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On Slowly Varying Sequences

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Abstract. In this paper we investigate the connection between the class SV_s of slowly varying sequences (in the sense of Karamata) and the slow equivalence, strong asymptotic equivalence, selection principles and game theory.

1. Introduction and results

Real functions $f, g : [a, +\infty) \longmapsto \mathbb{R}$, $(a > 0)$, are *mutually inversely asymptotic*, in denotation $f(x) \overset{*}{\sim} g(x)$, as $x \to +\infty$ (see e.g. [1, 5, 7]), if for each $\lambda > 1$, there is an $x_0 = x_0(\lambda) \geqslant a$ such that the inequality

$$f\left(\frac{x}{\lambda}\right) \leqslant g(x) \leqslant f(\lambda x), \tag{1}$$

is satisfied for each $x \geqslant x_0$.

In particular, real functions $f, g : [a, +\infty) \longmapsto (0, +\infty)$, $(a > 0)$, are *mutually slowly equivalent* (see e.g. [8]), in denotation $f(x) \overset{s}{\sim} g(x)$, as $x \to +\infty$, if

$$\lim_{x \to +\infty} \frac{f(\lambda x)}{g(x)} = 1 \tag{2}$$

and

$$\lim_{x \to +\infty} \frac{g(\lambda x)}{f(x)} = 1 \tag{3}$$

hold for each $\lambda > 1$.

Sequences of positive real numbers $(c_n)_{n \in \mathbb{N}}$ and $(d_n)_{n \in \mathbb{N}}$ are *mutually slowly equivalent*, in denotation $c_n \overset{s}{\sim} d_n$, as $n \to +\infty$, if

$$\lim_{n \to +\infty} \frac{c_{[\lambda n]}}{d_n} = 1 \tag{4}$$

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and

$$\lim_{n\to+\infty}\frac{d_{[\lambda n]}}{c_n}=1 \tag{5}$$

hold for each $\lambda > 1$.

A measurable real function $f : [a, +\infty) \longmapsto (0, +\infty)$, $(a > 0)$ is *slowly varying* in sense of Karamata (see e.g. [9]) if

$$\lim_{x\to+\infty}\frac{f(\lambda x)}{f(x)}=1, \tag{6}$$

holds for each $\lambda > 0$. The set of all these functions is denoted by SV_f. The class SV_f is very important in asymptotic analysis (see [12]).

A sequence of positive real numbers $c = (c_n)_{n\in\mathbb{N}}$ is *slowly varying* in sense of Karamata (see e.g. [1]) if

$$\lim_{n\to+\infty}\frac{c_{[\lambda n]}}{c_n}=1, \tag{7}$$

holds for each $\lambda > 0$. The set of all these sequences important in asymptotic analysis is denoted by SV_s (see [1]).

In this paper the set of all positive real sequences will be denoted with $\mathbb{S}$ (see e.g. [2]).

Proposition 1.1. *Let sequences $c = (c_n)_{n\in\mathbb{N}}$ and $d = (d_n)_{n\in\mathbb{N}}$ be elements from $\mathbb{S}$. If $c_n \overset{s}{\sim} d_n$, as $n \to +\infty$, then $c \in SV_s$ and $d \in SV_s$.*

Proposition 1.2. *Relation $\overset{s}{\sim}$ is a relation of equivalence in SV_s.*

The next definition is well-known definition of α_i-selection principles (see e.g. [11]).

Definition 1.3. Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty subfamilies of the set $\mathbb{S}$. The symbol $\alpha_i(\mathcal{A},\mathcal{B})$, $i \in \{2, 3, 4\}$, denotes the following selection hypotheses: for each sequence $(A_n)_{n\in\mathbb{N}}$ of elements from $\mathcal{A}$ there is an element $B \in \mathcal{B}$ such that:

1. $\alpha_2(\mathcal{A},\mathcal{B})$: the set $\mathrm{Im}(A_n) \cap \mathrm{Im}(B)$ is infinite for each $n \in \mathbb{N}$;
2. $\alpha_3(\mathcal{A},\mathcal{B})$: the set $\mathrm{Im}(A_n) \cap \mathrm{Im}(B)$ is infinite for infinitely many $n \in \mathbb{N}$;
3. $\alpha_4(\mathcal{A},\mathcal{B})$: the set $\mathrm{Im}(A_n) \cap \mathrm{Im}(B)$ is nonempty for infinitely many $n \in \mathbb{N}$,

where Im denotes the image of the corresponding set.

