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The Quasi Scott (Lawson) Topology and q-Continuous (q-Algebraic) Complete Lattices

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Abstract. Let L be a complete lattice. On L we define the so called quasi Scott topology, denoted by τ_{qSc}. This topology is always larger than or equal to the Scott topology and smaller than or equal to the strong Scott topology. Results concerning the above topology are given. Also, we introduce and investigate the notions of q-continuous and q-algebraic complete lattices. Finally, we give and examine the quasi Lawson topology on a complete lattice.

1. Preliminaries

Our reference for complete lattices are [2, 3, 8, 9]. We shall frequently denote complete lattices with their underlying sets and write L for $(L, \leqslant)$. The top element and the bottom element of a complete lattice L will be denoted by 1_L and 0_L, respectively.

In what follows we denote by L a complete lattice. By a *cover* of L we mean a subset C of L such that $\bigvee C = 1_L$. An element x of L is called *dense* (see [7]) if $x \wedge y \neq 0_L$ for all $y \in L \setminus \{0_L\}$. The set of dense elements of L is denoted by $\mathcal{D}(L)$. By a *quasicover* of L we mean a subset A of L such that $\bigvee A \in \mathcal{D}(L)$.

A subset D of L is called *directed* if for every $x, y \in D$ there exists $z \in D$ such that $x \leqslant z$ and $y \leqslant z$.

For every $x \in L$ and $A \subseteq L$ we consider the following subsets of L:

$$\downarrow x = \{y \in L : y \leqslant x\}, \quad \uparrow x = \{y \in L : x \leqslant y\}, \quad \text{and} \quad \uparrow A = \cup\{\uparrow x : x \in A\}.$$

A non-empty subset I of L is called an *ideal* if the following conditions hold:
(a) $I = \downarrow I$.
(b) I is a directed set.

The Scott topology τ_{Sc} on L (see, for example, [7]) is the family of all subsets U of L such that:
(a) $U = \uparrow U$.
(b) For every directed subset D of L the condition $\bigvee D \in U$ implies $D \cap U \neq \varnothing$.

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The strong Scott topology τ_{sSc} on L (see [12]) is the family of all subsets U of L such that:
(a) $U = \uparrow U$.
(b) For every directed subset D of L the condition $\bigvee D = 1_L$ implies $D \cap U \neq \emptyset$.

Let $x, y \in L$. We say that *x is way below y*, in symbols $x \ll y$, if for every directed subset D of L the relation $y \leqslant \bigvee D$ implies the existence of a $d \in D$ with $x \leqslant d$. Let $x, y, z, w \in L$. The following statements are true:
(1) $0_L \ll x$.
(2) If $x \ll y$, then $x \leqslant y$.
(3) If $x \leqslant y \ll z \leqslant w$, then $x \ll w$.
(4) If $x \ll z$ and $y \ll z$, then $x \vee y \ll z$.

For every $x \in L$ we consider the following subsets of L:

$$\twoheaddownarrow x = \{y \in L : y \ll x\} \text{ and } \twoheaduparrow x = \{y \in L : x \ll y\}.$$

A complete lattice L is called a *continuous lattice* if $x = \bigvee \twoheaddownarrow x$ for every $x \in L$. An element x of L is said to be *compact* if $x \ll x$. The subset of all compact elements is denoted by $K(L)$. A complete lattice L is called *algebraic* if $x = \bigvee(\downarrow x \cap K(L))$ for every $x \in L$.

Definitions and notations concerning topological spaces follow [6].

Many researchers are interested in continuous (algebraic) lattices, Scott (Lawson) topology, and their applications (see, for example, [1, 4, 10–25]). In section 2 we define and study the quasi Scott topology on a complete lattice. In section 3 we present results concerning the quasi Scott continuous functions. In sections 4 and 5 we introduce and investigate the notions of q-continuous and q-algebraic complete lattices. Finally, in section 6 we give and examine the quasi Lawson topology on a complete lattice.

2. The notion of quasi Scott topology

Notation 2.1. *Let L be a complete lattice. By $\tau_{qSc}(L)$ or briefly τ_{qSc} we denote the family of all subsets U of L such that:*
(a) $U = \uparrow U$.
(b) For every directed subset D of L the condition $\bigvee D \in \mathcal{D}(L) \cap U$ implies $D \cap U \neq \emptyset$.

Proposition 2.2. *Let L be a complete lattice. Then, $U \in \tau_{qSc}$ if and only if the following two conditions are satisfied:*
(a) $U = \uparrow U$.
(b) For every subset X of L the condition $\bigvee X \in \mathcal{D}(L) \cap U$ implies the existence of a finite subset A of X such that $\bigvee A \in U$.

Proof. Let $U \in \tau_{qSc}$. Obviously, $U = \uparrow U$. Also, let X be a subset of L such that $\bigvee X \in \mathcal{D}(L) \cap U$. We prove that there exists a finite subset A of X such that $\bigvee A \in U$. Consider the directed subset

$$X^+ = \{\bigvee A : A \text{ is a finite subset of } X\}$$

of L. Then, $\bigvee X^+ = \bigvee X$. Hence, $X^+ \cap U \neq \emptyset$. Thus, there exists a finite subset A of X such that $\bigvee A \in U$. The converse is immediate. □

The following proposition can be easily proved.

Proposition 2.3. *Let L be a complete lattice. Then, the following are true:*
(1) The family τ_{qSc} is a topology on L.
(2) $\tau_{Sc} \subseteq \tau_{qSc} \subseteq \tau_{sSc}$.
(3) τ_{qSc} is a T_0-topology.

Definition 2.4. The topology τ_{qSc} on a complete lattice L is called the *quasi Scott topology* on L.

Example 2.5. *(i) If L is finite complete lattice, then $\tau_{qSc} = \tau_{Sc} = \{U \subseteq L : U = \uparrow U\}$.*

(ii) If L is complete lattice such that $\mathcal{D}(L) = \{1_L\}$, then $\tau_{qSc} = \tau_{sSc}$.

(iii) Let $a > 0$ and $(a_n)_{n=1}^{\infty}$ be a strictly increasing sequence of positive real numbers such that $\lim_{n\to\infty} a_n = a$. We consider the complete lattice $(L, \leqslant)$, where

$$L = \{a_n : n = 1, 2, \dots\} \cup \{0, a, b, c\}$$

and $0 < b < c$, $0 < a_n < a_m < a < c$ for $n < m$. Then, $\mathcal{D}(L) = \{c\}$. We consider the subset $U = \{a, c\}$ of L. Then, $U = \uparrow U$. For the directed subset $D_0 = \{a_n : n = 1, 2, \dots\}$ of L we have $\bigvee D_0 = a \in U$ but $D_0 \cap U = \emptyset$. This means that $U \notin \tau_{Sc}$. Also, for every directed subset D of L with $\bigvee D = c$ we have $c \in D$ and, therefore, $c \in D \cap U$. Hence, $U \in \tau_{qSc}$. Thus, we have $\tau_{Sc} \neq \tau_{qSc}$.

