



The Lower Bound for the Modulus of the Derivatives and Jacobian of Harmonic Injective Mappings

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Dedicated to professor Matti Vuorinen on the occasion of his sixty-five years old birthday.

Abstract. We give the lower bound for the modulus of the radial derivatives and Jacobian of harmonic injective mappings from the unit ball onto convex domain in plane and space. As an application we show co-Lipschitz property of some classes of qch mappings. We also review related results in planar case using some novelty.

Throughout the paper we denote by Ω , G and D open subset of \mathbb{R}^n , $n \geq 1$.

Let $B^n(x, r) = \{z \in \mathbb{R}^n : |z - x| < r\}$, $S^{n-1}(x, r) = \partial B^n(x, r)$ (abbreviated $S(x, r)$) and let \mathbb{B}^n , $\mathbb{S} = \mathbb{S}^{n-1}$ stand for the unit ball and the unit sphere in \mathbb{R}^n , respectively. In particular, by \mathbb{D} we denote the unit disc \mathbb{B}^2 and $\mathbb{T} = \partial\mathbb{D}$ we denote the unit circle \mathbb{S}^1 in the complex plane.

For a domain D in \mathbb{R}^n with non-empty boundary, we define the distance function $d = d_D = \text{dist}(D)$ by $d(x) = d(x; \partial D) = \text{dist}(D)(x) = \inf\{|x - y| : y \in \partial D\}$; and if f maps D onto $D' \subset \mathbb{R}^n$, in some settings it is convenient to use short notation $d^* = d^*(x) = d_f(x)$ for $d(f(x); \partial D')$. It is clear that $d(x) = \text{dist}(x, D^c)$, where D^c is the complement of D in \mathbb{R}^n . Let G be an open set in \mathbb{R}^n . A mapping $f : G \rightarrow \mathbb{R}^m$ is differentiable at $x \in G$ if there is a linear mapping $f'(x) : \mathbb{R}^n \rightarrow \mathbb{R}^m$, called the derivative of f at x , such that

$$f(x+h) - f(x) = f'(x)h + |h|\varepsilon(x, h)$$

where $\varepsilon(x, h) \rightarrow 0$ as $h \rightarrow 0$. For a vector-valued function $f : G \rightarrow \mathbb{R}^m$, where $G \subset \mathbb{R}^n$, is a domain, we define

$$|f'(x)| = \max_{|h|=1} |f'(x)h| \quad \text{and} \quad l(f'(x)) = \min_{|h|=1} |f'(x)h|,$$

when f is differentiable at $x \in G$. Occasionally we use the notation $\Lambda_f(x)$ and $\lambda_f(x)$ instead of $|f'(x)|$ and $l(f'(x))$ (in particular in planar case) respectively.

For $x \in \mathbb{R}^n$, we use notation $r = |x|$. We say that Jacobian J of mapping on a domain Ω satisfies minimum principle if for every compact $F \subset \Omega$ we have $\inf_F J \geq \inf_{\partial F} J$. A C^1 (in particular diffeomorphism) mapping $f : \Omega \rightarrow \Omega^*$ is K -qc iff

$$|f'(x)|^n / K \leq |J(x, f)| \leq K \ell((f'(x))^n) \tag{1}$$

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holds for every $x \in \Omega$. For $\xi \in \mathbb{S}$, define

$$h_b(\xi) = h^*(\xi) = \lim_{r \rightarrow 1} h(r\xi)$$

when this limit exists.

The directional derivative of a scalar function $f(\mathbf{x}) = f(x_1, x_2, \dots, x_n)$ along a vector $\mathbf{v} = (v_1, \dots, v_n)$ is the function defined by the limit

$$\nabla_{\mathbf{v}} f(\mathbf{x}) = \lim_{h \rightarrow 0} \frac{f(\mathbf{x} + h\mathbf{v}) - f(\mathbf{x})}{h}.$$

If the function f is differentiable at x , then the directional derivative exists along any vector v , and one has

$$\nabla_{\mathbf{v}} f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{v},$$

where the ∇ on the right denotes the gradient and \cdot is the dot product. Intuitively, the directional derivative of f at a point x represents the rate of change of f with respect to time when it is moving at a speed and direction given by v . Instead of $\nabla_{\mathbf{v}} f$ we also write $D_{\mathbf{v}} f$. If $v = \frac{x}{|x|}$, $x \neq 0$, $g(t) = f(x + tv)$ and $D_{\mathbf{v}} f(x)$ exists, we define $\partial_r f(x) = D_{\mathbf{v}} f(x) = g'(0)$.

Let $\Omega \in \mathbb{R}^n$ and $\mathbb{R}^+ = [0, \infty)$ and $f, g : \Omega \rightarrow \mathbb{R}^+$. If there is a positive constant c such that $f(x) \leq c g(x)$, $x \in \Omega$, we write $f \leq g$ on Ω . If there is a positive constant c such that

$$\frac{1}{c} g(x) \leq f(x) \leq c g(x), \quad x \in \Omega,$$

we write $f \approx g$ (or $f \simeq g$) on Ω .

O.Martio [31] observed that, every quasiconformal harmonic mapping of the unit disk onto itself is co-Lipschitz. Then the subject was intensively studied by the participants of Belgrade Analysis Seminar, see for example [25, 34, 39, 42] and the literature cited there. In particular Kalaj and Mateljević, shortly KM-approach, study lower bound of Jacobian. The corresponding results for harmonic maps between surfaces were obtained previously by Jost and Jost-Karcher [17, 18]. We refer to this results shortly as JK- result (approach). Recently Iwaniec has communicated the proof of Rado-Kneser-Choquet theorem¹⁾ (shortly Theorem RKC), cf. [14], Iwaniec- Onninen cf. [15]. We refer to this communication shortly as IwOn-approach. It seems that there is some overlap between KM- results with [14, 15] and [17, 18] (we will shortly describe it in Section 1). Note only here that in planar case JK- result is reduced to Theorem RKC.

The author has begun to consider harmonic functions in the space roughly since 2006 trying to generalize theory in the plane, cf [1, 3, 27, 35, 36, 39].