We need also the definition of an interesting game related to the α_2 selection principle (see e.g [11]; see also [4]).

Definition 1.4. Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty subfamilies of the set $\mathbb{S}$. The symbol $G_{\alpha_2}(\mathcal{A},\mathcal{B})$ denotes the following infinitely long game for two players who play a round for each natural number n. In the first round the first player plays an arbitrary element $A_1 = (A_{1,j})_{j\in\mathbb{N}}$ from $\mathcal{A}$, and the second one chooses an elements from the subsequence $y_{r_1} = (A_{1,r_1(j)})_{j\in\mathbb{N}}$ of the sequence A_1. At the k^{th} round, $k \geq 2$), the first player plays an arbitrary element $A_k = (A_{k,j})_{j\in\mathbb{N}}$ from $\mathcal{A}$ and the second one chooses an elements from the subsequence $y_{r_k} = (A_{k,r_k(j)})_{j\in\mathbb{N}}$ of the sequence A_k, such that $\mathrm{Im}(r_k(j)) \cap \mathrm{Im}(r_p(j)) = \emptyset$ is satisfied, for each $p \leq k-1$. We will say that the second player wins a play $A_1, y_{r_1}; \ldots; A_k, y_{r_k}; \ldots$ if and only if all elements from the $Y = \cup_{k\in\mathbb{N}} \cup_{j\in\mathbb{N}} A_{k,r_k(j)}$, with respect to second index, form a subsequence of the sequence $y = (y_m)_{m\in\mathbb{N}} \in \mathcal{B}$.

Remark 1.5. Let sequence $c = (c_n)_{n\in\mathbb{N}} \in SV_s$. We will introduce the next set

$$[c]_s = \left\{ d = (d_n)_{n\in\mathbb{N}} \in SV_s \middle| \, c_n \overset{s}{\sim} d_n, \ as\ n \to +\infty \right\}.$$

Proposition 1.6. *The second player has a winning strategy in the game $G_{\alpha_2}([c]_s, [c]_s)$ for each fixed sequence $c \in SV_s$.*

Corollary 1.7. *The selection principle $\alpha_2([c]_s, [c]_s)$ is satisfied, where the sequence $c \in SV_s$ is given and fixed.*

Remark 1.8. (1) From Corollary 1.7 and [2] it follows that the selection principles $\alpha_i([c]_s, [c]_s)$ are satisfied for $i \in \{3, 4\}$, where the sequence $c \in SV_s$ is arbitrary pre-selected and fixed.

(2) From the proof of Proposition 1.6 we have that $c_n \sim d_n$, as $n \to +\infty$, is equal to $c_n \overset{s}{\sim} d_n$, as $n \to +\infty$, whenever sequences $c = (c_n)_{n\in\mathbb{N}}$ and $d = (d_n)_{n\in\mathbb{N}}$ belong to the class SV_s. (The symbol $\sim$ denotes strong asymptotic equivalence (see e.g. [1])).

(3) The assertion of Corollary 1.7 has already been given in [3], but in a different form. Actually, in [3] only the sketch of the proof of this corollary is given.

The following is the definition of one of classical selection principles (see e.g. [10]).

Definition 1.9. Let $\mathcal{A}$ and $\mathcal{B}$ be a nonempty subfamilies of the set $\mathbb{S}$. The symbol $S_{fin}(\mathcal{A}, \mathcal{B})$ denotes the next selection hypothesis: for each sequence $(A_n)_{n\in\mathbb{N}}$ from $\mathcal{A}$ there is a sequence $B \in \mathcal{B}$ which consists of some numbers from the double sequence $(A_n)_{n\in\mathbb{N}}$ such that sequences B and $(A_n)_{n\in\mathbb{N}}$ have finitely many common elements for each $n \in \mathbb{N}$.

