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Best Proximity Point and Best Proximity Coupled Point in a Complete Metric Space with (*P*)-Property

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Abstract. In this paper, we utilize the concept of (P)-property, weak (P)-property and the comparison function to introduce and prove an existence and uniqueness theorem of a best proximity point. Also, we introduce the notion of a best proximity coupled point of a mapping $F\colon X \times X \to X$. Using this notion and the comparison function to prove an existence and uniqueness theorem of a best proximity coupled point. Our results extend and improve many existing results in the literature. Finally, we introduce examples to support our theorems.

1. Introduction

Let A be a nonempty subset of a metric space (X, d). Let T be a mapping from X into X. A point $x \in X$ is called a best proximity point of T if $d(x, Tx) = d(A, x)$, where

$$d(A, x) := \inf\{d(a, x) : a \in A\}.$$

Note that if $x \in A$, then x is a fixed point of T. Thus the best proximity point plays a crucial role in fixed point theory, and many authors studied this notion. In [1], the existence of a best proximity point for a cyclic contraction map in a reflexive Banach space is proved. Also, the authors introduce a new class of mappings, the cyclic φ-contractions, and they prove convergence and existence results for those class of mappings. The notion of proximal pointwise contraction and results regarding the existence of a best proximity point on a pair of weakly compact convex subset of a Banach space are obtained in [2]. In [3], there are stated contraction type existence results for a best proximity point and an algorithm to find a best proximity point for a mapping in the context of a uniformly convex Banach space. In [4], there is introduced the notion of cyclic orbital Meir-Keeler contraction, and there are given sufficient conditions for the existence of fixed points and best proximity points of such a map. The proximity and best proximity pair theorems in hyperconvex metric spaces and in Hilbert spaces are presented in [5], providing optimal approximate solutions for the situation when a mapping does not have fixed points. Paper [6] applies a convergence theorem in order to prove the existence of a best proximity point, without the use of Zorns lemma. In [7], the authors study a mapping which satisfies a cyclical generalized contractive condition related to a

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pair of altering distance functions. Paper [8] introduces the class of p-cyclic φ-contractions, larger than the p-cyclic contraction mappings and presents convergence and existence results of best proximity points for mappings from this class are obtained. In [9], Sankar Raj studied a fixed point theorem for weakly contractive nonselfmappings based on the notion of (P)-property. For some interesting examples of pairs having the (P)-property, we address the reader to [9], [10], [11]. For some work in almost contraction see [12]-[20].

In this paper, we introduce the notion of the generalized almost (φ, θ)-contraction and the notion of a best proximity coupled point of a mapping $F\colon X \times X \to X$. Also, we utilize our notions to introduce and prove a best proximity point theorem and a best proximity coupled point theorem. Our results extend and improve many existing results in literature.

2. Preliminaries

To introduce our new results, it is fundamental to recall the definition of a best proximity point of a nonselfmapping T and the notion of (weak) (P)-property.

Let A and B be nonempty subsets of a metric space. To facilitate the arguments let

$$A_0 = \{a \in A : d(a,b) = d(A,B), \text{ for some } b \in B\},$$

$$B_0 = \{b \in B : d(a,b) = d(A,B), \text{ for some } a \in A\},$$

and

$$d(A,B) := \inf\{d(a,b) : a \in A, b \in B\}.$$

Definition 2.1 ([10]). Let A and B be two nonempty subsets of a metric space (X,d). An element $u \in A$ is said to be a *best proximity point* of the nonselfmapping $T\colon A \to B$ iff it satisfies the condition

$$d(u, Tu) = d(A,B).$$

Definition 2.2 ([9]). Let (A,B) be a pair of nonempty subsets of a metric space (X,d) with $A_0 \neq \emptyset$. Then, pair (A,B) is said to have the *weak (P)-property* if, for each $x_1, x_2 \in A$, and $y_1, y_2 \in B$, the following implication holds

$$\begin{pmatrix} d(x_1,y_1) = d(A,B) \\ d(x_2,y_2) = d(A,B) \end{pmatrix} \Rightarrow \quad d(x_1,x_2) \le d(y_1,y_2).$$

If we replace relation $d(x_1,x_2) \le d(y_1,y_2)$ by $d(x_1,x_2) = d(y_1,y_2)$ we obtain a less general notion, that of a pair endowed with the (P)-property.

In his elegant paper [10], Samet studied a nice best proximity point theorem of the form almost contraction for a pair of sets endowed with the (P)-property. Before we present the main result of Samet, we recall the following

Definition 2.3 ([13]). A map $\varphi\colon [0,+\infty) \to [0,+\infty)$ is called a *c-comparison function* if it satisfies:

1. φ is a monotone increasing,
2. $\sum_{n=0}^{+\infty} \varphi^n(t)$ converges for all $t \ge 0$.

If we replace the second condition by $\lim_{n\to+\infty} \varphi^n(t) = 0$, $\forall n \in \mathbb{N}$, we obtain the notion of comparison function, which is more general than the one of c-comparison function.

It is known that if φ is a comparison function, then $\varphi(t) < t$ for all $t > 0$ and $\varphi(0) = 0$.