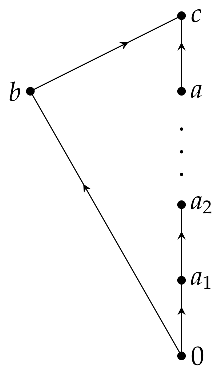

Figure 1: The lattice L in Example 2.5(iii)

(iv) Let $a > 0$ and $(a_n)_{n=1}^{\infty}$ be a strictly increasing sequence of positive real numbers such that $\lim_{n\to\infty} a_n = a$. We consider the complete lattice $(L, \leqslant)$, where

$$L = \{a_n : n = 1, 2, \dots\} \cup \{0, a, b\}$$

and $0 < a_n < a_m < a < b$ for $n < m$. Then, $\mathcal{D}(L) = L \setminus \{0\}$. We consider the subset $U = \{a, b\}$ of L. Then, $U = \uparrow U$. For the directed subset $D_0 = \{a_n : n = 1, 2, \dots\}$ of L we have $\bigvee D_0 = a \in \mathcal{D}(L) \cap U$ but $D_0 \cap U = \emptyset$. This means that $U \notin \tau_{qSc}$. Also, for every directed subset D of L with $\bigvee D = b$ we have $b \in D$ and, therefore, $b \in D \cap U$. Hence, $U \in \tau_{sSc}$. Thus, we have $\tau_{qSc} \neq \tau_{sSc}$.

Figure 2: The lattice L in Example 2.5(iv)

(v) Let X be a topological space and $\mathcal{O}(X)$ be the set of all open subsets of X with the inclusion as order. Then, the quasi Scott topology on $\mathcal{O}(X)$ is the family of all subsets $\mathbb{H}$ of $\mathcal{O}(X)$ such that:
(a) The conditions $U \in \mathbb{H}$, $V \in \mathcal{O}(X)$, and $U \subseteq V$ imply $V \in \mathbb{H}$.
(b) For every family $\{U_i : i \in I\} \subseteq \mathcal{O}(X)$ such that $\bigcup\{U_i : i \in I\}$ is a dense subset of X and $\bigcup\{U_i : i \in I\} \in \mathbb{H}$, there exists a finite subset J of I such that $\bigcup\{U_i : i \in J\} \in \mathbb{H}$.

The following proposition can be easily proved.

Proposition 2.6. *Let L be a complete lattice and $\tau^c_{qSc}(L) = \{L \setminus U : U \in \tau_{qSc}\}$. Then, the family $\tau^c_{qSc}(L)$ consists of all subsets F of L such that:*
(a) $F = \downarrow F$.
(b) For every directed subset D of L the conditions $\bigvee D \in \mathcal{D}(L)$ and $D \subseteq F$ imply $\bigvee D \in F$.

Proposition 2.7. *Let L be a complete lattice with the quasi Scott topology and $A \subseteq L$. If A is a complete lattice and $\mathcal{D}(A) \subseteq \mathcal{D}(L)$, then the subspace topology on A is contained in the quasi Scott topology on A.*

Proof. The subspace topology on A is the family

$$\tau_{qSc}(L)|_A = \{A \cap U : U \in \tau_{qSc}(L)\}.$$

We prove that $\tau_{qSc}(L)|_A \subseteq \tau_{qSc}(A)$. Let $U_A \in \tau_{qSc}(L)|_A$. Then, there exists $U \in \tau_{qSc}(L)$ such that $U_A = A \cap U$. Obviously, $U_A = \uparrow U_A$ with respect to A. Let D be a directed subset of A such that $\bigvee D \in \mathcal{D}(A) \cap U_A$. Since, $\mathcal{D}(A) \subseteq \mathcal{D}(L)$ and $U \in \tau_{qSc}(L)$, we have $D \cap U_A \neq \emptyset$. Thus, $U_A \in \tau_{qSc}(A)$ and, consequently, $\tau_{qSc}(L)|_A \subseteq \tau_{qSc}(A)$. □

Corollary 2.8. *Let L be a complete lattice with the quasi Scott topology and $A \in \tau_{qSc}(L)$. If A is a complete lattice and $\mathcal{D}(A) \subseteq \mathcal{D}(L)$, then the quasi Scott topology on A and the subspace topology on A coincide.*

Proof. By Proposition 2.7, $\tau_{qSc}(L)|_A \subseteq \tau_{qSc}(A)$. Moreover, since $A \in \tau_{qSc}(L)$, we have $\tau_{qSc}(A) \subseteq \tau_{qSc}(L)|_A$. Therefore, $\tau_{qSc}(A) = \tau_{qSc}(L)|_A$. □

Proposition 2.9. *Let L be a complete lattice and $x \in L$. Then,*

$$L \setminus \downarrow x = \{y \in L : y \nleqslant x\} \in \tau_{qSc}.$$

Proof. It is known that $L \setminus \downarrow x \in \tau_{Sc}$. By Proposition 2.3(2), $\tau_{Sc} \subseteq \tau_{qSc}$. Hence, $L \setminus \downarrow x \in \tau_{qSc}$. □

Let L be a complete lattice. For any net $(x_j)_{j \in J}$ the *lower limit* is defined as follows

$$\underline{\lim}_{j \in J} x_j = \sup_{j \in J} \inf_{i \geq j} x_i.$$

By $\mathcal{D}$ we denote the class of all those pairs $((x_j)_{j \in J}, x)$ consisting of a net $(x_j)_{j \in J}$ on L and an element $x \in \mathcal{D}(L)$ such that $x \leqslant \underline{\lim}_{j \in J} x_j$. If $((x_j)_{j \in J}, x) \in \mathcal{D}$, then we say that x is an *$\mathcal{D}$-limit* of $(x_j)_{j \in J}$ and we write briefly $x \equiv_{\mathcal{D}} \lim_{j \in J} x_j$.

Notation 2.10. *Let L be a complete lattice. By $\tau(\mathcal{D})$ we denote the family of all subsets U of L satisfying the following conditions:*
(a) $\uparrow u \in U$ for every $u \in (L \setminus \mathcal{D}(L)) \cap U$.
(b) If $x \equiv_{\mathcal{D}} \lim_{j \in J} x_j$ and $x \in U$, then there exists $j_0 \in J$ such that $x_j \in U$ for every $j \geq j_0$.

The following proposition can be easily proved.