In the following definition we define a new interesting two-person game.

Definition 1.10. Let $\mathcal{A}$ and $\mathcal{B}$ be a nonempty subfamilies of the set $\mathbb{S}$. By $G^{\bullet}_{fin}(\mathcal{A}, \mathcal{B})$ we denote the following infinitely long game for two players: In the first round the first player plays element $A_1 \in \mathcal{A}$, and the second player chooses k_1 ($k_1 \in \mathbb{N}$) elements from the sequence A_1, i.e. elements $b_{11}, b_{12}, \dots, b_{1k_1}$. At s^{th} round, $s \geqslant 2$, the first player chooses an element $A_s \in \mathcal{A}$, and the second player responses by choosing k^*_{s-1}, ($k^*_{s-1} \in \mathbb{N} \cup \{0\}$) elements from the sequence A_{s-1}, i.e. $b_{s-1k_{s-1}+1}, b_{s-1k_{s-1}+2}, \dots, b_{s-1k_{s-1}+k^*_{s-1}}$ and k_s^{th} element from the sequence A_s, say b_{sk_s}. If we form the sequence $(b_t)_{t\in\mathbb{N}}$ from such chosen elements

$$b_{11}, b_{12}, \dots, b_{1k_1}, \dots, b_{s-1k_{s-1}}, b_{s-1k_{s-1}+1}, b_{s-1,k_{s-1}+2}, \dots, b_{s-1k_{s-1}+k^*_{s-1}}, b_{sk_s}, \dots$$

then we say that the second player wins a play

$$A_1, b_{11}, b_{12}, \dots, b_{1k_1}; \cdots ; A_s, b_{s-1k_{s-1}}, b_{s-1k_{s-1}+1}, \cdots , b_{s-1k_{s-1}+k^*_{s-1}}, b_{sk_s}; \cdots$$

if the sequence $(b_t)_{t\in\mathbb{N}}$ belongs to $\mathcal{B}$.

Proposition 1.11. *The second player has a winning strategy in the game $G^{\bullet}_{fin}([c]_s, [c]_s)$ for each fixed $c \in SV_s$.*

An important game, denoted by $G^{*}_{fin}(\mathcal{A}, \mathcal{B})$, was considered in [6]. The game $G^{\bullet}_{fin}(\mathcal{A}, \mathcal{B})$ introduced in the previous definition is a special case of the game $G^{*}_{fin}(\mathcal{A}, \mathcal{B})$.

Corollary 1.12. *The second player has a winning strategy in the game $G^{*}_{fin}([c]_s, [c]_s)$ for each fixed $c \in SV_s$.*

Remark 1.13. From the previously mentioned we have that the selection principle $S_{fin}([c]_s, [c]_s)$ is satisfied for each fixed sequence $c \in SV_s$.

2. Proofs of the results

Proof of Proposition 1.1. Firstly, we have

$$\lim_{n\to+\infty} \frac{c_{[\lambda n]}}{c_n} = \lim_{n\to+\infty} \frac{c_{[\lambda n]}}{d_{([\lambda]-1)n}} \cdot \lim_{n\to+\infty} \frac{d_{([\lambda]-1)n}}{c_n} = \lim_{n\to+\infty} \frac{c_{[\frac{\lambda n}{([\lambda]-1)n} \cdot ([\lambda]-1)n]}}{d_{([\lambda]-1)n}} \cdot 1 = 1 \cdot 1 = 1,$$

since $[\lambda] - 1 > 1$ and $\frac{\lambda}{[\lambda]-1} > 1$, for $\lambda \geq 3$.