He realized some differences between theory in the plane and space and some difficulties to develop the space theory. It was observed that gradient mappings of harmonic functions are good candidate for generalization of the planar theory to space outlining some ideas and asking several open problems on the Belgrade Analysis seminar. Having studied Iwaniec’s lecture [14] recently the author has found an additional motivation to investigate in this direction. For the present state see also the recent arXiv papers of Astala-Manojlović [6], Božin-Mateljević [8] and Mateljević [40].

Suppose that F is mapping from a domain $G \subset \mathbb{R}^n$ (in particular, from the unit ball $\mathbb{B} \subset \mathbb{R}^n$) onto a bounded convex domain $D = F(G)$. To every $a \in \partial D$ we associate a nonnegative function $u = u_a = \underline{F}^a$. Since D is convex, for $a \in \partial D$, there is a supporting hyper-plane (a subspace of dimension $n - 1$) Λ_a defined by $\Lambda_a = \{w \in \mathbb{R}^n : (w - a, n_a) = 0\}$, where $n = n_a \in T_a \mathbb{R}^n$ is a unit vector such that $(w - a, n_a) \geq 0$ for every $w \in \overline{D}$. Define $u(z) = \underline{F}^a(z) = (F(z) - a, n_a)$, cf. [21, 32–34]. Our approach here is also based on function \underline{F}^a .

¹⁾V. Manojlović informed me about Iwaniec’s lecture [14].

We provide explicit interior lower bounds on the Jacobian in terms of the regularity of the domains and the boundary map in Section 1. For the convenience of the reader in Section 4 we mainly collect some results which we used in Section 1 and give a few additional results.

In Section 5, we outline short review of results from [17] for Harmonic Maps Between surfaces concerning lower bounds on the Jacobian, and the existence of harmonic diffeomorphisms which solve a Dirichlet problem.

In Section 2, estimates for the modulus of the derivatives of harmonic univalent mappings in space are given.

In Section 3 we generalize and develop the arguments used in planar theory of harmonic mappings to gradient mappings of harmonic functions in domains of \mathbb{R}^3 . For example, we can consider the proof of Theorem 3.6 (which is not based on an approximation argument) as a suitable generalization of the proof of Theorem 1.5.

In Section 6 we discuss shortly some results from [33, 36, 50] related to the subject and ask a few question.

1. Estimates for the modulus of the derivatives of harmonic univalent planar mappings from below

1.1.

For univalent harmonic maps between surfaces, estimates of Jacobian from below in terms of the geometric data involved are given in Jost [17] (see Corollary 8.1, Theorem 8.1) and for univalent euclidean harmonic maps in [21, 24, 34, 39]. In this subsection, we consider convex codomains and give short review of a few result from [32, 33]. It seems that Theorem 1.5 is a new result. For a function h , we use notation $\partial h = \frac{1}{2}(h'_x - ih'_y)$ and $\bar{\partial} h = \frac{1}{2}(h'_x + ih'_y)$; we also use notations Dh and $\bar{D}h$ instead of ∂h and $\bar{\partial} h$ respectively when it seems convenient.

Throughout this paper, if h is a complex harmonic function on simple connected planar domain, we will write h in the form $h = f + \bar{g}$, where f and g are holomorphic. Note that every complex valued harmonic function h on simply connected domain D is of this form.

Recall by \mathbb{D} we denote the unit disc and $\mathbb{T} = \partial\mathbb{D}$ we denote the unit circle, and we use notation $z = re^{i\theta}$. For a function h we denote by h'_r, h'_x and h'_y (or sometimes by $\partial_r h, \partial_x h$ and $\partial_y h$) partial derivatives with respect to r, x and y respectively. Let $h = f + \bar{g}$ be harmonic, where f and g are analytic. Then $\partial h = f'$, $h'_r = f'_r + \bar{g}'_r, f'_r = f'(z)e^{i\theta}$ and $J_h = |f'|^2 - |g'|^2$. If h is univalent, then $|g'| < |f'|$ and therefore $|h'_r| \leq |f'_r| + |g'_r|$ and $|h'_r| < 2|f'_r|$.

Theorem 1.1 ([33]). *Suppose that*

(a) *h is a euclidean harmonic mapping from \mathbb{D} onto a bounded convex domain $D = h(\mathbb{D})$, which contains the disc $B(h(0); R_0)$. Then*

(i.1) *$d(h(z), \partial D) \geq (1 - |z|)R_0/2, z \in \mathbb{D}$.*

(i.2) *Suppose that $\omega = h^*(e^{i\theta})$ and $h^*_r = h'_r(e^{i\theta})$ exist at a point $e^{i\theta} \in \mathbb{T}$, and there exists the unit inner normal $n = n_\omega$ at $\omega = h^*(e^{i\theta})$ with respect to ∂D .*

*Then $(h^*_r, n) \geq c_0$, where $c_0 = \frac{R_0}{2}$.*

(i.3) *In addition to the hypothesis stated in the item i.2), suppose that h'_b exists at the point $e^{i\theta}$. Then $|J_h| = |(h^*_r, N)| = |(h^*_r, n)||N| \geq c_0|N|$, where $N = ih'_b$ and the Jacobian is computed at the point $e^{i\theta}$ with respect to the polar coordinates.*

(i.4) *If in addition to the hypothesis (a) suppose that h is an euclidean univalent harmonic mapping from an open set G which contains $\bar{\mathbb{D}}$. Then $|f'| \geq \frac{R}{4}$ on \mathbb{D} .*

A generalization of this result to several variables has been communicated at Analysis Belgrade Seminar, cf. [35].

Note that (i.4) is a corollary of (i.2).

Outline of proof of (i.1). To every $a \in \partial D$ we associate a nonnegative harmonic function $u = u_a$. Since D is convex, for $a \in \partial D$, there is a supporting line Λ_a defined by $(w - a, n_a) = 0$, where $n = n_a$ is a unimodular

complex number such that $(w - a, n_a) \geq 0$ for every $w \in \bar{D}$. Define $u(z) = (h(z) - a, n_a)$ and $d_a = d(h(0), \Lambda_a)$. Then $u(0) = (h(0) - a, n_a) = d(h(0), \Lambda_a)$ and therefore, by the mean value theorem,

$$\frac{1}{2\pi} \int_0^{2\pi} u(e^{it}) dt = u(0) = d_a = d(h(0), \Lambda_a).$$

Since $u = u_a$ is a nonnegative harmonic function, for $z = re^{i\varphi} \in \mathbb{D}$, we obtain

$$u(z) \geq \frac{1-r}{1+r} \frac{1}{2\pi} \int_0^{2\pi} u(e^{it}) dt.$$

Hence $u(re^{i\varphi}) \geq d_a(1-r)/2$, and therefore $|h(z) - a| \geq d_a(1-r)/2 \geq (1-r)R_0/2$. Thus $|h(z) - a| \geq (1-r)R_0/2$ for every $a \in \partial D$ and therefore we obtain (1): $d(h(z), \partial D) \geq (1-r)R_0/2$. \square

Note that if D is a convex domain, then in general for $b \in \partial D$ there is no inner normal. However, there is a supporting line Λ_b defined by $(w - b, n_b) = 0$, where $n = n_b$ is a unimodular complex number such that $(w - b, n_b) \geq 0$ for every $w \in \bar{D}$.