Proposition 2.11. *Let L be a complete lattice. Then, the family $\tau(\mathcal{D})$ is a topology on L.*

Proposition 2.12. *For the topologies τ_{qSc} and $\tau(\mathcal{D})$ on a complete lattice L we have $\tau_{qSc} = \tau(\mathcal{D})$.*

Proof. We prove that $\tau(\mathcal{D}) \subseteq \tau_{qSc}$. Let $U \in \tau(\mathcal{D})$. First we prove $U = \uparrow U$. It suffices to prove that $\uparrow u \in U$ for every $u \in \mathcal{D}(L) \cap U$. Let $u \in \mathcal{D}(L) \cap U$ and $x \in L$ such that $u \leqslant x$. We consider the net $(x_j)_{j\in J}$, where $x_j = x$, $j \in J$. Then,

$$u \leqslant x = \underline{\lim}_{j\in J} x_j.$$

From the definition of $\tau(\mathcal{D})$ we conclude that there exists $j_0 \in J$ such that $x_j \in U$ for every $j \geq j_0$. This means that $x \in U$.

Now, let D be a directed subset of L such that $\bigvee D \in \mathcal{D}(L) \cap U$. Consider the net $(x_d)_{d\in D}$ with $x_d = d$. Then,

$$\inf_{a \geqslant d} x_a = d$$

and, hence,

$$\underline{\lim}_{d\in D} x_d = \bigvee D.$$

Therefore,

$$\bigvee D \equiv_{\mathcal{D}} \lim_{d\in D} x_d.$$

By assumption there exists $d_0 \in D$ such that $x_d = d \in U$ for every $d \geqslant d_0$. Thus, $D \cap U \neq \emptyset$ and, therefore, $\tau(\mathcal{D}) \subseteq \tau_{qSc}$.

We prove that $\tau_{qSc} \subseteq \tau(\mathcal{D})$. Let $U \in \tau_{qSc}$. We take a net $(x_j)_{j\in J}$ and $x \in U$ with $x \equiv_{\mathcal{D}} \lim_{j\in J} x_j$. Then, $x \leqslant \underline{\lim}_{j\in J} x_j$. Consider the directed subset

$$D = \{\inf_{i\geq j} x_i : j \in J\}$$

of L. Then, $x \leqslant \bigvee D$. Since $U = \uparrow U$, we have $\bigvee D \in U$. Moreover, since $x \in \mathcal{D}(L)$ and $x \leqslant \bigvee D$, we have $\bigvee D \in \mathcal{D}(L)$. By assumption there exists $d_0 \in D$ such that $d_0 \in D \cap U$. By the definition of D, $d_0 = \inf_{i\geq j_0} x_i$ for some $j_0 \in J$. Hence, $d_0 \leqslant x_i$ for all $i \geq j_0$. Since $U = \uparrow U$ and $d_0 \in U$, we have $x_i \in U$ for all $i \geq j_0$. Thus, $\tau_{qSc} \subseteq \tau(\mathcal{D})$. □

3. Quasi Scott continuous functions

Definition 3.1. Let L_1 and L_2 be two complete lattices. A function $f : L_1 \to L_2$ is called *quasi Scott continuous* if for every $V \in \tau_{qSc}(L_2)$ we have $f^{-1}(V) \in \tau_{qSc}(L_1)$.

Proposition 3.2. *Let $f : L_1 \to L_2$ be a quasi Scott continuous function. Then, f is monotone.*

Proof. Let $x, y \in L_1$ with $x \leqslant y$. We show that $f(x) \leqslant f(y)$. Suppose that $f(x) \nleqslant f(y)$ and set

$$V = L_2 \setminus \downarrow f(y).$$

Then, $f(x) \in V$. By Proposition 2.9, $V \in \tau_{qSc}(L_2)$. Hence, $f^{-1}(V) \in \tau_{qSc}(L_1)$ and, consequently, $\uparrow f^{-1}(V) = f^{-1}(V)$. Since $x \in f^{-1}(V)$ and $x \leqslant y$, we have $y \in f^{-1}(V)$ or $f(y) \in V$ which is a contradiction. Thus, $f(x) \leqslant f(y)$. □

The following proposition can be easily proved.

Proposition 3.3. *Let $f : L_1 \to L_2$ be a function. The following conditions are equivalent:*
(1) f is quasi Scott continuous.
(2) For every $F \in \tau^c_{qSc}(L_2)$ we have $f^{-1}(F) \in \tau^c_{qSc}(L_1)$.

Proposition 3.4. *Let $f : L_1 \to L_2$ be a quasi Scott continuous function. The following statements are true:*
(1) For every directed subset D of L_1 with $\bigvee D \in \mathcal{D}(L_1)$ we have

$$f(\bigvee D) = \bigvee f(D).$$

(2) For every net $(x_j)_{j\in J}$ with $\underline{\lim}_{j\in J} x_j \in \mathcal{D}(L_1)$ we have

$$f(\underline{\lim}_{j\in J} x_j) \leqslant \underline{\lim}_{j\in J} f(x_j).$$

Proof. (1) Let D be a directed subset of L_1 with $\bigvee D \in \mathcal{D}(L_1)$. We prove that $f(\bigvee D) = \bigvee f(D)$. By Proposition 3.2, $\bigvee f(D) \leqslant f(\bigvee D)$. So, it suffices to prove that $f(\bigvee D) \leqslant \bigvee f(D)$. We set

$$x = \bigvee D \text{ and } a = \bigvee f(D).$$

We will show that $f(x) \leqslant a$. Suppose that $f(x) \nleqslant a$. We consider the set

$$V = L_2 \setminus \downarrow a.$$

Then, $f(x) \in V$. By Proposition 2.9, $V \in \tau_{qSc}(L_2)$. Hence, $U = f^{-1}(V) \in \tau_{qSc}(L_1)$. Also, $x \in \mathcal{D}(L_1) \cap U$. Hence, there exists $d \in D$ such that $d \in U$. It follows that $f(d) \in L_2 \setminus \downarrow a$, that is, $f(d) \nleqslant a = \bigvee f(D)$ which is a contradiction.

(2) Let $(x_j)_{j\in J}$ be a net and $x = \underline{\lim}_{j\in J} x_j \in \mathcal{D}(L_1)$. We prove that $f(x) \leqslant \underline{\lim}_{j\in J} f(x_j)$. Consider the directed subset

$$D = \{\inf_{i\geq j} x_i : j \in J\}$$

of L_1. Then, $x = \bigvee D$. Let $d \in D$. Then, there exists $j_0 \in J$ such that $d = \inf_{i\geq j_0} x_i$. By Proposition 3.2,

$$f(d) = f(\inf_{i\geq j_0} x_i) \leqslant \inf_{i\geq j_0} f(x_i).$$

Therefore,

$$\underline{\lim}_{j\in J} f(x_j) = \sup_{j\in J} \inf_{i\geq j} f(x_i) \geqslant f(d).$$

It follows that $\underline{\lim}_{j\in J} f(x_j) \geqslant f(d)$ for every $d \in D$. Hence, $\bigvee f(D) \leqslant \underline{\lim}_{j\in J} f(x_j)$. ¿From (1) we conclude

$$f(\underline{\lim}_{j\in J} x_j) = f(\bigvee D) = \bigvee f(D) \leqslant \underline{\lim}_{j\in J} f(x_j).$$