Now, let us observe the function $c_{[x]}$, for $x \geqslant 1$, where x is real number. Let $\varepsilon > 1$. We will prove that there exists an interval $[A, B] \subsetneq (3,4)$, $(A < B)$, depending on ε, such that the inequality $\frac{1}{\varepsilon} < \frac{c_{[\lambda n]}}{c_n} < \varepsilon$ holds uniformly for $\lambda \in [A, B]$, for sufficiently large $n \in \mathbb{N}$. Hence, we will define n_λ, $(n_\lambda \in \mathbb{N})$ as follows

$$n_\lambda = \begin{cases} 1, & \text{if } \frac{1}{\varepsilon} < \frac{c_{[\lambda n]}}{c_n} < \varepsilon, \text{ for each } n \in \mathbb{N}; \\ 1 + \max\left\{ n \in \mathbb{N} \,\middle|\, \frac{c_{[\lambda n]}}{c_n} \geqslant \varepsilon \text{ or } \frac{c_{[\lambda n]}}{c_n} \leqslant \frac{1}{\varepsilon} \right\}, & \text{otherwise,} \end{cases}$$

for each $\lambda \in (3,4)$. Note that $1 \leqslant n_\lambda < +\infty$.
Also, we will define a sequence $(A_k)_{k\in\mathbb{N}}$ of sets $A_k = \{\lambda \in (3,4) \mid n_\lambda > k\}$, $k \in \mathbb{N}$. This is a non-increasing sequence which satisfies that $\bigcap_{k=1}^{+\infty} A_k = \emptyset$. Not all sets from this sequence are dense in $(3,4)$, i.e. there exists a set A_k for some $k \in \mathbb{N}$ which is not dense in $(3,4)$. To prove the previously mentioned we must, firstly, emphasize that at least one of the two following inequalities is true: $\frac{1}{\varepsilon} \geqslant \frac{c_{[(n_\lambda-1)\lambda]}}{c_{n_\lambda-1}}$ or $\frac{c_{[(n_\lambda-1)\lambda]}}{c_{n_\lambda-1}} \geqslant \varepsilon$, for each $\lambda \in A_k$ and for fixed $k \in \mathbb{N}$. Also, there exists $\delta_\lambda > 0$ for which at least one of the following inequalities is true: $\frac{1}{\varepsilon} \geqslant \frac{c_{[(n_\lambda-1)t]}}{c_{n_\lambda-1}} = \frac{c_{[(n_\lambda-1)\lambda]}}{c_{n_\lambda-1}}$ or $\frac{c_{[(n_\lambda-1)\lambda]}}{c_{n_\lambda-1}} = \frac{c_{[(n_\lambda-1)t]}}{c_{n_\lambda-1}} \geqslant \varepsilon$, for each $t \in [\lambda, \lambda + \delta_\lambda)$. Since, from inequality $n_t \geqslant (n_\lambda - 1) + 1 > k$ we obtain that $t \in A_k$, for this k. Moreover, from $\lambda \in A_k$ we have that $(\lambda, \lambda + \delta_k) \subsetneq A_k$. Therefore, if the set A_k is dense in the interval $(3,4)$, then the set $\operatorname{Int} A_k$ is also dense in the interval $(3,4)$. If we assume that each set A_k, $(k \in \mathbb{N})$ is dense in $(3,4)$ we obtain that $(\operatorname{Int} A_k)_{k\in\mathbb{N}}$ is a sequence of dense and open sets in $(3,4)$, also, and all of these sets are of the second category in $(3,4)$. Consequently, $\bigcap_{k=1}^{+\infty} \operatorname{Int} A_k$ is a dense set in $(3,4)$, so it is nonempty. That is a contradiction. Hence, there is a set A_{n_0}, for some $n_0 \in \mathbb{N}$, which is not dense in $(3,4)$ and there is an interval $[A, B] \subsetneq (3,4)(A < B)$ such that $[A, B] \subsetneq (3,4) \setminus A_{n_0} = \{\lambda \in (3,4) \mid n_\lambda \leqslant n_0\}$. Now, we have that $n_\lambda \leqslant n_0$, for each $\lambda \in [A, B]$, and from that it follows $\frac{1}{\varepsilon} < \frac{c_{[\lambda n]}}{c_n} < \varepsilon$, for each $n \geqslant n_0 \geqslant n_\lambda$ and $\lambda \in [A, B]$. Also, it holds that $\frac{c_{[\lambda x]}}{c_{[x]}} = \frac{c_{[t[\mu[x]]]}}{c_{[\mu[x]]}} \cdot \frac{c_{[\mu[x]]}}{c_{[x]}}$, for each $\lambda \geq 12$ and sufficiently large $x \geqslant x_0$, where $t = t(x) \in [A, B]$ and $\mu = \frac{2\lambda}{A+B}$.
Finally, we obtain that inequalities $\varliminf_{x\to+\infty} \frac{c_{[\lambda x]}}{c_{[x]}} \geqslant \frac{1}{\varepsilon} \cdot 1 = \frac{1}{\varepsilon}$ and $\varlimsup_{x\to+\infty} \frac{c_{[\lambda x]}}{c_{[x]}} \leqslant \varepsilon \cdot 1 = \varepsilon$ are true, for each $\lambda \geq 12$, where $\varepsilon > 1$ is arbitrary and pre-selected. Therefore, we have that $\lim_{x\to+\infty} \frac{c_{[\lambda x]}}{c_{[x]}} = 1$ is satisfied, for each $\lambda \geq 12$, and the function $c_{[x]}$, $x \geqslant 1$ is the element of the class SV_f (see e.g. [1]). The sequence $(c_n)_{n\in\mathbb{N}}$ is the restriction of this function to $\mathbb{N}$, so it is an element of the class SV_s. The proof for the sequence $(d_n)_{n\in\mathbb{N}}$ is analogous. This completes the proof. □