Note that proof of theorem can also be based on Harnack’s theorem (see also [48], Lemma 15.3.7) or Hopf Lemma.

We use the notation $\lambda_f = l_f(z) = |\partial f(z)| - |\bar{\partial} f(z)|$ and $\Lambda_f(z) = |\partial f(z)| + |\bar{\partial} f(z)|$, if $\partial f(z)$ and $\bar{\partial} f(z)$ exist.

Theorem 1.2 ([32]). (ii.1) Suppose that $h = f + \bar{g}$ is a Euclidean orientation preserving harmonic mapping from \mathbb{D} onto bounded convex domain $D = h(\mathbb{D})$, which contains a disc $B(h(0); R_0)$. Then $|f'| \geq R_0/4$ on \mathbb{D} .

(ii.2) Suppose, in addition, that h is qc. Then $l_h \geq (1-k)|f'| \geq (1-k)R_0/4$ on \mathbb{D}

(ii.3) In particular, h^{-1} is Lipschitz.

A proof of the theorem can be based on Theorem 1.1 and

(b): the approximation of a convex domain with smooth convex domains,

which is based on the hereditary property of convex functions: if an analytic function maps the unit disk univalently onto a convex domain, then it also maps each concentric subdisk onto a convex domain. Now we outline an approximation argument for convex domain G . Let ϕ be conformal mapping of \mathbb{D} onto G , $\phi'(0) > 0$, $G_n = \phi(r_n \mathbb{D})$, $r_n = \frac{n}{n+1}$, $D_n = h^{-1}(G_n)$; and φ_n conformal mapping of \mathbb{D} onto D_n , $\varphi_n(0) = 0$, $\varphi_n'(0) > 0$ and $h_n = h \circ \varphi_n$. Since $D_n \subset D_{n+1}$ and $\cup D_n = \mathbb{D}$, we can apply the Carathéodory theorem; φ_n tends to z , uniformly on compacts, whence $\varphi_n'(z) \rightarrow 1$ ($n \rightarrow \infty$). By hereditary property G_n is convex.

Since the boundary of D_n is an analytic Jordan curve, the mapping φ_n can be continued analytically across \mathbb{T} , which implies that h_n has a harmonic extension across \mathbb{T} . An application of Theorem 1.1 (i.4) to h_n gives the proof.

Example 1.3. (ii.4) $f(z) = (z - 1)^2$ is univalent on \mathbb{D} . Since $f'(z) = 2(z - 1)$ it follows that $f'(z)$ tends 0 if z tends 1. This example shows that we can not drop the hypothesis that $f(\mathbb{D})$ is a convex domain in Theorem 1.1 (i.4).

(ii.5) $g(z) = \sqrt{z + 1}$ is univalent on \mathbb{D} is not Lipschitz on \mathbb{D} .

(ii.6) $g^{-1}(w) = w^2 - 1$ and $(g^{-1})'(w) = 2w$ tends to 0 if $w \in g(\mathbb{D})$ tends to 0.

1.2.

Hall, see [9] p. 66-68, proved the following:

Lemma 1.4 (Hall lemma). (ii.7) For all harmonic univalent mappings f of the unit disk onto itself with $f(0) = 0$, $|a_1|^2 + |b_1|^2 \geq c_0 = \frac{27}{4\pi^2}$,

where $a_1 = Df(0)$, $b_1 = \bar{D}f(0)$ and $c_0 = \frac{27}{4\pi^2} = 0.6839 \dots$

(ii.8) If in addition f is orientation preserving, then $|a_1| \geq \sigma_0$, where $\sigma_0 = \frac{3\sqrt{3}}{2\sqrt{2}\pi}$.

Hence, one can derive:

(I0) There is a constant $c > 0$ such that if h is harmonic planar mapping and $B(hx, R) \subset h(B(x, r))$, then $r\Lambda_h(x) \geq cr$.

Now we give another proof and generalization of the part (ii.3) of Theorem 1.2, which is not based on the approximation of a convex domain with smooth convex domains.

Theorem 1.5. Suppose that (a.1): D and D_* are simply-connected hyperbolic domains in \mathbb{R}^2 with non-empty boundary and that f is a euclidean harmonic univalent mapping from D onto D_* . Then

(iii.1) $d\Lambda_f \geq d_*$ on D , where $w = f(z)$, $d = \text{dist}(z, \partial D)$ and $d_*(w) = \text{dist}(w, \partial D_*)$.

(iii.2) If in addition to hypothesis (a.1) we suppose that (a.2): D is a $C^{1,\alpha}$, $0 < \alpha < 1$, domain and D_* is convex bounded domain, then there is a constant $c > 0$ such that $\Lambda_f \geq c$ on D .

By Theorem 4.9(Kellogg) we can reduce the proof to the case $D = \mathbb{D}$.

Proof. Let $z \in D$ and ϕ_0 conformal of \mathbb{B} onto D such that $\phi_0(0) = z$ and $F = h \circ \phi_0$. Since $\Lambda_F = \Lambda_f|\phi_0'|$, by (I0) version of Hall lemma, $\Lambda_F \geq d_*$, $|\phi_0'| \asymp d$. Hence we find (iii.1): $d\Lambda_f \geq d_*$.

Now let $z = \phi(z')$ be a conformal of \mathbb{B} onto D and $d' = \text{dist}(z')$. Then $d'|\phi'| \asymp d$ and by Theorem 4.9(Kellogg) (iii.3): $d' \asymp d$.