Works involving either (c)-comparison functions or comparison functions are, for instance, [14] and [20].

In the following, denote $[0,+\infty) \times [0,+\infty) \times [0,+\infty) \times [0,+\infty)$ by $[0,+\infty)^4$.

Let Θ be the set of all continuous functions $\theta\colon [0,+\infty)^4 \to [0,+\infty)$ such that

$$\theta(0,t,s,u) = 0 \text{ for all } t,s,u \in [0,+\infty)$$

and

$$\theta(t,s,0,u) = 0 \text{ for all } t,s,u \in [0,+\infty).$$

Example 2.4 ([10]). Define $\theta_1, \theta_2, \theta_3 \colon [0, +\infty)^4 \to [0, +\infty)$ by the formulas

$$\theta_1(t, s, u, v) = \tau \inf\{t, s, u, v\}, \quad \tau > 0,$$

$$\theta_2(t, s, u, v) = \tau \ln(1 + tsuv), \quad \tau > 0,$$

and

$$\theta_3(t, s, u, v) = \tau tsuv, \quad \tau > 0.$$

Then $\theta_1, \theta_2, \theta_3 \in \Theta$.

Samet [10] introduced the following definition.

Definition 2.5 ([10]). Let φ be a c-comparison function, and $\theta \in \Theta$. A mapping $T \colon A \to B$ is called an *almost* (φ, θ)*-contraction* if, for each $x, y \in A$,

$$\begin{aligned} d(Tx, Ty) \quad \le \quad & \varphi(d(x, y)) + \theta(d(y, Tx) - d(A, B), d(x, Ty) - d(A, B), \\ & d(x, Tx) - d(A, B), d(y, Ty) - d(A, B)). \end{aligned}$$

The main result of Samet is

Theorem 2.6 ([10]). *Let A and B two closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Suppose that $T \colon A \to B$ satisfies the following conditions:*
1) T is an almost (φ, θ)-contraction;
2) $TA_0 \subseteq B_0$;
3) Pair (A, B) has the P-property.
Then, there exists a unique element $x* \in A$ such that

$$d(x^*, Tx^*) = d(A, B).$$

Moreover, for any fixed element $x_0 \in A_0$, any iterative sequence (x_n) satisfying

$$d(x_{n+1}, Tx_n) = d(A, B)$$

converges to x^*.

3. Main Results

Our first aim in the paper is to introduce and prove a best proximity point theorem for a more general case. For this instance, we introduce the notion of a generalized almost (φ, θ)-contraction, as follows

Definition 3.1. Let φ be a comparison function, and $\theta \in \Theta$. Mapping $T \colon A \to B$ is called a *generalized almost* (φ, θ)*-contraction* if, for each $x, y \in A$,

$$\begin{aligned} d(Tx, Ty) \quad \le \quad & \varphi(d(x, y)) + \theta(d(y, Tx) - d(A, B), d(x, Ty) - d(A, B), \\ & d(x, Tx) - d(A, B), d(y, Ty) - d(A, B)). \end{aligned}$$

Our first result is

Theorem 3.2. *Consider A and B two closed subsets of a complete metric space (X, d) for which A_0 is nonempty. Let $T \colon A \to B$ be a mapping which satisfies the following conditions:*
1) *T is a generalized almost (φ, θ)-contraction;*
2) *$TA_0 \subseteq B_0$;*
3) *Pair (A, B) has the weak P-property.*
Then, there exists a unique best proximity point of T, $x^* \in A$.

Proof. Consider $x_0 \in A_0$. Since $TA_0 \subseteq B_0$, then $Tx_0 \in B_0$, and there is $x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A, B)$. By continuing this procedure, we obtain a sequence $(x_n) \subseteq A_0$,

$$d(x_{n+1}, Tx_n) = d(A, B), \quad \forall n \in \mathbb{N} \cup \{0\}.$$

If there is $n \in \mathbb{N} \cup \{0\}$, for which $d(x_{n+1}, x_n) = 0$, it follows

$$d(A, B) \leq d(x_n, Tx_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, Tx_n) = d(x_{n+1}, Tx_n) = d(A, B),$$

hence $d(A, B) = d(x_n, Tx_n)$, so x_n is a best proximity point of T.

Without loss of generality, in the following we may assume that $d(x_n, x_{n+1}) > 0$, for each $n \in \mathbb{N} \cup \{0\}$.

(A, B) satisfies the weak (P)-property, so $d(x_n, x_{n+1}) \leq d(Tx_{n-1}, Tx_n)$, $n \in \mathbb{N}$.