Proof of Proposition 1.2.

1. (Reflexivity) The asymptotic relation $\lim_{n\to+\infty} \frac{c_{[\lambda n]}}{c_n} = 1$ is satisfied, for each sequence $c = (c_n)_{n\in\mathbb{N}} \in SV_s$ and $\lambda > 1$. Hance, $c_n \overset{s}{\sim} c_n$, as $n \to +\infty$.
2. (Symmetry) Relation $\overset{s}{\sim}$ is symmetric in $\mathbb{S}$, therefore it is symmetric in $SV_s \subsetneq \mathbb{S}$, also.
3. (Transitivity) Let us assume that $c_n \overset{s}{\sim} d_n$, as $n \to +\infty$, and $d_n \overset{s}{\sim} e_n$, as $n \to +\infty$ are satisfied, for given sequences $c = (c_n)_{n\in\mathbb{N}}, d = (d_n)_{n\in\mathbb{N}}$ and $e = (e_n)_{n\in\mathbb{N}}$ from the class SV_s. Therefore, we obtain that $\lim_{n\to+\infty} \frac{c_{[\lambda n]}}{e_n} = \lim_{n\to+\infty} \frac{c_{[\lambda n]}}{d_{[\sqrt{\lambda} n]}} \cdot \lim_{n\to+\infty} \frac{d_{[\sqrt{\lambda} n]}}{e_n} = \lim_{n\to+\infty} \frac{c_{[\frac{\lambda n}{[\sqrt{\lambda} n]} \cdot [\sqrt{\lambda} n]]}}{d_{[\sqrt{\lambda} n]}} \cdot 1 = 1 \cdot 1 = 1$, for each $\lambda > 1$, since $d_{[\sqrt{\lambda} n]} \sim c_{[\sqrt{\lambda} n]}$, as $n \to +\infty$, and $\lim_{n\to+\infty} \frac{c_{[tn]}}{c_n} = 1$ is uniform limit, for each $t \in [a, b] \subsetneq (0, +\infty)$, $(a < b)$, (see e.g. [1]) and consequently for each $t \in [\frac{\sqrt{\lambda}+1}{2}, \sqrt{\lambda}]$, and for some $\lambda > 1$, which is arbitrary pre-selected and fixed. In an analogous way it can be proved that $\lim_{n\to+\infty} \frac{e_{[\lambda n]}}{c_n} = 1$, for each $\lambda > 1$. Hence, we obtain

that $c_n \overset{s}{\sim} e_n$, as $n \to +\infty$. Finally, we will prove that $d_{[\sqrt{\lambda}n]} \sim c_{[\sqrt{\lambda}n]}$ is satisfied, as $n \to +\infty$, for $\lambda > 1$. Namely, it holds that $\lim_{n\to+\infty} \frac{d_{[\sqrt{\lambda}n]}}{c_{[\sqrt{\lambda}n]}} = \lim_{n\to+\infty} \frac{d_{[\sqrt{\lambda}n]}}{d_n} \cdot \lim_{n\to+\infty} \frac{d_n}{c_{[\sqrt{\lambda}n]}} = 1 \cdot 1 = 1$, for $\lambda > 1$. This completes the proof. □