Since D_* convex, $d_* \geq d'$ and therefore by (iii.3) we find $d_* \geq d$. Hence by (iii.1) $\Lambda_f \geq c$ on D . \square

The following result is an immediate corollary of Theorem 1.5.

Theorem 1.6. If in addition to hypothesis (a.1) and (a.2) of Theorem 1.5 f is qc, then f^{-1} is Lipschitz on D_* .

Corollary 1.7. In particular, if f is conformal, then f^{-1} is Lipschitz on D_* .

Proof. Using that f is qc, it follows that $\lambda_f \geq \Lambda_f \geq c$ and the rest of the proof is routine. \square

Let G be simply connected hyperbolic planar domain and $\rho = \rho_G^{hyp}$ hyperbolic density; we also use short notation $\rho_{hyp} = \rho_G$. Using the uniformization theorem, one can define hyperbolic density for a hyperbolic planar domain.

If G is a planar domain with non-empty boundary, f a C^1 complex valued mapping $d = d_G$ and $d_* = d_{f(G)}$, we define $H := |\partial f|_{hyp}^{qv} = \frac{d(z)}{d_*(f(z))}|f_z|$. In addition if G is hyperbolic planar domain, $\rho = \rho_G$ and $\rho_* = \rho_{f(G)}$, we define $H := |\partial f|_{hyp} = \frac{\rho_*(f(z))}{\rho(z)}|f_z|$.

Theorem 1.8. If G is simply connected, then

(iv.1) $d_G(w) \leq \rho_G^{-1} \leq 8d_G(w)$.

There is an absolute constant c such that under the hypothesis (a.1) of Theorem 1.5 if f in addition orientation preserving, we have

(iv.2) $H := |\partial f|_{hyp}^{qv} = \frac{d(z)}{d_*(f(z))}|f_z| \geq c$.

(iv.3) $H := |\partial f|_{hyp} \geq c/8$.

Proof. Let $\phi : \mathbb{D} \rightarrow G$ be conformal. Then $\rho(w)|\phi'(z)| = \rho(z)$ and $d_G(w) \leq d_{\mathbb{D}}(z)|\phi'(z)| \leq 4d_G(w)$. Hence $d_G(w) \leq \rho_G^{-1} \leq 8d_G(w)$.

Set $w = \phi(z)$, $\zeta = f(w)$ and $F = f \circ \phi$. Since $F_z = f_w\phi'(z)$, by an application of Hall lemma, (ii.8), to F , we find $|f_w(w)||\phi'(z)| \geq d_*\sigma_0$. If $w \in G$, we can choose a conformal mapping $\phi_0 : \mathbb{D} \rightarrow G$ such that $w = \phi_0(0)$. Then $|\phi_0'(0)| \leq 4d$ and therefore $4d|f_w(w)| \geq d_*\sigma_0$. Hence we get (iv.2) with $c = \sigma_0/4$. \square

The part (iv.3) has also been proved by Kalaj.

1.3. The minimum principle for the Jacobian

(I1) Let $h : \Omega \rightarrow \mathbb{C}$ be a harmonic map whose Jacobian determinant $J = |h_z|^2 - |h_{\bar{z}}|^2$ is positive everywhere in Ω . Then $-\ln J$ is subharmonic; More precisely, cf. [14, 15],

$$-\frac{1}{4} \Delta \ln J = -(\ln J)_{z\bar{z}} = \frac{|h_{z\bar{z}}\bar{h}_z - \overline{h_{z\bar{z}}h_z}|^2}{J^2}.$$

Note that in [30] it is proved previously that

(I2) $X = \log \frac{1}{J_h}$ is a subharmonic function.

We left to the reader to check the following fact (I3-I6):

If F is an analytic function, then $|F|_{z\bar{z}}^2 = |F'|^2$. Hence

(I3) $J(h)_z = f''\bar{f}' - g''\bar{g}'$, $J(h)_{z\bar{z}} = |f''|^2 - |g''|^2$.

(I4) In general, $J(h)$ nether subharmonic nor superharmonic.

If $\tau : D \rightarrow I$, $I = (a, b)$, $\chi : I \rightarrow R$, then

(I5) $(\chi \circ \tau)_{z\bar{z}} = (\chi'' \circ \tau)\tau_z\bar{\tau}_z + (\chi' \circ \tau)\tau_{z\bar{z}} = (\chi'' \circ \tau)|\tau_z|^2 + (\chi' \circ \tau)\tau_{z\bar{z}}$.

If we set $\chi(x) = x^{-1}$ and $\tau = J$, we find

(I6) $-(J^{-1})_{z\bar{z}}J^3 = |\tau_z|^2 + |B + C|^2$.

If we set $\chi = \log$, $\tau = J$, $B = f'g''$ and $C = g'f''$, then

(I7) $-(\ln J)_{z\bar{z}}J^2 = |B + C|^2$.

(I8) Suppose that F and H are analytic function in a domain G such that $|F|^2 = m_0 + |H|^2$ on G , where m_0 is a positive constant. Then $F_z\bar{F} = H_z\bar{H}$ and therefore

$H'/F' = \bar{F}/\bar{H}$. Hence $H'/F' = a_0$, where a_0 is a constant and therefore $H/F = a_0z + a_1$.

$|F|^2 = m_0 + |a_0z + a_1|^2|F|^2$ on G . Without loss of generality we can suppose that $0 \in G$ and that $F = b_0 + b_nz^n + o(z^n)$, $n \geq 1$.

This leads to a contradiction and so $F = b_0$ and therefore $H = c_0$ on G , where b_0 and c_0 are constants.

Proposition 1.9. (a.2) If h is harmonic on U and $J(h)$ attains minimum (different from 0) at interior point a , then $J(h)$ is constant function, and h is affine.

(b.2) Functions $\log \frac{1}{J_h}$ and $\frac{1}{J_h}$ are subharmonic function.

(c.2) In particular, $\log J_h$ is a superharmonic function.

Corollary 1.10 (Minimum Principle). Let $h : \Omega \rightarrow \mathbb{C}$ be a harmonic map whose Jacobian determinant J is positive everywhere in Ω . Then $\inf_F J \geq \inf_{\partial F} J$ for every compact $F \subset \Omega$.

Example $h = f + \bar{g}$, where $f = 4z$, $g = z^2/2$, shows that that analog statement is not valid for maximum; $J = 16 - (x^2 + y^2)$ attains maximum 16 at $(0, 0)$.