Using the almost (φ, θ)-contraction property of T, we have

$$\begin{aligned} & d(x_n, x_{n+1}) \leq d(Tx_{n-1}, Tx_n) \\ \leq\ & \varphi(d(x_{n-1}, x_n)) + \theta(d(x_n, Tx_{n-1}) - d(A, B), d(x_{n-1}, Tx_n) - d(A, B), \\ & \qquad d(x_{n-1}, Tx_{n-1}) - d(A, B), d(x_n, Tx_n) - d(A, B)) \\ =\ & \varphi(d(x_{n-1}, x_n)) + \theta(0, d(x_{n-1}, Tx_n) - d(A, B), \\ & \qquad d(x_{n-1}, Tx_{n-1}) - d(A, B), d(x_n, Tx_n) - d(A, B)) \\ =\ & \varphi(d(x_{n-1}, x_n)), \quad n \in \mathbb{N} \cup \{0\}. \end{aligned}$$

Applying repeatedly this inequality, and using the monotone of φ, we get

$$d(x_n, x_{n+1}) \leq \varphi^n(d(x_0, x_1)), \quad n \in \mathbb{N} \cup \{0\}.$$

But φ is a comparison function, so, taking $n \to +\infty$, we obtain $\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0$.

Taking into account the inequalities

$$d(A, B) \leq d(x_n, Tx_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, Tx_n),$$

and letting $n \to +\infty$, we obtain

$$\lim_{n\to+\infty} d(x_n, Tx_n) = d(A, B). \tag{1}$$

Let $\varepsilon > 0$. Since $\lim_{n\to+\infty} d(x_n, x_{n+1})$ there exists $n_0 \in \mathbb{N}$ such that for each $n > n_0$, we have

$$d(x_n, x_{n+1}) < \frac{1}{2}(\varepsilon - \varphi(\varepsilon)). \tag{2}$$

We shall prove that $d(x_n, x_m) < \varepsilon$, for each $m > n > n_0$ by induction on m.

For $m = n + 1$, we obtain

$$d(x_n, x_{n+1}) < \frac{1}{2}(\varepsilon - \varphi(\varepsilon)) < \varepsilon.$$

Suppose the inequality is satisfied for $m = k$, and we shall prove that the relation holds for $m = k + 1$. The triangular inequality leads us to

$$d(x_n, x_{k+1}) \quad \leq \quad d(x_n, x_{n+1}) + d(x_{n+1}, x_{k+1}). \tag{3}$$

Since $d(x_{n+1}, Tx_n) = d(A, B)$, and $d(x_{k+1}, Tx_k) = d(A, B)$, applying the weak (P)-property, it follows that $d(x_{n+1}, x_{k+1}) \leq d(Tx_n, Tx_k)$. The almost (φ, θ)-contraction property of T, we obtain

$$\begin{aligned} & d(x_{n+1}, x_{k+1}) \leq d(Tx_n, Tx_k) \\ \leq\ & \varphi(d(x_n, x_k)) + \theta(d(x_k, Tx_n) - d(A, B), d(x_n, Tx_k) - d(A, B), \\ & \qquad d(x_n, Tx_n) - d(A, B), d(x_k, Tx_k) - d(A, B)) \end{aligned} \tag{4}$$

Since θ is a continuous function and $\lim_{n\to+\infty} d(x_n, Tx_n) = d(A,B)$, we have

$$\limsup_{n\to+\infty} \theta(d(x_k, Tx_n) - d(A,B), d(x_n, Tx_k) - d(A,B),$$
$$d(x_n, Tx_n) - d(A,B), d(x_k, Tx_k) - d(A,B)) = 0.$$

Thus, we may consider that n_0 is large enough so for each $n > n_0$,

$$\theta(d(x_k, Tx_n) - d(A,B), d(x_n, Tx_k) - d(A,B),$$
$$d(x_n, Tx_n) - d(A,B), d(x_k, Tx_k) - d(A,B)) < \frac{1}{2}(\varepsilon - \varphi(\varepsilon)) \tag{5}$$

Using inequalities (2), (4), and (5) into (3), we get

$$d(x_n, x_{k+1}) \leq \frac{1}{2}(\varepsilon - \varphi(\varepsilon)) + \varphi(\varepsilon) + \frac{1}{2}(\varepsilon - \varphi(\varepsilon)),$$

hence $d(x_n, x_{k+1}) < \varepsilon$, and we proved that $d(x_n, x_m) < \varepsilon$, $m > n > n_0$. We got that (x_n) is a Cauchy sequence in A, which is a closed subset of (X,d), a complete metric space. Therefore, there exists $x \in A$ such that $\lim_{n\to+\infty} x_n = x^*$.

Using the triangle inequality, it follows

$$d(x^*, Tx^*) \leq d(x^*, x_n) + d(x_n, Tx_n) + d(Tx^*, Tx_n). \tag{6}$$

Letting $n \to +\infty$ in the inequality

$$d(Tx^*, Tx_n) \leq \varphi(d(x^*, x_n)) + \theta(d(x_n, Tx^*) - d(A,B), d(x^*, Tx_n) - d(A,B),$$
$$d(x_n, Tx_n) - d(A,B), d(x^*, Tx^*) - d(A,B)),$$

it follows $\lim_{n\to+\infty} d(Tx_n, Tx^*) = 0$. Taking $n \to +\infty$ in relation (6), it follows that $d(x^*, Tx^*) = d(A,B)$, so x^* is a best proximity point of T.