Proof of Proposition 1.6. Let $c = (c_n)_{n\in\mathbb{N}}$ be an arbitrary and fixed sequence from SV_s and let $[c]_s = \{d = (d_n)_{n\in\mathbb{N}} \in SV_s |\, d_n \overset{s}{\sim} c_n,\ as\ n \to +\infty\}$.

(1st step) Let $c = (c_n)_{n\in\mathbb{N}} \in SV_s$ and $d = (d_n)_{n\in\mathbb{N}} \in SV_s$, and $c_n \overset{s}{\sim} d_n$, as $n \to +\infty$. Hence, we obtain $\lim_{n\to+\infty} \frac{c_n}{d_n} = \lim_{n\to+\infty} \frac{c_{[\lambda n]}}{d_n} \cdot \lim_{n\to+\infty} \frac{c_n}{c_{[\lambda n]}} = 1 \cdot \lim_{n\to+\infty} \frac{1}{\frac{c_{[\lambda n]}}{c_n}} = 1 \cdot 1 = 1$ for each $\lambda > 1$ i.e. $c_n \sim d_n$, as $n \to +\infty$. Inversely, let $c = (c_n)_{n\in\mathbb{N}} \in SV_s$ and $d = (d_n)_{n\in\mathbb{N}}$ and $c_n \sim d_n$, as $n \to +\infty$. We have that $\lim_{n\to+\infty} \frac{c_{[\lambda n]}}{d_n} = \lim_{n\to+\infty} \frac{c_{[\lambda n]}}{c_n} \cdot \lim_{n\to+\infty} \frac{c_n}{d_n} = 1 \cdot 1 = 1$ is satisfied, for $\lambda > 1$. In the similar way, we can prove that $\lim_{n\to+\infty} \frac{d_{[\lambda n]}}{c_n} = 1$ holds, for $\lambda > 1$, so we obtain $c_n \overset{s}{\sim} d_n$ as $n \to +\infty$.

(2nd step)(1st round) Let sequence $c = (c_n)_{n\in\mathbb{N}} \in SV_s$ and the class $[c]_s$ be given. Also, let σ be the strategy of the second player. First player chooses the sequence $x_1 = (x_{1,j})_{j\in n} \in [c]_s$ arbitrary. Then the second player chooses the subsequence $\sigma(x_1) = (x_{1,k_1(j)})_{j\in\mathbb{N}}$ of the sequence x_1 where $\mathrm{Im}(k_1)$ is the set of natural numbers which are divisible with 2 and not divisible with 2^2.

(i^{th} round, $i \geqslant 2$) The first player chooses the sequence $x_i = (x_{i,j})_{j\in n} \in [c]_s$ arbitrary. Then the second player chooses the subsequence $\sigma(x_i) = (x_{i,k_i(j)})_{j\in\mathbb{N}}$ of the sequence x_i, so that $\mathrm{Im}(k_i)$ is the set of natural numbers greater or equal to j_i, so that they are divisible with 2^i, and not divisible with 2^{i+1}, and j_i exists in $\mathbb{N}$ (because of the 1ststep of this proof) and it is given by: $1 - \frac{1}{2^i} \leqslant \frac{x_{1,j}}{x_{i,j}} \leqslant 1 + \frac{1}{2^i}$, for each $j \geqslant j_i$. Now, we will observe the set $Y = \cup_{i\in\mathbb{N}} \cup_{j\in\mathbb{N}} X_{i,k_i}(j)$ of positive real numbers indexed by two indexes. Elements of the set Y we can consider as the subsequence of the sequence $y = (y_m)_{m\in\mathbb{N}}$ given by:

$$y_m = \begin{cases} x_{i,k_i(j)}, & \text{if } m = k_i(j) \text{ for some } i, j \in \mathbb{N}; \\ x_{1,m}, & \text{otherwise.} \end{cases}$$

By the construction of the sequence y we have that $y \in \mathbb{S}$. Also, the intersection between y and x_i, $(i \in \mathbb{N})$ is an infinite set of common elements. Let us prove that $y_m \sim x_{1,m}$ as $m \to +\infty$.

