p forms a topology on X and $\gamma \subset \gamma_p$.*

It is important to note that $x = p - lim\ x_\alpha$ in γ implies $x = p - lim\ x_\alpha$ in γ_p. Really, if we have $x \neq p - lim\ x_\alpha$ in γ_p for some x and some τ-sequence (x_α), then there exists some $W \in \gamma_p$ such that $x \in W$ and $\{\alpha : x_\alpha \in W\} \notin p$. Obviously that $x \neq p - lim\ x_\alpha$ in γ too, otherwise for the sequentially open set W we would get that $\{\alpha : x_\alpha \in W\} \in p$ which is in contradiction with $\{\alpha : x_\alpha \in W\} \notin p$.

Proposition 2. *Topological space (X, γ_p) is p-sequential.*

Proof. Let A be a nonclosed subset in (X, γ_p), then $O = X \setminus A$ is not open in (X, γ_p), i.e. O is not p-sequentially open in (X, γ) which implies that there are some point $z \in O$ and some τ-sequence (z_α) p-converging to z such that $\{\alpha : z_\alpha \in O\} \notin p$ which implies that $\{\alpha : z_\alpha \in A\} \in p$. We put $x_\alpha = z_\alpha$ for $z_\alpha \in A$ and $x_\alpha = y$ for some $y \in A$ if $z_\alpha \notin A$. Now it is easy to verify that $z = p - lim\ x_\alpha$ for a τ-sequence $(x_\alpha) \subset A$. So (X, γ_p) is a p-sequential space. □

As usually by symbol $t(X, \gamma)$ we denote the tightness of a topological space (X, γ)

Proposition 3. *The intersection of any family of topologies each of tightness not greater than τ has the tightness not greater than τ too.*

Proof. Let $\gamma = \cap\{\gamma_\alpha : \alpha < k\}$ where each γ_α is a topology on a set X such that $t(X, \gamma_\alpha) \leq \tau$ for any $\alpha < k$. For each $A \subset X$ we put $A_1 = \cup\{[A]_{\gamma_\alpha} : \alpha < k\}$. Suppose we have constructed A_α for any ordinal $\alpha < \beta$ where $\beta < \tau^+$. Now we construct A_β and there are two cases:

1. $\beta = \alpha_0 + 1$ for some α_0 then
   $A_\beta = (A_{\alpha_0})_1$.
2. β is a limit ordinal then $A_\beta = \cup\{A_\alpha : \alpha < \beta\}$.

Finally we put $A_{\tau^+} = \cup\{A_\alpha : \alpha < \tau^+\}$

Using the fact that $\tau \cdot \tau = \tau$ we can state that $[A]_\gamma = A_{\tau^+}$ and applying transfinite induction on ordinals $\alpha < \tau^+$ one can see that for any $z \in [A]_\gamma$ there is $B \subset A$ such that $|B| \leq \tau$ and $z \in [B]_\gamma$ □

Proposition 4. *The tightness of a p-sequential space is not greater than τ.*

Proof. For each subset A of X let $A_1 = \{x : x = p - lim\ x_\alpha$ for some τ-sequence $(x_\alpha) \subset A\}$.

Like in the proof of the previous proposition we put $A_\beta = (A_\alpha)_1$ for $\beta = \alpha + 1$ and for a limit ordinal β let $A_\beta = \cup\{A_\alpha : \alpha < \beta\}$. It is easily seen that $A_{\tau^+} = (A_{\tau^+})_1$ and thus $[A] = A_{\tau^+}$ which due to the $\tau \cdot \tau = \tau$ imply the required result. □

The topology γ_p is called a p-sequential leader of γ. Let $\gamma_\tau = \cap\{\gamma_p : p \in \beta\tau \setminus \tau\}$ i.e. γ_τ is the intersection of all p-sequential leaders in (X, γ). The following theorem is a corollary of the propositions 3 and 4.