□

Proposition 3.5. *Let $f : L_1 \to L_2$ be a monotone function. If $f(x) \in \mathcal{D}(L_2)$ for any $x \in \mathcal{D}(L_1)$ and $f(\bigvee D) = \bigvee f(D)$ for every directed subset D of L_1 with $\bigvee D \in \mathcal{D}(L_1)$, then f is quasi Scott continuous.*

Proof. Let $V \in \tau_{qSc}(L_2)$. We prove that $f^{-1}(V) \in \tau_{qSc}(L_1)$. Since the function f is monotone, $\uparrow f^{-1}(V) = f^{-1}(V)$. Now let D be a directed subset of L_1 with $\bigvee D \in \mathcal{D}(L_1) \cap f^{-1}(V)$. Then, $f(\bigvee D) \in \mathcal{D}(L_2)$ and $f(\bigvee D) = \bigvee f(D) \in V$. Also, since f is monotone, $f(D)$ is a directed subset of L_2. Hence, there exists $y \in f(D) \cap V$. It follows that there exists $x \in D$ such that $y = f(x)$ and $x \in f^{-1}(V)$. Thus, $D \cap f^{-1}(V) \neq \varnothing$ and, therefore, $f^{-1}(V) \in \tau_{qSc}(L_1)$. □

4. q-continuous complete lattices

Definition 4.1. Let L be a complete lattice and $x, y \in L$. We say that x *is quasi way below* y, in symbols $x \ll_q y$, if the following two conditions are satisfied:
(a) $x \leqslant y$.
(b) For every directed subset D of L the relations $y \leqslant \bigvee D$ and $\bigvee D \in \mathcal{D}(L)$ imply the existence of a $d \in D$ with $x \leqslant d$.

Proposition 4.2. *Let L be a complete lattice and $x, y \in L$. Then, $x \ll_q y$ if and only if the following two conditions are satisfied:*
(a) $x \leqslant y$.
(b) For every subset X of L the relations $y \leqslant \bigvee X$ and $\bigvee X \in \mathcal{D}(L)$ imply the existence of a finite subset A of X such that $x \leqslant \bigvee A$.

Proof. Let $x \ll_q y$. Obviously, $x \leqslant y$. Let X be a subset of L such that $y \leqslant \bigvee X$ and $\bigvee X \in \mathcal{D}(L)$. Consider the directed subset

$$X^+ = \{\bigvee A : A \text{ is a finite subset of } X\}$$

of L. Then, $\bigvee X^+ = \bigvee X$. Hence, there exists a finite subset A of X such that $x \leqslant \bigvee A$. The converse is immediate. □

The following proposition can be easily proved.

Proposition 4.3. *Let L be a complete lattice and $x, y, z, w \in L$. The following statements are true:*
(1) If $x \ll y$, then $x \ll_q y$. Particularly, $0_L \ll_q y$.
(2) If $x \ll_q y$ and $y \ll_q z$, then $x \ll_q z$.
(3) If $x \leqslant y \ll_q z \leqslant w$, then $x \ll_q w$.
(4) If $x \ll_q z$ and $y \ll_q z$, then $x \vee y \ll_q z$.

Proposition 4.4. *Let L be a complete lattice and $x, y \in L$. Then, the following two statements are equivalent:*
(1) $x \ll_q y$.
(2) $x \leqslant y$ and for every ideal I of L the relations $y \leqslant \bigvee I$ and $\bigvee I \in \mathcal{D}(L)$ imply $x \in I$.

Proof. (1) implies (2): Suppose that $x \ll_q y$ and let I be an ideal of L such that $y \leqslant \bigvee I$ and $\bigvee I \in \mathcal{D}(L)$. Since $x \ll_q y$ and I is directed, there exists $z \in I$ such that $x \leqslant z$. Now, since $I = \downarrow I$, we have $x \in I$.

(2) implies (1): Let D be a directed subset of L such that $y \leqslant \bigvee D$ and $\bigvee D \in \mathcal{D}(L)$. We prove that there exists $d \in D$ with $x \leqslant d$. Set $I = \downarrow D$. We observe that I is an ideal and $\bigvee I = \bigvee D$. Hence, $x \in I$ and, therefore, there exists $d \in D$ such that $x \leqslant d$. □

For every $x \in L$ we consider the following subsets of L:

$$\Downarrow_q x = \{y \in L : y \ll_q x\} \text{ and } \Uparrow_q x = \{y \in L : x \ll_q y\}.$$

Remark 4.5. *If the condition (b) of Definition 4.1 satisfied and $y \in \mathcal{D}(L)$, then $x \ll y$ and, hence, $x \leqslant y$. It follows that if $y \in \mathcal{D}(L)$, then $\Downarrow_q y = \Downarrow y$.*

Proposition 4.6. *Let L be a complete lattice and $x \in L$. Then,* $\mathrm{Int}_{\tau_{qSc}}(\uparrow x) \subseteq \Uparrow_q x$ (By $\mathrm{Int}_{\tau_{qSc}}(\uparrow x)$ we denote the interior of $\uparrow x$ in the topology τ_{qSc}).

Proof. Let $y \in \mathrm{Int}_{\tau_{qSc}}(\uparrow x) \subseteq \uparrow x$. Then, $x \leqslant y$. Now, let D be a directed subset of L such that $y \leqslant \bigvee D$ and $\bigvee D \in \mathcal{D}(L)$. Since $y \in \mathrm{Int}_{\tau_{qSc}}(\uparrow x)$, $y \leqslant \bigvee D$, and $\mathrm{Int}_{\tau_{qSc}}(\uparrow x) = \uparrow \mathrm{Int}_{\tau_{qSc}}(\uparrow x)$, we have $\bigvee D \in \mathrm{Int}_{\tau_{qSc}}(\uparrow x)$. So,

$$\bigvee D \in \mathcal{D}(L) \cap \mathrm{Int}_{\tau_{qSc}}(\uparrow x).$$

Since $\mathrm{Int}_{\tau_{qSc}}(\uparrow x) \in \tau_{qSc}$, there exists $d \in D$ such that $d \in \mathrm{Int}_{\tau_{qSc}}(\uparrow x) \subseteq \uparrow x$. So, $x \leqslant d$. By the above we have $x \ll_q y$. Thus, $y \in \Uparrow_q x$. □

Definition 4.7. A complete lattice L is called *q-continuous* if

$$x = \bigvee \Downarrow_q x = \bigvee \{y \in L : y \ll_q x\} \text{ for every } x \in L.$$

We note that the notion of q-continuous complete lattice is quite different from the well known notion of quasi continuous complete lattice (see, for example, [8]).

Remark 4.8. *If L is q-continuous, then by Proposition 4.3(4) for all $x \in L$, the subset $\Downarrow_q x$ of L is directed.*

Example 4.9. *(i) Every continuous complete lattice L is q-continuous. Indeed, let $x \in L$. Then,*

$$\{y \in L : y \ll x\} \subseteq \{y \in L : y \ll_q x\} \subseteq \{y \in L : y \leqslant x\}.$$

Therefore,

$$x = \bigvee\{y \in L : y \ll x\} \leqslant \bigvee\{y \in L : y \ll_q x\} \leqslant \bigvee\{y \in L : y \leqslant x\} = x$$

which means that $x = \bigvee \twoheaddownarrow_q x$.