We shall focus now on the uniqueness of the best proximity point of T. Suppose there are $x^* \neq y^*$ two best proximity points of T. We obtain

$$\begin{aligned} d(x^*, y^*) &\leq d(Tx^*, Ty^*) \\ &\leq \varphi(d(x^*, y^*)) + \theta(d(y^*, Tx^*) - d(A,B), d(x^*, Ty^*) - d(A,B), \\ &\qquad d(x^*, Tx^*) - d(A,B), d(y^*, Ty^*) - d(A,B)) \\ &= \varphi(d(x^*, y^*)) + \theta(d(y^*, Tx^*) - d(A,B), d(x^*, Ty^*) - d(A,B), \\ &\qquad 0, d(y^*, Ty^*) - d(A,B)) \\ &\leq \varphi(d(x^*, y^*)), \end{aligned}$$

which is impossible, since $x^* \neq y^*$. The uniqueness part has been proved now. □

Let us take the particular case of $\varphi\colon [0,+\infty) \to [0,+\infty)$, $\varphi(t) = kt$, where $k \in [0,1)$, and

$$\theta\colon [0,+\infty)^4 \to [0,+\infty), \quad \theta(t_1, t_2, t, 3, t_4) = L\min\{t_1, t_2, t_3, t_4\},$$

for some $L \geq 0$. We obtain the following corollary.

Corollary 3.3. *Let A and B be two closed subsets of a complete metric space (X,d) for which A_0 is nonempty. Let $T\colon A \to B$ be a mapping which satisfies the following conditions:*

1) $TA_0 \subseteq B_0$;

2) Pair (A,B) has the weak (P)-property.

Suppose there exist $k \in [0,1)$ and $L \geq 0$ such that

$$d(Tx, Ty) \leq kd(x,y) + L\min\{d(y, Tx) - d(A,B), d(x, Ty) - d(A,B),$$
$$d(x, Tx) - d(A,B), d(y, Ty) - d(A,B)\}$$

holds for all $x, y \in A$. Then, there exists a unique best proximity point of T, $x^* \in A$.

By considering $A = B$ in Theorem 3.2, we get the next corollary

Corollary 3.4. *Let A be a closed subsets of a complete metric space (X, d). Let $T\colon A \to A$ be a mapping such that*

$$d(Tx, Ty) \le \varphi(d(x, y)) + \theta(d(y, Tx), d(x, Ty), d(x, Tx), d(y, Ty))$$

holds for all $x, y \in A$. Then T has a unique fixed point $u \in A$; that is $Tu = u$.

Our second aim in this paper is to present a best proximity coupled point of a mapping $T\colon X \times X \to X$. Before we present our second result we introduce the following definition.

Definition 3.5. Let A and B be closed subsets of a metric space (X, d). An element $(u, v) \in X \times X$ is called a *best proximity coupled point* of a mapping $F\colon X \times X \to X$ if $u \in A$, $v \in B$ and $d(u, F(u, v)) = d(A, B)$ and $d(v, F(v, u)) = d(A, B)$.

Theorem 3.6. *Let A and B be two closed subsets of a complete metric space (X, d) for which A_0 and B_0 are nonempty. Let $F\colon X \times X \to X$ be a continuous mapping which satisfies the following conditions:*
1) *$F(A_0 \times B_0) \subseteq B_0$;*
2) *$F(B_0 \times A_0) \subseteq A_0$;*
3) *Pair (A, B) has the (P)-property.*
Also, suppose there exist functions φ and $\theta \in \Theta$ such that

$$\begin{aligned} & d(F(x, y), F(u, v)) \\ \le\ & \varphi(\max\{d(x, u), d(y, v)\}) + \theta(d(u, F(x, y)) - d(A, B), d(v, F(y, x)) - d(A, B), \\ & d(x, F(x, y)) - d(A, B), d(y, F(y, x)) - d(A, B)) \end{aligned} \tag{7}$$

holds for all $x, y, u, v \in X$.

Then, there exists a unique best proximity coupled point of F of the form (u, u).

Proof. Choose $x_0 \in A_0$ and $y_0 \in B_0$. Since $F(x_0, y_0) \in B_0$, we choose $x_1 \in A$ such that $d(x_1, F(x_0, y_0)) = d(A, B)$. Also, since $F(y_0, x_0) \in A_0$ we choose $y_1 \in B$ such that $d(y_1, F(y_0, x_0)) = d(B, A)$. As $F(x_1, y_1) \in B_0$, we choose $x_2 \in A$ such that $d(x_2, F(x_1, y_1)) = d(A, B)$. Also, since $F(y_1, x_1) \in A_0$ we choose $y_2 \in B$ such that $d(y_2, F(y_1, x_1)) = d(B, A)$. Continuing this process, we construct two sequences (x_n) in A and (y_n) in B such that

$$d(x_{n+1}, F(x_n, y_n)) = d(A, B)$$

and

$$d(y_{n+1}, F(y_n, x_n)) = d(B, A)$$

hold for all $n \in \mathbb{N} \cup \{0\}$.

Suppose there exists $n \in \mathbb{N}$ such that $d(x_n, x_{n+1}) = 0$ and $d(y_n, y_{n+1}) = 0$. Thus

$$\begin{aligned} d(A, B) &\le d(x_n, F(x_n, y_n)) \\ &\le d(x_n, x_{n+1}) + d(x_{n+1}, F(x_n, y_n)) \\ &= d(A, B). \end{aligned}$$

Thus we have $d(A, B) = d(x_n, F(x_n, y_n))$. Similarly, we obtain $d(A, B) = d(y_n, F(y_n, x_n))$. Therefore, (x_n, y_n) is a best proximity coupled point of F.