Theorem 1. *The tightness of a topological space (X, γ_τ) does not exceed τ.*

Theorem 2. *For a topological space (X, γ) $t(X, \gamma) \leq \tau$ iff $\gamma = \gamma_\tau$.*

Proof. We need only to prove the necessity, i.e. that the condition $t(X, \gamma) \leq \tau$ implies $\gamma = \gamma_\tau$. It is sufficient to demonstrate that $\gamma_\tau \subset \gamma$. To this end we take any nonopen set in the topology γ, say M. Then $A = X \setminus M$ is a nonclosed set in γ and there are some subset $B \subset A$ with $|B| \leq \tau$ and some point $y \in M$ such that $y \in [B]_\gamma$. Considering B as a τ-sequence (x_α) one can find some $q \in \beta\tau \setminus \tau$ such that (x_α) q-converges to y in γ. Then (x_α) q-converges to y in γ_q too. Since $\gamma_\tau \subset \gamma_q$ it follows that (x_α) q-converges to y in γ_τ. Thus M is not open in γ_τ implying $\gamma_\tau \subset \gamma$. □

Theorem 3. *Let (X, γ) be a Hausdorff compact topological space of density τ with tightness greater than τ. Then (X, γ_τ) is a Hausdorff τ-ultracompact not a τ-bounded space of density τ.*

Proof. Let X_0 be a dense subset in X of power τ. From the proof of the theorem 2 it is clear that two closure operators $[]_\gamma$ and $[]_{\gamma_\tau}$ coincide on subsets of power no more than τ. So we can see that (X, γ_τ) is a τ-ultracompact space and it contains X_0 as its dense subset. Since $t(X, \gamma_\tau) \leq \tau$ then the topology γ_τ is strictly stronger than γ and hence (X, γ_τ) is not a compact space which in its turn implies that it is not τ-bounded. Thus (X, γ_τ) is a τ-ultracompact not a τ-bounded space of density τ. □

It is known that the Stone-Čech compactification of any discrete space of power $\tau \geq \aleph_0$ has a tightness more than τ so we get the following result.

Corollary 1. *For every infinite cardinal τ there is a Hausdorff τ-ultracompact not a τ-bounded space space of density τ.*

Corollary 2. *The notions of τ-ultracompactness and τ-boundedness are not equivalent in the class of Hausdorff spaces.*

Proposition 5. *The topology γ_τ is the least one among all topologies of tightness not greater than τ and each containing the given topology γ.*

Proof. Let σ be any topology with tightness not greater than τ and containing γ. Assume that A is a nonclosed set in σ. Then it is nonclosed in γ. Fix $x \in [X] \setminus X$ then there is some $B \subset A$, $|B| \leq \tau$ such that $x \in [B]_\sigma$ and consequently $x \in [B]_\gamma$. Now we can represent B as a τ-sequence q-converging in γ to x for some $q \in \beta\tau \setminus \tau$ and hence q-converging to x in γ_q. So this τ-sequence q-converges to x in γ_τ implying that A is a nonclosed set in γ_τ which proves that $\gamma_\tau \subset \sigma$. □

The closure operator in the topological space (X, γ_τ) can be described more clearly using the following τ-closure operator on (X, γ): let $A \subset X$ then we put $[A]_\tau = \{x : \exists B \subset A \text{ such that } |B| \leq \tau \text{ and } x \in [B]_\gamma\}$. This operator is well-known and generates some topology, say $\gamma\prime_\tau$, of tightness not greater than τ with $\gamma\prime_\tau \supset \gamma$ and coinciding with the origin topology γ provided the tightness of the space (X, γ) does not exceed τ.

Proposition 6. *In any topological space (X, γ) the topologies γ_τ and $\gamma\prime_\tau$ coincide.*

Proof. From the previous proposition we get that $\gamma_\tau \subset \gamma\prime_\tau$ but the converse inclusion can be obtained using the same arguments as in the proof of the proposition 5. □

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