Particularly, every complete chain and every finite complete lattice are q-continuous.

(ii) Consider the complete lattice $(L, \leqslant)$, where

$$L = \{0, 1, 2, \ldots\} \cup \{a, b, c, d\},$$

$0 < 1 < 2 < \ldots < b$, $0 < c < d$, and $0 < a < b < d$. Then, $\mathcal{D}(L) = \{d\}$. We observe that $\twoheaddownarrow_q n = \{0, \ldots, n\}$, $n = 0, 1, 2, \ldots$, $\twoheaddownarrow_q a = \{0, a\}$, $\twoheaddownarrow_q b = \{0, 1, 2, \ldots\} \cup \{a, b\}$, $\twoheaddownarrow_q c = \{0, c\}$, and $\twoheaddownarrow_q d = L$. Therefore, L is q-continuous. We prove that L is not continuous. It suffices to prove that $a \notin \twoheaddownarrow a$. Indeed, consider the directed subset $D = \{0, 1, 2, \ldots\}$ of L. Then, $a \leqslant b = \bigvee D$ but there not exists $n \in D$ such that $a \leqslant n$. Thus, $\twoheaddownarrow a = \{0\}$ and, hence, $\bigvee \twoheaddownarrow a \neq a$. This means that L is not continuous.

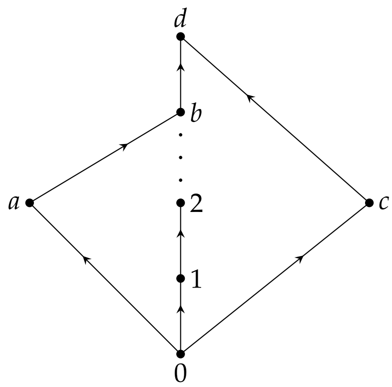

Figure 3: The lattice L in Example 4.9(ii)

(iii) Let X be a topological space and $\mathcal{O}(X)$ be the set of all open subsets of X with the inclusion as order (see Example 2.5(v)). It follows that the complete lattice $\mathcal{O}(X)$ is q-continuous if and only if for every $x \in X$ and for every open neighborhood U of x there exists an open neighborhood V of x satisfying the following conditions:
(a) $V \subseteq U$.
(b) For every family $\{U_i : i \in I\} \subseteq \mathcal{O}(X)$ such that $\bigcup\{U_i : i \in I\}$ is a dense subset of X and $U \subseteq \bigcup\{U_i : i \in I\}$, there exists a finite subset J of I such that $V \subseteq \bigcup\{U_i : i \in J\}$.

Definition 4.10. (see [12]) A complete lattice L is called *weakly continuous* if

$$x = \bigvee\{y \in L : y \ll_w x\} \text{ for every } x \in L.$$

We write $x \ll_w y$, if the following two conditions are satisfied:
(a) $x \leqslant y$.
(b) For every directed subset D of L the relation $\bigvee D = 1_L$ implies the existence of a $d \in D$ with $x \leqslant d$.

Example 4.11. *(i) Every q-continuous complete lattice L is weakly continuous. Indeed, let $x \in L$. Then,*

$$\{y \in L : y \ll_q x\} \subseteq \{y \in L : y \ll_w x\} \subseteq \{y \in L : y \leqslant x\}.$$

Therefore,

$$x = \bigvee\{y \in L : y \ll_q x\} \leqslant \bigvee\{y \in L : y \ll_w x\} \leqslant \bigvee\{y \in L : y \leqslant x\} = x$$

which means that $x = \bigvee\{y \in L : y \ll_w x\}$.

(ii) Consider the complete lattice $(L, \leqslant)$, *where* $L = [1,2] \cup \{0,a,b\}$, $0 < x < y < b$ *for every* $x, y \in [1,2]$ *with* $x \neq y$, *and* $1 < a < 2$. *Then,* $\mathcal{D}(L) = L \setminus \{0\}$. *We observe that* $\Downarrow_q a = \{0,1\}$ *and, hence,* $a \neq \bigvee \Downarrow_q a$. *This means that L is not q-continuous. Also, we observe that L is weakly continuous.*

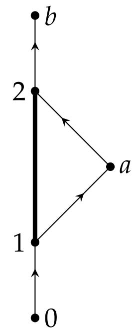

Figure 4: The lattice L in Example 4.11(ii)

Proposition 4.12. *Let L be a q-continuous complete lattice. Then, the following statements are true:*
(1) If $x \ll_q y \leqslant \bigvee D$, *where* $\bigvee D \in \mathcal{D}(L)$, *for a directed subset D of L, then there exists* $d \in D$ *with* $x \ll_q d$.
(2) If $x \ll_q z$, *where* $z \in \mathcal{D}(L)$, *then there exists* $y \in L$ *such that* $x \ll_q y \ll_q z$.

Proof. (1) Let $x, y \in L$ and let D be a directed subset of L such that $x \ll_q y \leqslant \bigvee D$ and $\bigvee D \in \mathcal{D}(L)$. We set

$$I = \bigcup\{\Downarrow_q d : d \in D\}.$$

We observe that I is an ideal and $\bigvee I = \bigvee D$. By Proposition 4.4, $x \in I$ which means that $x \ll_q d$ for some $d \in D$.

(2) Let $x \ll_q z$, where $z \in \mathcal{D}(L)$. Set $D = \Downarrow_q z$. Then, D is a directed subset of L. Since L is q-continuous, $z = \bigvee \Downarrow_q z = \bigvee D$. From (1) there exists $y \in D$ with $x \ll_q y$. Hence, $x \ll_q y \ll_q z$. □

Corollary 4.13. *Let L be a q-continuous complete lattice and* $x \in L$. *Then,* $\Uparrow_q x \in \tau_{qSc}$.

Proof. By Proposition 4.3(3), $\Uparrow_q x = \uparrow(\Uparrow_q x)$. Let D be a directed subset of L such that $\bigvee D \in \mathcal{D}(L) \cap \Uparrow_q x$. We prove that $D \cap \Uparrow_q x \neq \emptyset$. Indeed, by Proposition 4.12(1), there exists $d \in D$ with $x \ll_q d$. Hence, $d \in D \cap \Uparrow_q x$. □

Proposition 4.14. *Let L be a q-continuous complete lattice and* $x \in L$. *Then,* $\Uparrow_q x = \mathrm{Int}_{\tau_{qSc}}(\uparrow x)$.