So, we may assume that $d(x_n, x_{n+1}) > 0$ or $d(y_n, y_{n+1}) > 0$.

Since pair (A, B) has the (P)-property, $d(x_n, F(x_{n-1}, y_{n-1})) = d(A, B)$, and $d(x_{n+1}, F(x_n, y_n)) = d(A, B)$, we have

$$d(x_n, x_{n+1}) = d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)).$$

By (7), we obtain

$$\begin{aligned}
& d(x_n, x_{n+1}) \\
= \;& d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\
\leq \;& \varphi(\max\{d(x_{n-1}, x_n), d(y_{n-1}, y_n)\} + \theta(d(x_n, F(x_{n-1}, y_{n-1})) - d(A, B), \\
& d(y_n, F(y_{n-1}, x_{n-1})) - d(A, B), d(x_{n-1}, F(x_{n-1}, y_{n-1})) - d(A, B), \\
& d(y_{n-1}, F(y_{n-1}, x_{n-1})) - d(A, B)) \\
= \;& \varphi(\max\{d(x_{n-1}, x_n), d(y_{n-1}, y_n)\}. \qquad (8)
\end{aligned}$$

Also, since pair (A, B) has the (P)-property, $d(y_n, F(y_{n-1}, x_{n-1})) = d(A, B)$, and $d(y_{n+1}, F(y_n, x_n)) = d(A, B)$, we have

$$d(y_n, y_{n+1}) = d(F(y_{n-1}, x_{n-1}), F(y_n, x_n)).$$

Again by (7), we get

$$\begin{aligned}
& d(y_n, y_{n+1}) \\
= \;& d(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \\
\leq \;& \varphi(\max\{d(y_{n-1}, y_n), d(x_{n-1}, x_n)\}) + \theta(d(y_n, F(y_{n-1}, x_{n-1})) - d(A, B), \\
& d(x_n, F(x_{n-1}, y_{n-1})) - d(A, B), d(y_{n-1}, F(y_{n-1}, x_{n-1})) - d(A, B), \\
& d(x_{n-1}, F(x_{n-1}, y_{n-1})) - d(A, B)) \\
= \;& \varphi(\max\{d(y_{n-1}, y_n), d(x_{n-1}, x_n)\}). \qquad (9)
\end{aligned}$$

Combining (8) and (9), we get

$$\max\{d(x_n, x_{n+1}), d(y_n, y_{n+1})\} \leq \varphi(\max\{d(x_{n-1}, x_n), d(y_{n-1}, y_n)\}). \qquad (10)$$

Repeating (10) n-times, we obtain

$$\begin{aligned}
\max\{d(x_n, x_{n+1}), d(y_n, y_{n+1})\} &\leq \varphi(\max\{d(x_{n-1}, x_n), d(y_{n-1}, y_n)\}) \\
&\leq \varphi^2(\max\{d(x_{n-2}, x_{n-1}), d(y_{n-2}, y_{n-1})\}) \\
&\;\;\vdots \\
&\leq \varphi^n(\max\{d(x_0, x_1), d(y_0, y_1)\}).
\end{aligned}$$

Thus

$$\lim_{n\to+\infty} d(x_n, x_{n+1}) = \lim_{n\to+\infty} d(y_n, y_{n+1}) = 0.$$

On other hand,

$$\begin{aligned}
d(A, B) &\leq d(x_n, F(x_n, y_n)) \\
&\leq d(x_n, x_{n+1}) + d(x_{n+1}, F(x_n, y_n)) \\
&= d(x_n, x_{n+1}) + d(A, B).
\end{aligned}$$

Letting $n \to +\infty$ in the above inequalities, we get

$$\lim_{n\to+\infty} d(x_n, F(x_n, y_n)) = d(A, B).$$

Similarly, one can show that

$$\lim_{n\to+\infty} d(y_n, F(y_n, x_n)) = d(A, B).$$

Consider $\epsilon > 0$. Since $\varphi^n(\max\{d(x_0, x_1), d(y_0, y_1)\} \to 0$ as $n \to +\infty$, there exists $n_0 \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) < \frac{1}{2}(\epsilon - \varphi(\epsilon))$$

and

$$d(y_n, y_{n+1}) < \frac{1}{2}(\epsilon - \varphi(\epsilon))$$

hold for all $n \geq n_0$.