In general, $\min J$ is attained at the boundary.

Proof of (a.2). Suppose that h is orientation preserving and $J(h) = |f'|^2 - |g'|^2$ attains min (different from 0) at interior point a ; then $|f'|^2 - |g'|^2 \geq J(h, a) = m$, $|f'|^2 \geq m + |g'|^2$ and therefore $1 \geq s(z)$, where

$$s = \frac{m}{|f'|^2} + \frac{|g'|^2}{|f'|^2}.$$

Since s is subharmonic and $s(a) = 1$, s is a constant, ie. $s = 1$. Hence $|f'|^2 = m + |g'|^2$, ie. $J = m$. By (I8), f' and g' are constant functions and therefore h is affine.

Proof of (b.2). Hence, since \exp is a convex increasing function, it follows that $\exp \circ X = \frac{1}{J_h}$ is also a subharmonic function.

Although $\chi(x) = e^{-x}$ is convex the conclusion that $\chi \circ X = J(h)$ is a subharmonic function is not true in general. Note that here χ is a decreasing function. In general, the minimum modulus principle for complex-valued harmonic functions is not valid; see the following examples:

Example 1.11. 1. If $f(z) = x + i$, then $|f(z)|^2 = x^2 + 1$ and $|f|$ attains minimum which is 1 for every points on y axis
 2. If $f_c(z) = x + i(x^2 - y^2 + c)$, then $J_f = -2y$. Let $d : \mathbb{C} \rightarrow \mathbb{R}$ is given by $d(z) = |z|$, $z \in \mathbb{C}$, $g = f_{-1}$, $C(x) = x + i(x^2 - 1)$ and $D = \{(x, y) : y < x^2 - 1\}$. Since $g(\mathbb{C}) = D$ and $0 \notin D$, then d attains minimum on $\text{tr}(C)$ at some point w_0 and there is a real point x_0 such that $g(x_0) = w_0$, g maps \mathbb{C} onto D and $|g|$ attains minimum at x_0 .

Let $c < 0$ and $D_c = \{(x, y) : y < x^2 - |c|\}$. Then $f_c(\mathbb{C}) = D_c$ and $0 \notin D_c$. At first sight someone can guess that $|f_c(0)| = |c|$ is the minimum value for $|f_c|$ if $c < 0$? We leave to the interested reader to show that $|c|$ is not the minimum value.

1.4. Outline of proof of Theorem RCK given in [14, 15]

For $f : \mathbb{T} \rightarrow \mathbb{C}$, we define \underline{f} on $[0, 2\pi]$ by $\underline{f}(t) = f(e^{it})$. Let γ be a closed Jordan curve, $G = \text{Int}(\gamma)$, $f_0 : S^1 \xrightarrow{\text{onto}} \gamma$ a monotone map and $F = P[f_0]$.

Theorem 1.12 (T. Radó H. Kneser G. Choquet, Theorem RKC). If G is convex, then F is a homeomorphism of \mathbb{D} onto G .

Iwaniec-Onninen [14, 15] presented a new analytic proof of RKC-Theorem. The approach is based on the following steps.

c1) Prove the theorem if f_0 is diffeomorphism and G is a smooth strictly convex domain, using the minimum principle for the Jacobian determinant and explicit interior lower bounds on the Jacobian in terms of the regularity of the domains and the boundary map.

c2) Let G be still a smooth strictly convex domain, but $f : S^1 \xrightarrow{\text{onto}} \gamma$ an arbitrary monotone map. This map can easily be shown to be a uniform limit of diffeomorphisms. Now the Poisson extensions F_j are harmonic diffeomorphisms in \mathbb{D} , converging uniformly on $\overline{\mathbb{D}}$ to F and $J_F > 0$, cf also [24, 35].

c3) the approximation of a convex domain with smooth convex domains.

c4) there is a conformal map ϕ of \mathbb{D} onto G ; by variation of boundary values deform this conformal map into a harmonic diffeomorphism.

It is convenient to give kinematic description of $f : S^1 \xrightarrow{\text{onto}} \gamma$, to view it as motion of an object along γ in which S^1 is labeled as a clock. As time runs from 0 to 2π the motion $t \rightarrow \underline{f}(t)$ begins at the point $\underline{f}(0) = 0$ and terminates at the same point $\underline{f}(2\pi) = 0$. The velocity vector $v(t) = \underline{f}'(t)$ is tangent to γ at $s = s(t) = \int_0^t |v(\tau)| d\tau$. We call $|v(t)|$ the speed. Let $z = z(s)$ be the length parametrization of γ . Since $|z'(s)| = 1$, we have $z(s) = e^{i\varphi(s)}$, where $\varphi(s)$ referred to as the tangential angle, is uniquely determined by the arc parameter s because G is smooth and strictly convex. The derivative is exactly the curvature of γ ; that is $\kappa(s) = \varphi'(s)$. The speed $|v(t)|$, being positive, uniquely represents unique diffeomorphism $f : S^1 \xrightarrow{\text{onto}} \gamma$. An explicit formula for f involves the curvature of γ .

Theorem 1.13 ([14, 15]). Let $f : S^1 \xrightarrow{\text{onto}} \gamma$ be a C^∞ -diffeomorphism and $F : \mathbb{D} \rightarrow \mathbb{C}$ its continuous harmonic extension. Then $F : \mathbb{D} \xrightarrow{\text{onto}} G$ is a C^1 -smooth diffeomorphism whose Jacobian determinant satisfies:

$$J_F \geq \frac{km^3}{2\pi KM} \text{ everywhere in } \mathbb{D} \tag{2}$$

provided $0 < k \leq \min \kappa(s) \leq \max \kappa(s) \leq K$ and $0 < m \leq \min v(t) \leq \max v(t)(s) \leq M$, where $v(t) = \underline{f}'(t)$.

In [14, 15] the strategy is used to prove first The Lower Bound of the Jacobian along S^1 . Then the proof is reduced to showing inequality (2) at the boundary of the disk, as one may have expected from the Minimum Principle.

1.5.

We also can use Theorem 1.14 below, which yields appriori estimate, instead of Theorem 1.13 in the procedure of proof of Theorem RKC.