Proof. By Proposition 4.6 it suffices to prove that $\Uparrow_q x \subseteq \mathrm{Int}_{\tau_{qSc}}(\uparrow x)$. By Corollary 4.13, $\Uparrow_q x \in \tau_{qSc}$. Moreover, $\Uparrow_q x \subseteq \uparrow x$. Therefore, $\Uparrow_q x \subseteq \mathrm{Int}_{\tau_{qSc}}(\uparrow x)$. □

Proposition 4.15. *Let L be a q-continuous complete lattice and* $x \in \mathcal{D}(L)$. *Then, the family* $\{\Uparrow_q u : u \ll_q x\}$ *is a neighborhood basis of x with respect to* τ_{qSc}.

Proof. Let $U \in \tau_{qSc}$ such that $x \in U$. We set $D = \Downarrow_q x$. Since L is q-continuous, the subset $\Downarrow_q x$ of L is directed (see Remark 4.8) and $x = \bigvee D$. Moreover, $\bigvee D \in \mathcal{D}(L)$. Hence, $D \cap U \neq \emptyset$. It follows that there exists $u \in U$ such that $u \ll_q x$. By Corollary 4.13, $\Uparrow_q u \in \tau_{qSc}$. We prove that $\Uparrow_q u \subseteq U$. Indeed, let $y \in \Uparrow_q u$. Then, $u \ll_q y$ and, therefore, $u \leqslant y$. Since $U = \uparrow U$ and $u \in U$, we have $y \in U$. Thus, $\Uparrow_q u \subseteq U$. □

Corollary 4.16. *Let L be a q-continuous complete lattice with $\mathcal{D}(L) = L \setminus \{0_L\}$. Then, the family $\{\Uparrow_q x : x \in L\}$ is a base for the quasi Scott topology τ_{qSc} on L.*

Example 4.17. *The condition $\mathcal{D}(L) = L \setminus \{0_L\}$ cannot be omitted in Corollary 4.16. Indeed, let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be two strictly increasing sequences of positive real numbers such that $\lim_{n\to\infty} a_n = a > 0$ and $\lim_{n\to\infty} b_n = b > 0$. We consider the complete lattice $(L, \leqslant)$, where*

$$L = \{a_1, a_2, \ldots\} \cup \{b_1, b_2, \ldots\} \cup \{0, a, b, c, d\}$$

and $0 < b_1 < c < d$, $0 < a_n < a_m < a < b < d$, $0 < b_n < b_m < b < d$ for $n < m$. Then, $\mathcal{D}(L) = \{b, d\}$. We observe that

$$\Downarrow_q a_n = \{0, a_1 \ldots, a_n\},\ \Downarrow_q b_n = \{0, b_1 \ldots, b_n\},\ n = 1, 2, \ldots,$$

$$\Downarrow_q a = \{0, a_1, a_2, \ldots\},\ \Downarrow_q b = \{0, b_1, b_2, \ldots\},\ \Downarrow_q c = \{0, b_1, c\},\ \text{and}\ \Downarrow_q d = L.$$

Indeed, we will show, for example, that $\Downarrow_q a = \{0, a_1, a_2, \ldots\}$. It suffices to prove that $a \notin \Downarrow_q a$. We consider the directed subset $D = \{b_1, b_2, \ldots\}$ of L. Then,

$$a \leqslant b = \bigvee D \in \mathcal{D}(L)$$

but there not exists $b_n \in D$ such that $a \leqslant b_n$.

By the above we have $a_n = \bigvee \Downarrow_q a_n$ for $n = 1, 2, \ldots$, $b_n = \bigvee \Downarrow_q b_n$ for $n = 1, 2, \ldots$, $a = \bigvee \Downarrow_q a$, $b = \bigvee \Downarrow_q b$, $c = \bigvee \Downarrow_q c$, and $d = \bigvee \Downarrow_q d$. Therefore, L is a q-continuous complete lattice.

Now, we consider the subset $U = \{b_2, b_3, \ldots\} \cup \{a, b, d\}$ of L. Then, $U \in \tau_{qSc}$ and $a \in U$. This means that U is an open neighborhood of a in the topology τ_{qSc}. We observe that there not exists $x \in L$ such that $a \in \Uparrow_q x \subseteq U$. Thus, the family $\{\Uparrow_q x : x \in L\}$ is not a base for the quasi Scott topology τ_{qSc} on L.

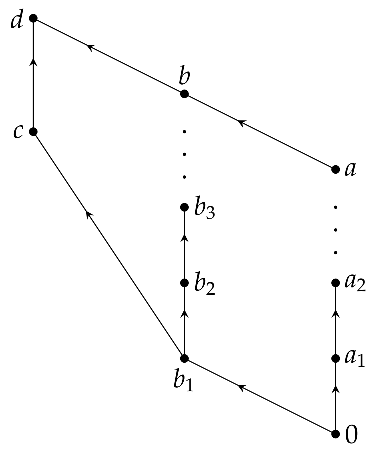

Figure 5: The lattice L in Example 4.17

Definition 4.18. Let L be a complete lattice. A subset B of L is called a *q-basis* for L if for every $x \in L$ there is a directed subset D_x of B such that $D_x \subseteq \Downarrow_q x$ and $\bigvee D_x = x$.

Proposition 4.19. *A complete lattice is q-continuous if and only if it has a q-basis.*

Proof. Let L be a q-continuous complete lattice. We prove that L is a q-basis of L. Indeed, let $x \in L$. We set $D_x = \Downarrow_q x$. Then, D_x is a directed subset of L and $\bigvee D_x = x$.

Conversely, let L be a complete lattice with B as its q-basis and $x \in L$. We prove that $x = \bigvee \Downarrow_q x$. Indeed, there is a directed subset D_x of B such that $D_x \subseteq \Downarrow_q x$ and $\bigvee D_x = x$. Hence, $x = \bigvee D_x \leqslant \bigvee \Downarrow_q x$. Moreover, $\bigvee \Downarrow_q x \leqslant x$. Thus, $x = \bigvee \Downarrow_q x$. □

5. q-algebraic complete lattices

Definition 5.1. Let L be a complete lattice. An element x of L is said to be *q-compact* if $x \ll_q x$. The subset of all q-compact elements is denoted by $K_q(L)$.

The following proposition can be easily proved.

Proposition 5.2. *Let L be a complete lattice. The following statements are true:*
(1) $0_L \in K_q(L)$.
(2) If $x, y \in K_q(L)$, *then* $x \vee y \in K_q(L)$.
(3) If $x \leqslant k \leqslant y$ *with* $k \in K_q(L)$, *then* $x \ll_q y$.

Proposition 5.3. *Let L be a complete lattice and* $x \in K_q(L)$. *Then,* $\uparrow x \in \tau_{qSc}$.