Now, we use the induction on m to prove that

$$\max\{d(x_n, x_m), d(y_n, y_m)\} < \epsilon \;\; \forall \;\; m > n \geq n_0. \tag{11}$$

Note that (11) holds for $m = n + 1$ because $\max\{d(x_n, x_m), d(y_n, y_m)\} < \frac{1}{2}(\epsilon - \varphi(\epsilon)) < \epsilon$ holds for all $n \geq n_0$. Assume inequality (11) holds for $m = k$. Now, we prove relation (11) for $m = k + 1$. By using the triangular inequality, we have

$$d(x_n, x_{k+1}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{k+1}). \tag{12}$$

Since pair (A, B) has the (P)-property, $d(x_{n+1}, F(x_n, y_n)) = d(A, B)$, and

$$d(x_{k+1}, F(x_k, y_k)) = d(A, B)$$

we have

$$d(x_{n+1}, x_{k+1}) \leq d(F(x_n, y_n), F(x_k, y_k)).$$

Using the contraction condition (7), we have

$$\begin{aligned} & d(x_{n+1}, x_{k+1}) \\ = \;& d(F(x_n, y_n), F(x_k, y_k)) \\ \leq \;& \varphi(\max\{d(x_n, x_k), d(y_n, y_k)\}) + \theta(d(x_k, F(x_n, y_n)) - d(A, B), \\ & d(y_k, F(y_n, x_n)) - d(A, B), d(x_n, F(x_n, y_n)) - d(A, B), d(y_n, F(y_n, x_n)) - d(A, B)), \end{aligned} \tag{13}$$

and

$$\begin{aligned} & d(y_{n+1}, y_{k+1}) \\ = \;& d(F(y_n, x_n), F(y_k, x_k)) \\ \leq \;& \varphi(\max\{d(x_n, x_k), d(y_n, y_k)\}) + \theta(d(y_k, F(x_n, x_n)) - d(A, B), \\ & d(x_k, F(x_n, y_n)) - d(A, B), d(y_n, F(x_n, x_n)) - d(A, B), d(x_n, F(x_n, y_n)) - d(A, B)), \end{aligned} \tag{14}$$

Using the properties of θ, and the fact that $\lim_{n\to+\infty} d(x_n, F(x_n, y_n)) = d(A, B)$, and $\lim_{n\to+\infty} d(y_n, F(y_n, x_n)) = d(A, B)$ we have

$$\begin{aligned} & \limsup_{n\to+\infty} \theta(d(x_k, F(x_n, y_n)) - d(A, B), d(y_k, F(y_n, x_n)) - d(A, B), \\ & d(x_n, F(x_n, y_n)) - d(A, B), d(y_n, F(y_n, x_n)) - d(A, B)) = 0, \end{aligned}$$

and

$$\begin{aligned} & \limsup_{n\to+\infty} \theta(d(y_k, F(y_n, x_n)) - d(A, B), d(x_k, F(x_n, y_n)) - d(A, B), \\ & d(y_n, F(y_n, x_n)) - d(A, B), d(x_n, F(x_n, y_n)) - d(A, B)) = 0, \end{aligned}$$

Thus for n_0 large enough, we have

$$\begin{aligned} & \theta(d(x_k, F(x_n, y_n)) - d(A, B), d(y_k, F(y_n, x_n)) - d(A, B), \\ & d(x_n, F(x_n, y_n)) - d(A, B), d(y_n, F(y_n, x_n)) - d(A, B)) < \frac{1}{2}(\epsilon - \varphi(\epsilon)). \end{aligned} \tag{15}$$

and

$$\begin{aligned}&\theta(d(y_k, F(y_n, x_n)) - d(A, B), d(x_k, F(x_n, y_n)) - d(A, B),\\ &d(y_n, F(y_n, x_n)) - d(A, B), d(x_n, F(x_n, y_n)) - d(A, B)) < \frac{1}{2}(\epsilon - \varphi(\epsilon)).\end{aligned} \tag{16}$$

From relation (11)-(16), we get

$$\max\{d(x_n, x_{k+1}, d(y_n, y_{k+1}))\} \leq \frac{1}{2}(\epsilon - \varphi(\epsilon)) + \varphi(\epsilon) + \frac{1}{2}(\epsilon - \varphi(\epsilon)) < \epsilon. \tag{17}$$

Thus (11) holds for $m = k + 1$. Thus (11) holds for all $m \geq n \geq n_0$. Thus (x_n) and (y_n) are Cauchy sequences in A and B respectively. Since (X, d) is complete, there exist $u, v \in X$ such that

$$\lim_{n\to+\infty} x_n = u$$

and

$$\lim_{n\to+\infty} y_n = v.$$

Since A and B are closed, we get $u \in A$ and $v \in B$.

Letting $n \to +\infty$ in

$$d(x_{n+1}, F(x_n, y_n)) = d(A, B)$$

and using the continuity of F, we get

$$d(u, F(u, v)) = d(A, B).$$

Similarly, we get

$$d(v, F(v, u)) = d(A, B).$$

Thus, (u, v) is a best proximity coupled point of F. Now, we show that $u = v$. Using the (P)-property of pair (A, B), we get

$$d(u, v) = d(F(u, v), F(v, u)).$$

Using inequality (7), we get

$$\begin{aligned}&d(u, v) = d(F(u, v), F(v, u))\\ \leq\ &\varphi(\max\{d(u, v), d(v, u)\}) + \theta(d(v, F(u, v)) - d(A, B),\\ &d(u, F(v, u)) - d(A, B), d(u, F(u, v)) - d(A, B), d(v, F(v, u)) - d(A, B))\\ =\ &\varphi(d(u, v)) + \theta(d(v, F(u, v)) - d(A, B), d(u, F(v, u)) - d(A, B), 0, 0)\\ =\ &\varphi(d(u, v)).\end{aligned}$$

Since $\varphi(t) < t$ for all $t > 0$, we conclude that $d(u, v) = 0$. Thus $u = v$.