Let $A(\gamma)$ be the family of C^1 -diffeomorphism $f : \mathbb{S}^1 \xrightarrow{\text{onto}} \gamma$. Set $v(t) = v_f(t) = |f'(t)|$.

Let d be diameter of G , $b \in G$, $B = B(b; d/2)$, γ_1 the part of γ out of B and $I = \{t : f(t) \in \text{tr}(\gamma_1)\}$. Then $d/2 \leq |\gamma_1| \leq M|I|$.

Theorem 1.14. *Let $f \in A(\gamma)$ and $0 < m \leq \min v(t) \leq \max v(t)(s) \leq M$. Then*

$$J_F \geq \frac{dm}{8\pi M} \text{ on } \mathbb{D}. \tag{3}$$

Using approach outlined in [33, 34], we can prove

Lemma 1.15. *The inequality (3) holds everywhere in \mathbb{S}^1 .*

Naturally, the interior estimate at (3) would follow from the already established estimate at the boundary (via the minimum principle) if we knew that the Jacobian of F was positive in \mathbb{D} .

But, by Theorem RKC the Jacobian of F is positive in \mathbb{D} . Theorem 1.16 below yields better estimate. Using Theorem 1.1, the part (i.3), and the minimum principle for Jacobian one can derive:

Theorem 1.16. d1) *Let Ω be a convex Jordan domain, $f : \mathbb{T} \rightarrow \partial\Omega$ absolutely continuous homeomorphism which preserves orientation, and let $w = P[f]$ be a harmonic function between the unit disk and Ω , such that $w(0) = 0$,
d2) $|f'(t)| \geq m$, for almost every $0 \leq t \leq 2\pi$,
d3) $\underline{f'}$ is Dini's continuous.*

Then the following results hold (iv.1) : $J_w(z) > m \text{ dist}(0, \partial\Omega)/2$, for every $z \in \mathbb{D}$, and (iv.2) w is bi-Lipschitz.

Let X be a compact subset of a metric space with metric d_1 (such as \mathbb{R}^n) and let $f : X \rightarrow Y$ be a function from X into another metric space Y with metric d_2 . The modulus of continuity of f is $\omega_f(t) = \sup_{d_1(x,y) \leq t} d_2(f(x), f(y))$, $t > 0$. The function f is called Dini-continuous if

$$\int_0^1 \frac{\omega_f(t)}{t} dt < \infty.$$

Dini continuity is a refinement of continuity. Every Dini continuous function is continuous. Every Lipschitz continuous function is Dini continuous. Note that under the above hypothesis $\underline{f'}$ has continuous extension to $[0, 2\pi]$ and partial derivatives of w have continuous extension to $\overline{\mathbb{D}}$ and one can show that w is bi-Lipschitz. We can use an approximation argument to prove (iv.1) for $C^{1,\alpha}$ domains Ω without the hypothesis d3). Moreover, an application of H^p theory shows that the following result, due to Kalaj, holds in general:

Theorem 1.17 (Theorem 2.8, Corollary 2.9 [23]). *Under hypothesis d1) and d2) of Theorem 1.16, J_w^* exists a.e. on \mathbb{T} and (iv.1) holds.*

Under hypothesis d1) of Theorem 1.16, Theorem RKC states that (v.1) $J_w > 0$ on \mathbb{D} .

Question 1. Can we modify approach in [14, 15] to give analytic proof of Theorem 1.17 (of course without appeal to Theorem RKC (moreprecisely to (v.1)))?

2. Estimates for the modulus of the derivatives of harmonic univalent mappings in space

Definition 2.1. Let G be a subset of the Euclidean space \mathbb{R}^n and let

$$\varphi: G \rightarrow \mathbb{R} \cup \{+\infty\}$$

be an lower semi-continuous function. Then, φ is called q -superharmonic, $0 < q \leq 1$, if for any closed ball $\overline{B(x, r)}$ of center x and radius r contained in G and every real-valued continuous function h on $\overline{B(x, r)}$ that is harmonic in $B(x, r)$ and satisfies $\varphi(y) \geq qh(y)$ for all y on the boundary $\partial B(x, r)$ of $B(x, r)$ we have $\varphi(y) \geq h(y)$ for all $y \in B(x, r)$. If $q = 1$ we say superharmonic instead of 1- superharmonic.

Definition 2.2. Let G be a domain in \mathbb{R}^n . Suppose that $f : G \rightarrow \mathbb{R}^n$ is a C^1 function and there is a constant $c > 0$ such that for every $x \in G$

(a) if $B(x, r) \subset G$ and $B(fx, R) \subset f(B(x, r))$, then $r\Lambda_f(x) \geq cR$.

(a') if $B(fx, R) \subset f(\overline{B(x, r)})$, then $r\Lambda_f(x) \geq cR$.

We say that f has H -property (respectively weak H -property) if (a) (respectively (a')) holds.

By Hall lemma planar euclidean harmonic mappings have H -property.

We say that $\underline{F} \subset HQC_K(G, G')$ has H -property if F is closed with respect to uniform convergence, and for $f \in \underline{F}$, $J(f)$ has no zeros in G .

If $\underline{F} \subset HQC_K(G, G')$ has H -property, then $f \in \underline{F}$ has weak H -property.

By Lemma 3.8, there is a constant $c > 0$ such that $d(x)\Lambda_f(x) \geq cd(fx)$, $x \in G$.

Lemma 2.3. Let F_K be a family of harmonic K - q c mapping $f : \mathbb{B} \rightarrow \mathbb{R}^n$ such that for $f \in F_K$, $J(f)$ has no zeros, F_K is closed with respect to uniform convergence, $f(\mathbb{B}) \supset \mathbb{B}$ and $f(0) = 0$. Then there is a constant $c > 0$ such that if $f \in F_K$ is harmonic K - q c mapping $f(\mathbb{B}) \supset \mathbb{B}$, $f(0) = 0$, then $\Lambda_f(0) \geq c$.

Contrary there is a sequence $f_n \in F_K$ such that $\Lambda_{f_n}(0) \rightarrow 0$. Sequence f_n forms a normal family and there is a subsequence of f_n which converges uniformly to a limit $f_0 \in F_K$; this is a contradiction.

Suppose that F is mapping from the unit ball $\mathbb{B} \subset \mathbb{R}^n$ into \mathbb{R}^m and suppose that $\omega = F^*(x)$ and $(\partial_r F)^*(x)$ exist at a point $x \in \mathbb{S}$. Then

(A1) $F'_r(x)$ exists and $(\partial_r F)^*(x) = F'_r(x)$.