Proof. Obviously, $\uparrow x = \uparrow(\uparrow x)$. Now, let D be a directed subset of L such that $\bigvee D \in \mathcal{D}(L) \cap \uparrow x$. Then, $x \leqslant \bigvee D$. Since $x \in K_q(L)$, there exists $d \in D$ such that $x \leqslant d$. Hence, $d \in \uparrow x$. It follows that $D \cap \uparrow x \neq \emptyset$. □

Definition 5.4. A complete lattice L is called *q-algebraic* if

$$x = \bigvee(\downarrow x \cap K_q(L)) \text{ for every } x \in L.$$

Remark 5.5. *If L is q-algebraic, then by Proposition 5.2(3) for all* $x \in L$, *the subset* $\downarrow x \cap K_q(L)$ *of L is directed.*

Example 5.6. *(i) Every algebraic complete lattice L is q-algebraic. Indeed, let* $x \in L$. *Then,*

$$\downarrow x \cap K(L) \subseteq \downarrow x \cap K_q(L) \subseteq \downarrow x.$$

Therefore,

$$x = \bigvee(\downarrow x \cap K(L)) \leqslant \bigvee(\downarrow x \cap K_q(L)) \leqslant \bigvee \downarrow x = x$$

which means that $x = \bigvee(\downarrow x \cap K_q(L))$.

Particularly, every finite linearly ordered set is q-algebraic.

(ii) Consider the complete lattice $(L, \leqslant)$, *where* $L = [0,1] \cup \{a, b\}$, $x < y < b$ *for every* $x, y \in [0,1]$ *with* $x \neq y$, *and* $0 < a < 1$. *Since* $K(L) = \{0, a, b\}$, *the complete lattice L is not algebraic. We observe that* $K_q(L) = L$. *Hence,*

$$\bigvee(\downarrow x \cap K_q(L)) = \bigvee \downarrow x = x$$

for every $x \in L$ *which means that L is q-algebraic.*

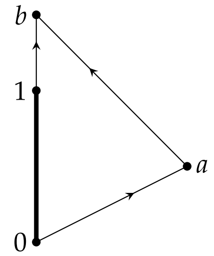

Figure 6: The lattice L in Example 5.6(ii)

Proposition 5.7. *Let L be a q-algebraic complete lattice and* $x \in \mathcal{D}(L)$. *Then, the family*

$$\{\uparrow a : a \leqslant x \text{ and } a \in K_q(L)\}$$

is a neighborhood basis of x with respect to τ_{qSc}.

Proof. Let $U \in \tau_{qSc}$ such that $x \in U$. We set $D = \downarrow x \cap K_q(L)$. Since L is q-algebraic, the subset $\downarrow x \cap K_q(L)$ of L is directed (see Remark 5.5) and $x = \bigvee D$. Moreover, $\bigvee D \in \mathcal{D}(L)$. Hence, $D \cap U \neq \emptyset$. It follows that there exists $a \in U$ such that $a \leqslant x$ and $a \in K_q(L)$. By Proposition 5.3, $\uparrow a \in \tau_{qSc}$. We prove that $\uparrow a \subseteq U$. Indeed, let $y \in \uparrow a$. Then, $a \leqslant y$. Since $U = \uparrow U$ and $a \in U$, we have $y \in U$. Thus, $\uparrow a \subseteq U$. □

Corollary 5.8. *Let L be a q-algebraic complete lattice with $\mathcal{D}(L) = L \setminus \{0_L\}$. Then, the family $\{\uparrow x : x \in K_q(L)\}$ is a base for the quasi Scott topology τ_{qSc} on L.*

Remark 5.9. *The condition $\mathcal{D}(L) = L \setminus \{0_L\}$ cannot be omitted in Corollary 5.8. Indeed, we consider the continuous complete lattice L in Example 4.17. We observe that $K_q(L) = L \setminus \{a, b\}$. Therefore, L is q-algebraic. Let*

$$U = \{b_2, b_3, \ldots\} \cup \{a, b, d\}.$$

Then, $U \in \tau_{qSc}$ and $a \in U$. This means that U is an open neighborhood of a in the topology τ_{qSc}. Since $a \notin K_q(L)$, there not exists $x \in K_q(L)$ with $a \in \uparrow x \subseteq U$. Thus, the family $\{\uparrow x : x \in K_q(L)\}$ is not a base for the quasi Scott topology τ_{qSc} on L.

Proposition 5.10. *Every q-algebraic complete lattice is q-continuous.*

Proof. Let L be a q-algebraic complete lattice and let $x \in L$. We prove that $x = \bigvee \twoheaddownarrow_q x$. We set

$$D = \downarrow x \cap K_q(L).$$

Then, D is directed and $x = \bigvee D$. Let $y \in D$. Then, $y \ll_q y \leqslant x$ and, therefore, $y \in \twoheaddownarrow_q x$. This means that $D \subseteq \twoheaddownarrow_q x$ and, hence, $x = \bigvee D \leqslant \bigvee \twoheaddownarrow_q x$. Moreover,

$$\bigvee \twoheaddownarrow_q x = \bigvee \{y \in L : y \ll_q x\} \leqslant \bigvee \{y \in L : y \leqslant x\} = x$$

which means that $x = \bigvee \twoheaddownarrow_q x$. □

Example 5.11. *Let $L = [0, 1]$ with the usual order. Obviously L is q-continuous. Since $K_q(L) = \{0\}$, the complete lattice L is not q-algebraic.*

Proposition 5.12. *Let L be a q-algebraic complete lattice and $x, y \in L$. The following statements are equivalent:*
(1) $x \leqslant y$.
(2) $\downarrow x \cap K_q(L) \subseteq \downarrow y \cap K_q(L)$.

Proof. (1) implies (2). It is obvious.

(2) implies (1). Suppose that $\downarrow x \cap K_q(L) \subseteq \downarrow y \cap K_q(L)$. Then,

$$\bigvee(\downarrow x \cap K_q(L)) \leqslant \bigvee(\downarrow y \cap K_q(L)).$$

Since L is q-algebraic,

$$x = \bigvee(\downarrow x \cap K_q(L)) \text{ and } y = \bigvee(\downarrow y \cap K_q(L)).$$

Thus, $x \leqslant y$. □

Corollary 5.13. *Let L be a q-algebraic complete lattice and $x, y \in L$. The following statements are equivalent:*
(1) $x \leqslant y$.
(2) For every $U \in \tau_{qSc}$, $x \in U$ implies $y \in U$.

Proof. (1) implies (2). It is obvious since $U = \uparrow U$.

(2) implies (1). By Proposition 5.12 it suffices to prove that

$$\downarrow x \cap K_q(L) \subseteq \downarrow y \cap K_q(L).$$

Let $a \in \downarrow x \cap K_q(L)$ and set $U = \uparrow a$. Then, $x \in U$ and $a \in K_q(L)$. By Proposition 5.3, $U \in \tau_{qSc}$. Hence, $y \in U$ which means that $a \leqslant y$ or $a \in \downarrow y$. □

6. The quasi Lawson topology

Recall the notion of the Lawson topology (see [8]). Let L be a complete lattice. The *lower topology* on L, denoted here by τ_l, is the topology, which has as a subbasis the family of all sets of the form $L \setminus \uparrow x$, $x \in L$. The topology $\tau_l \vee \tau_{Sc}$ is called the *Lawson topology* and is denoted here by τ_L.