To prove the uniqueness of the best proximity coupled point of F, we assume that w is another best proximity coupled point of F; that is, $d(u, F(u, u)) = d(A, B)$ and $d(w, F(w, w)) = d(A, B)$. Using the (P)-property of pair (A, B), we get $d(u, w) = d(F(u, u), F(w, w))$. Now using (7), we get

$$\begin{aligned}&d(u, w) = d(F(u, u), F(w, w))\\ \leq\ &\varphi(d(u, w)) + \theta(d(w, F(u, u)) - d(A, B),\\ &d(w, F(u, u)) - d(A, B), d(u, F(u, u)) - d(A, B), d(u, F(u, u)) - d(A, B))\\ =\ &\varphi(d(u, v)) + \theta(d(w, F(u, u)) - d(A, B), d(w, F(u, u)) - d(A, B), 0, 0)\\ =\ &\varphi(d(u, w)).\end{aligned}$$

Again, since $\varphi(t) < t$ for all $t > 0$, we conclude that $d(u, w) = 0$. Thus $u = w$. □

Define $\varphi\colon [0,+\infty) \to [0,+\infty)$ via $\varphi(t) = kt$, where $k \in [0,1)$ and

$$\theta\colon [0,+\infty)^4 \to [0,+\infty), \quad \theta(t_1,t_2,t,3,t_4) = L\min\{t_1,t_2,t_3,t_4\},$$

for some $L \geq 0$. The following results are corollaries of Theorem 3.6.

Corollary 3.7. *Let A and B be two closed subsets of a complete metric space (X,d) for which A_0 and B_0 are nonempty. Let $F\colon X\times X \to X$ be a continuous mapping which satisfies the following conditions:*
1) $F(A_0 \times B_0) \subseteq B_0$*;*
2) $F(B_0 \times A_0) \subseteq A_0$*;*
3) *The pair (A,B) has the (P)-property.*
Also, suppose there exist $k \in [0,1)$ and $L \geq 0$ such that

$$\begin{aligned} & d(F(x,y),F(u,v)) \\ \leq\ & k\max\{d(x,u),d(y,v)\} + L\min\{d(u,F(x,y)) - d(A,B), d(v,F(y,x)) - d(A,B), \\ & d(x,F(x,y)) - d(A,B), d(y,F(y,x)) - d(A,B)\} \end{aligned}$$

holds for all $x,y,u,v \in X$ Then, there exists a unique best proximity coupled point of F of the form (u,u).

Take $B = A$ in Theorem 3.6, we have the following result.

Corollary 3.8. *Let A a closed subsets of a complete metric space (X,d). Let $F\colon X\times X \to X$ be a continuous mapping with $F(A\times A) \subseteq A$. Suppose there exists a comparison function φ and $\theta \in \Theta$ such that*

$$\begin{aligned} & d(F(x,y),F(u,v)) \\ \leq\ & \varphi(\max\{d(x,u),d(y,v)\}) + \theta(d(u,F(x,y)), d(v,F(y,x)), \\ & d(x,F(x,y)), d(y,F(y,x))) \end{aligned}$$

holds for all $x,y,u,v \in X$ Then F has a unique coupled fixed point of the form (u,u); that is $F(u,u) = u$.

4. Examples and concluding remark

Now we shall provide an example to substantiate our Theorem 3.2. Function φ which will be used here is a comparison, but not a c-comparison, proving that Theorem 2.6 from the work of Samet [10] cannot be applied in our case.

Example 4.1. Consider

$$X = \left\{0,1,\frac{1}{2},\frac{1}{3},\dots\right\}, \quad A = \left\{0,\frac{1}{2},\frac{1}{4},\dots\right\}, \quad B = \left\{0,\frac{1}{3},\frac{1}{5},\dots\right\}.$$

We endow X with the metric

$$d\colon X\times X \to X, \quad d(x,y) = \begin{cases} 0, & \text{if } x = y; \\ \max\{x,y\}, & \text{if } x \neq y. \end{cases}$$

Let $T\colon X \to X$, $Tx = \dfrac{x}{1+x}$, $\theta\colon [0,+\infty)^4 \to [0,+\infty)$, $\theta(t,s,u,v) = \inf\{t,s,u,v\}$, and $\varphi\colon [0,+\infty) \to [0,+\infty)$, $\varphi t = \dfrac{t}{1+t}$. Then

1. $TA_0 \subseteq B_0$.
2. Pair (A,B) has the (P)-property.
3. T is an almost (φ,θ)-contraction.

Proof. Here, $A_0 = \{0\}$, $B_0 = \{0\}$ and $d(A, B) = 0$. So the proofs of (1) and (2) are clear.

We spill the proof of (3) into three cases.

Case 1. $x = \dfrac{1}{n}$, $y = \dfrac{1}{m}$, $n < m$ and n, m are even (the situation $n > m$ is similar to this one).