Proof of (A1). By Lagrange theorem, there is $t_k \in [r, 1)$ such that $F_k(rx) - F_k(x) = -(F_k)'_r(t_k x)(1 - r)$. Hence, since $(\partial_r h)^*(x)$ exist at a point $x \in \mathbb{S}$, if r tends 1, then $(F_k)'_r(t_k x)$ tends $(F_k)'_r(x)$.

The next theorem concerns harmonic maps onto a convex domain. See Appendix 6 for short discussion about it where the part (i.1) is stated as Theorem 6.4.

Theorem 2.4. (a1) Suppose that h is a euclidean harmonic mapping from the unit ball $\mathbb{B} \subset \mathbb{R}^n$ onto a bounded convex domain $D = h(\mathbb{B})$, which contains the ball $B(h(0); R_0)$. Then

(i.1) $d(h(z), \partial D) \geq (1 - |z|)\bar{c}_n R_0$, $z \in \mathbb{B}$, where $\bar{c}_n = \frac{1}{2^{n-1}}$.

(i.2) For every $x \in \mathbb{S}$ and for $0 < r < 1$, there is $t \in [r, 1)$ such that $|h'_r(tx)| \geq c_0$.

(i.3) If h is K - q c and (a2) h has H -property or (a3) for some k , $|h'_{x_k}|^n$ is q -super harmonic, then h is co-Lipschitz on \mathbb{B} .

(a4) Suppose, in addition, to (a1) that h is K - q c and D is C^2 domain. Then

(i.4) $h'_r(x)$ exists and $(\partial_r h)^*(x) = h'_r(x) \geq c_0$ for almost everywhere $x \in \mathbb{S}$.

Proof of (i.1). To every $a \in \partial D$ we associate a nonnegative harmonic function $u = u_a$. Since D is convex, for $a \in \partial D$, there is a supporting hyper-plane Λ_a defined by $(w - a, n_a) = 0$ where $n = n_a \in T_a \mathbb{R}^n$ is a unit vector such that $(w - a, n_a) \geq 0$ for every $w \in \overline{D}$. Define $u(z) = (h(z) - a, n_a)$ and $d_a = d(h(0), \Lambda_a)$. Then $u(0) = (h(0) - a, n_a) = d(h(0), \Lambda_a)$. Let $a_0 \in \Lambda_a$ be the point such that $d_a = |h(0) - a_0|$. Then from geometric interpretation it is clear that $d_a \geq R_0$.

By Harnack's inequality, $\bar{c}_n(1 - r)u(0) \leq u(x)$, $x \in \mathbb{B}$ and $r = |x|$, where $\bar{c}_n = \frac{1}{2^{n-1}}$. In particular, $\bar{c}_n d(x)R_0 \leq u(x) \leq |h(x) - a|$ for every $a \in \partial D$. Hence, for a fixed x , $d_h(x) = \inf_{a \in \partial D} |h(x) - a| \geq \bar{c}_n R_0 d(x)$ and therefore we obtain (i.1). □

Proof of (i.2). Set $u(z) = (h(z) - h(x), n_{h(x)})$ and $c_0 = \bar{c}_n R_0$. Since $u(x) = 0$, then (i.5) for $x \in \mathbb{S}$, $u(rx) - u(x) \geq c_0(1 - r)$.

By Lagrange theorem, there is $t \in [r, 1)$ such that $u(rx) - u(x) = u'_r(tx)(1 - r)$. □

Proof of (i.3). Suppose for example (b1). By Theorems 2.4 and 2.6, $\underline{J}(x) \geq \frac{d_x}{d} \geq c$ for every $x \in \mathbb{B}$. By (a3),

$$|h'_{x_k}(x)|^n \geq q \frac{1}{|\underline{B}_x|} \int_{\underline{B}_x} |h'_{x_k}(z)|^n dz, x \in \mathbb{B}. \text{ Hence } |h'_{x_k}(x)| \geq \underline{J}(x) \geq c$$

and $\lambda(h'(x)) \geq c$ and therefore h is co-Lipschitz on \mathbb{B} . □

Proof of (i.4). First, suppose that $h'_r(x)$ exists for some $x \in \mathbb{S}$. By (i.5), $(\frac{h(rx)-h^*(x)}{1-r}, n) \geq c_0$, where $n = n_a$ and $a = h^*(x)$. Hence, since $h'_r(x)$ exists, it follows

$$(i.6) \ (h'_r(x), n) \geq \bar{c}_n R_0.$$

By a result of D. Kalaj [25], partial derivatives of h are bounded and therefore $h'_r(x)$ exists for almost everywhere $x \in S$ and therefore by (i.2) and (i.6), $(\partial_r h)^*(x) = h'_r(x) \geq c_0$ for almost everywhere $x \in \mathbb{S}$. □

In [27] it is proved the following theorem: a K quasiconformal harmonic mapping of the unit ball \mathbb{B}^n ($n > 2$) onto itself is Euclidean bi-lipschitz, providing that $u(0) = 0$ and that $K < 2^{n-1}$, where n is the dimension of the space. It is an extension of a similar result for hyperbolic harmonic mappings with respect to hyperbolic metric (see Tam and Wan, (1998)). The proof makes use of Möbius transformations in the space, and of a recent result which states that, harmonic quasiconformal self-mappings of the unit ball are Lipschitz continuous; this result first has been proved by the first author and then generalized also by the second author.

Introduce the quantity

$$a_f(x) = a_{f,G}(x) := \exp\left(\frac{1}{n|B_x|} \int_{B_x} \log J_f(z) dz\right), \ x \in G,$$

associated with a quasiconformal mapping $f : G \rightarrow f(G) \subset \mathbb{R}^n$; here J_f is the Jacobian of f ; while $\mathbf{B}_x = \mathbf{B}_{x,G}$ stands for the ball $B(x; d(x, \partial G))$; and $|B_x|$ for its volume. Astala and Gehring [5] observed that for certain distortion property of quasiconformal mappings the function a_f , defined the above, plays analogous role as $|f'|$ when $n = 2$ and f is conformal; and they establish quasiconformal version of the well-know result due to Koebe, cited here as Lemma 2.5:

Lemma 2.5 ([5]). *Suppose that G and G' are domains in \mathbb{R}^n : If $f : G \rightarrow G'$ is K -quasiconformal, then*

$$\frac{1}{c} \frac{d(f(x), \partial G')}{d(x, \partial G)} \leq a_{f,G}(x) \leq c \frac{d(f(x), \partial G')}{d(x, \partial G)}, \ x \in G,$$

where c is a constant which depends only on K and n .