Definition 6.1. Let L be a complete lattice. The topology $\tau_l \vee \tau_{qSc}$ is called the *quasi Lawson topology* and is denoted by τ_{qL} or $\tau_{qL}(L)$. That is the quasi Lawson topology has as a subbasis the sets U, with $U \in \tau_{qSc}$, together with the sets $L \setminus \uparrow x$, $x \in L$.

Remark 6.2. *Let L be a complete lattice. Then, the following are true:*
(1) $\tau_{Sc} \subseteq \tau_{qSc} \subseteq \tau_{sSc}$.
(2) $\tau_{Sc} \subseteq \tau_L \subseteq \tau_{qL}$.
(3) $\tau_{Sc} \subseteq \tau_{qSc} \subseteq \tau_{qL}$.

The relations between the topologies are summarized in the following diagram, where "$\rightarrow$" means "$\subseteq$".

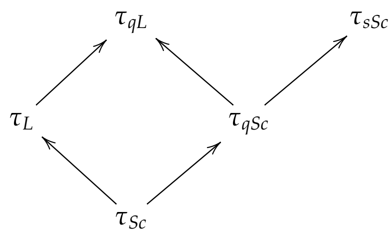

Figure 7: Relations between the topologies τ_{Sc}, τ_{qSc}, τ_{sSc}, τ_L, τ_{qL}

The proof of the following proposition is a straightforward verification of the relation $\tau_L \subseteq \tau_{qL}$ and the separation axioms of τ_L.

Proposition 6.3. *(1) For any complete lattice,* τ_{qL} *is* T_1.
(2) For any continuous complete lattice, τ_{qL} *is Hausdorff.*

Proposition 6.4. *Let L be a complete lattice. The following statements are true:*
(1) The sets $U \setminus \uparrow F$*, where* $U \in \tau_{qSc}$ *and F is a finite subset of L, form a basis for* τ_{qL}.
(2) If $U = \uparrow U$*, then* $U \in \tau_{qL}$ *if and only if* $U \in \tau_{qSc}$.

Proof. (1) It is obvious.

(2) Obviously, $\tau_{qSc} \subseteq \tau_{qL}$. Let $U \in \tau_{qL}$ such that $U = \uparrow U$. We prove that $U \in \tau_{qSc}$. Let D be a directed subset of L such that $\bigvee D \in \mathcal{D}(L) \cap U$. By (1) there exist $V \in \tau_{qSc}$ and a finite subset F of L such that $\bigvee D \in V \setminus \uparrow F \subseteq U$. Therefore, $D \cap V \neq \emptyset$. Let $d \in D \cap V$. Since $\bigvee D \notin \uparrow F$, we have $d \notin \uparrow F$. It follows that $d \in V \setminus \uparrow F \subseteq U$ which means that $D \cap U \neq \emptyset$. Thus, $U \in \tau_{qSc}$. □

Example 6.5. *Consider the complete lattice L given in Example 2.5(iii). By Proposition 6.4(2) and from the similar proposition for the topologies* τ_{Sc} *and* τ_L *(see Proposition III-1.6.(i) of [8]) we have* $\tau_{qSc} \neq \tau_L$ *and* $\tau_{qL} \neq \tau_L$.

Remark 6.6. *For a complete lattice L the topology* τ_L *is always compact. But, in general, the topology* τ_{qL} *is not compact (see Example 6.8).*

Proposition 6.7. *Let L be a complete lattice. Every cover*

$$\{U_i \in \tau_{qSc} : i \in I\} \bigcup \{L \setminus \uparrow x_j : j \in J\}$$

of L, where $\bigvee\{x_j : j \in J\} \in \mathcal{D}(L)$*, contains a finite subcover.*

Proof. Let $\{U_i \in \tau_{qSc} : i \in I\} \bigcup \{L \setminus \uparrow x_j : j \in J\}$ be an open cover of L such that $\bigvee\{x_j : j \in J\} \in \mathcal{D}(L)$. We set $x = \bigvee\{x_j : j \in J\}$. Then, $x \in \mathcal{D}(L)$ and

$$\bigcup\{L \setminus \uparrow x_j : j \in J\} = L \setminus \bigcap\{\uparrow x_j : j \in J\} = L \setminus \uparrow x.$$

Since $x \notin L \setminus \uparrow x$, there exists $i_0 \in I$ such that $x \in U_{i_0}$. By Proposition 2.2 there exist a finite subset $\{j_1, \ldots, j_n\}$ of J such that

$$x_{j_1} \vee \ldots \vee x_{j_n} \in U_{i_0}.$$

Moreover, since $U_{i_0} \in \tau_{qSc}$, we have $U_{i_0} = \uparrow U_{i_0}$. Hence,

$$L = U_{i_0} \bigcup (L \setminus \uparrow x_{j_1}) \bigcup \ldots \bigcup (L \setminus \uparrow x_{j_n}).$$

□

Example 6.8. *Consider the complete lattice $(L, \leqslant)$, where*

$$L = \{0, 1, 2, \ldots\} \cup \{a, b, c\},$$

$0 < 1 < 2 < \ldots < b < c$, and $0 < a < c$. It is known that the topology τ_L is compact. But the topology τ_{qL} is not compact. Indeed, the cover $\{L \setminus \uparrow n : n = 1, 2, \ldots\} \cup \{b, c\}$ of L is open with respect to the topology τ_{qSc} and does not contain a finite subcover.

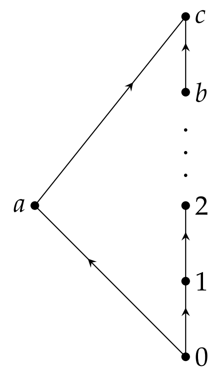

Figure 8: The lattice L in Example 6.8

Proposition 6.9. *Let L be a q-continuous complete lattice. Then, τ_{qL} is Hausdorff.*

Proof. Let $x, y \in L$ with $x \neq y$. Without loss of generality suppose that $x \nleqslant y$. Since L is q-continuous, $x = \bigvee \Downarrow_q x$. We show that there exists $z \in L$ such that $z \ll_q x$ and $z \nleqslant y$. Indeed, suppose that for every $z \ll_q x$ we have $z \leqslant y$. Then, y is an upper bound of $\Downarrow_q x$ and, hence,

$$x = \bigvee \Downarrow_q x \leqslant y$$

which is a contradiction. We set

$$U = \Uparrow_q z \text{ and } V = L \setminus \uparrow z$$

for some $z \in L$ such that $z \ll_q x$ and $z \nleqslant y$. Then, $V \in \tau_l \subseteq \tau_{qL}$. Also, by Corollary 4.13, $U \in \tau_{qSc} \subseteq \tau_{qL}$. We observe that $x \in U$, $y \in V$, and $U \cap V = \emptyset$. Thus, τ_{qL} is Hausdorff. □

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