Let $\varepsilon \in (0, 1)$. Let us choose the smallest natural number i satisfying $\frac{1}{2^i} < \varepsilon$. For each $k \in \{1, 2, \ldots, i - 1\}$ there is $j^*_k \in \mathbb{N}$ so that inequality $1 - \varepsilon \leqslant \frac{x_{1,j}}{x_{k,j}} \leqslant 1 + \varepsilon$ is satisfied, for each $j \geqslant j^*_k$. Let $j^* = \max\{j^*_1, \ldots, j^*_{i-1}\}$. Therefore, the inequality $1 - \varepsilon \leqslant \frac{x_{1,m}}{y_m} \leqslant 1 + \varepsilon$ is satisfied, for each $m \geqslant j^*$. Then, from $x_{1,m} \sim y_m$, as $m \to +\infty$ we obtain $y_m \sim c_m$, as $m \to +\infty$, since $\varepsilon \in (0, 1)$ is arbitrary. From the 1ststep of this proof we obtain that $y_m \overset{s}{\sim} c_m$, as $m \to +\infty$, i.e. $y \in [c]_s$. This completes the proof. □

Proof of Proposition 1.11. Let σ be the strategy of the second player.

(1st round) Let the first player choose an arbitrary sequence $x_1 = (x_{1,j})_{j\in\mathbb{N}}$ from the class $[c]_s$. Then the second player plays $\sigma(x_1) = x_{1,1}, x_{1,2}, \ldots, x_{1,k_1}$, where $1 - \frac{1}{2} \leqslant \frac{c_k}{x_{1,k}} \leqslant 1 + \frac{1}{2}$ holds, for each $k \geqslant k_1$. This is possible according to the 1ststep of the proof of Proposition 1.6.

(i^{th} round, $i \geqslant 2$) Let the first player choose an arbitrary sequence $x_i = (x_{i,j})_{j\in\mathbb{N}}$ from the class $[c]_s$. Then the second player plays $\sigma(x_i) = x_{i-1,k_{i-1}+1}, x_{i-1,k_{i-1}+2}, \ldots, x_{i-1,k_{i-1}+k^*_{i-1}}, x_{i,k_i}$, where $1 - \frac{1}{2^i} \leqslant \frac{c_k}{x_{i,k}} \leqslant 1 + \frac{1}{2^i}$ holds, for $k \geqslant k_i$, and $k_i = 1 + k_{i-1} + k^*_{i-1}$. Thus, the second player forms the sequence $y = (y_m)_{m\in\mathbb{N}}$ given by $x_{1,1}, \ldots, x_{1,k_1}, \ldots, x_{2,k_2}, \ldots, x_{i,k_i}, \ldots$ which belongs to $\mathbb{S}$ and has a finite number of elements in common with each of the sequences $x_i, i \in \mathbb{N}$. Let $\varepsilon \in (0,1)$. Then $\frac{1}{2^i} < \varepsilon$ holds, for some $i \in \mathbb{N}$. Therefore, the inequality $1 - \varepsilon \leqslant \frac{c_m}{y_m} \leqslant 1 + \varepsilon$ holds, for each $m \geqslant 1 + k_1 + k^*_1 + k^*_2 + \cdots + k^*_{i-1}$, and we have that $c_m \sim y_m$, as $m \to +\infty$ is true. From the 1^{st}step of the proof of Proposition 1.6, we obtain $y \in [c]_s$. This means that the second player wins using the strategy σ. This completes the proof. □

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