We obtain

$$\begin{aligned}
& \varphi\Big(d\Big(\frac{1}{n}, \frac{1}{m}\Big)\Big) + \theta\Big(d\Big(\frac{1}{m}, \frac{1}{n+1}\Big), d\Big(\frac{1}{n}, \frac{1}{m+1}\Big), d\Big(\frac{1}{n}, \frac{1}{n+1}\Big), d\Big(\frac{1}{m}, \frac{1}{m+1}\Big)\Big) \\
= \;& \varphi\Big(\frac{1}{n}\Big) + \theta\Big(\frac{1}{n+1}, \frac{1}{n}, \frac{1}{n}, \frac{1}{m}\Big) \\
= \;& \frac{1}{n+1} + \frac{1}{m} \\
\geq \;& \frac{1}{n+1} = d\Big(\frac{1}{n+1}, \frac{1}{m+1}\Big) \\
= \;& d\Big(T\frac{1}{n}, T\frac{1}{m}\Big),
\end{aligned}$$

so the almost (φ, θ)-contraction inequality is satisfied.

Case 2. $x = y = 0$. This case is straightforward.

Case 3. $x = 0$, and $y = \dfrac{1}{m}$, where m is even (which is similar to $y = 0$, and $x = \dfrac{1}{m}$).

We get

$$\begin{aligned}
d\Big(0, T\frac{1}{m}\Big) &= d\Big(0, \frac{1}{m+1}\Big) = \frac{1}{m+1} \\
&\leq \varphi\Big(\frac{1}{m}\Big) = \varphi\Big(d\Big(0, \frac{1}{m}\Big)\Big) \\
&\leq \varphi\Big(d\Big(0, \frac{1}{m}\Big)\Big) + \theta\Big(d\Big(\frac{1}{m}, 0\Big), d\Big(\frac{1}{m}, 0\Big), d(0, 0), d\Big(\frac{1}{m}, \frac{1}{m+1}\Big)\Big).
\end{aligned}$$

Therefore, T is an almost (φ, θ)-contraction. This end the proof of part (3).

By using Theorem 3.2, we conclude that T has a best proximity point in A, $x^* = 0$. □

Example 4.2. Let $X = \{0, 2, 3, 4, 5\}$, define a metric $d : X \times X \to X$ by $d(x, y) = \frac{1}{2}|x - y|$. Take $A = \{0, 3\}$ and $B = \{2, 4, 5\}$. Define a mapping $T : A \to B$ by $T0 = 5$ and $T3 = 4$. Also, define $\varphi : [0, +\infty) \to [0, +\infty)$ by $\varphi(t) = \frac{t}{1+t}$ and $\theta : [0, +\infty)^4 \to [0, +\infty)$, by $\theta(t_1, t_2, t_3, t_4) = \inf\{t_1, t_2, t_3, t_4\}$. Then

1. $TA_0 \subseteq B_0$.
2. Pair (A, B) has the weak (P)-property.
3. T is a generalized almost (φ, θ)-contraction.

Proof. Here $A_0 = \{3\}$, $B_0 = \{2, 4\}$ and $d(A, B) = \frac{1}{2}$. Thus $TA_0 \subseteq B_0$. To prove that (A, B) has the weak P-property, let $d(x_1, y_1) = d(A, B)$ and $d(x_2, y_2) = d(A, B)$. Then $d(x_1, y_1) = \frac{1}{2}$ and $d(x_2, y_2) = \frac{1}{2}$. Thus $(x_1, y_1), (x_2, y_2) \in \{(3, 2), (3, 4)\}$. Therefore $d(x_1, x_2) = 0 \leq d(y_1, y_2)$. Hence pair (A, B) has the weak (P)-property. To prove (3), let $x, y \in A$. We have only the following cases:

Case 1: $x = y$. Here $d(Tx, Ty) = 0$ and hence

$$\begin{aligned}
d(Tx, Ty) \leq \;& \varphi(d(x, y)) + \theta(d(y, Tx) - d(A, B), d(x, Ty) - d(A, B), \\
& d(x, Tx) - d(A, B), d(y, Ty) - d(A, B)).
\end{aligned}$$

Case 2: $x \neq y$. Here $(x = 0 \bigwedge y = 3) \bigvee (x = 3 \bigwedge y = 0)$. Without loss of generality, we assume $x = 1$ and $y = 3$. and hence

$$\begin{aligned} d(T0, T3) &= d(5,4) = \frac{1}{2} \\ &= \varphi(1) \\ &\leq \varphi(d(0,3)) \\ &\leq \varphi(d(x,y)) + \theta(d(y,Tx) - d(A,B), d(x,Ty) - d(A,B), \\ &\qquad d(x,Tx) - d(A,B), d(y,Ty) - d(A,B)). \end{aligned}$$

Thus T is a generalized almost (φ, θ)-contraction. By Theorem 3.2, we conclude that T has a unique best proximity point in A. Here $x^* = 3$ is the best proximity point of T. □

Remark 4.3. Theorem 2.6 of [10] is a special case of our result Theorem 3.2.

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