Our next result concerns the quantity

$$\underline{E}_{f,G}(x) := \frac{1}{|B_x|} \int_{B_x} J_f(z) dV(z), x \in G,$$

associated with a quasiconformal mapping $f : G \rightarrow f(G) \subset \mathbb{R}^n$; here $dV(z) = dz$ is the Euclidean volume element $dz_1 dz_2 \cdots dz_n$ and $z = (z_1 \cdots z_n)$ and J_f is the Jacobian of f ; while $\underline{B}_x = \underline{B}_{x,G}$ stands for the ball $B(x, d(x, \partial G)/2)$ and $|B_x|$ for its volume.

Define

$$\underline{J}_f = \underline{J}_{-f,G} = \sqrt[n]{\underline{E}_{f,G}}.$$

Theorem 2.6 ([39]). Suppose that G and G' are domains in \mathbb{R}^n : If $f : G \rightarrow G'$ is K -quasiconformal, then

$$\frac{1}{c} \frac{d(f(x), \partial G')}{d(x, \partial G)} \leq J_{-f,G}(x) \leq c \frac{d(f(x), \partial G')}{d(x, \partial G)}, \quad x \in G,$$

where c is a constant which depends only on K and n .

If G and G' are domains in \mathbb{R}^n , by $QCH(G, G')$ (respectively $QCH_K(G, G')$) we denote the set of Euclidean harmonic quasiconformal mappings (respectively K -qc) of G onto G' .

If D is a domain in \mathbb{R}^n , by $QCH(D)$ we denote the set of Euclidean harmonic quasiconformal mappings of D onto itself.

Definition 2.7. Let $Q_H(G)$ be a family of harmonic mappings h from G into \mathbb{R}^n such that

(b1) $J(h)$ has no zeros in G , and

(b2) which is closed with respect to uniform convergence on compact subsets and

(b3) for every sequence x_n which tends to a $x_0 \in \partial G$, $h_n \in Q_H$, where $h_n(x) = \frac{1}{d_n} h((d_n x + x_n))$ and $d_n = d(x_n) = \text{dist}(x_n)$. If G is the unit ball \mathbb{B} we write Q_H instead of $Q_H(G)$.

Using a criteria for normality of a qc family, one can establish criteria when a subfamily Q of $QCH(G)$, for which $K_O(f) < 3^{n-1}$ for every $f \in Q$, is $Q_H(G)$ -family.

Definition 2.8. Let $f : G \rightarrow G'$ be a C^1 function. We say that f has Jacobian non zero normal family-property if

(b1) $J(f)$ has no zeros in G , and

(b2) for every sequence x_n which tends to a $x_0 \in \partial G$, (f_n) forms a normal family, where $f_n(x) = \frac{1}{d_n} f((d_n x + x_n))$, and $d_n = d(x_n) = \text{dist}(x_n)$ and $d_n^* = d(f(x_n))$ and

(b3) for every limit f_0 of (f_n) in sense of the uniform convergence, f_0 is a C^1 function and $J(f_0)$ has no zeros in G . In this setting, we say that the sequence (f_n) is associated sequence to the sequence (x_n) and that f_0 is the associated limit.

Theorem 2.9. (a) Suppose that h is a euclidean harmonic K -qc mapping from the unit ball $\mathbb{B} \subset \mathbb{R}^n$ onto a bounded convex domain $D = h(\mathbb{B})$, which contains the ball $B(h(0); R_0)$.

(i.7) If h has Jacobian non zero normal family-property (or $h \in Q_H$), then h is co-Lipschitz on \mathbb{B} .

(i.8) If $\log J_h$ is q -superharmonic, then $J_h \geq c$.

Proof. (i.7): Suppose that there is sequence x_n such that $|h'_r(x_n)| \rightarrow 0$ and set $d_n = d(x_n)$. Since by Theorems 2.4 and 2.6, $J(x_n) \geq \frac{d_n}{d} \geq c$, there is a point $y_n \in \underline{B}(x_n)$ such that $|h'_r(y_n)| \geq c$. Apply a normal family argument on $h_n(x) = \frac{1}{d_n} h((d_n x + x_n))$.

(i.8) follows from Theorem 2.4,(i.8), and Lemma 2.5. \square

Using the Thom splitting lemma, we can prove

Proposition 2.10. Suppose that f is real-valued function defined at a neighborhood $U(x_0)$ of a point $x_0 \in \mathbb{R}^n$, f has partial derivatives up to the order 3 at x_0 and that $f : U(x_0) \rightarrow \mathbb{R}^n$ is injective, where $U(x_0)$ is a neighborhood of x_0 in \mathbb{R}^n . If $\partial_k f(x_0) = 0$, then $\partial_{ij}^2 f(x_0) = 0$, $i, j = 1, 2, \dots, n$, that is $\text{Hess}(f)(x_0) = [0]$.

Frequently we use notation $X = (x, y, z) \in \mathbb{R}^3$. If we work in \mathbb{R}^n it is convenient to switch the notation to $x = (x^1, x^2, \dots, x^n) \in \mathbb{R}^n$.

Example 2.11. Let $a \neq 0$. A radial mapping f_a in n -space is given by: $f(X) = f_a(X) = |X|^{a-1} X$, where $X \in \mathbb{R}^n$. Prove

(i.2) $K_I(f) = |a|$, $K_O(f) = |a|^{n-1}$ if $|a| \geq 1$; in particular $K(f_3) = K_O(f_3) = 3^{n-1}$;

$K_I(f) = |a|^{1-n}$, $K_O(f) = |a|^{-1}$ if $|a| \leq 1$.

In particular, for $n = 3$, $X = (x, y, z) \in \mathbb{R}^3$,

$K_I(f) = |a|$, $K_O(f) = |a|^2$ if $|a| \geq 1$;

Now we consider 3-space.

(i.3) For $a = 3$ set $g = f_3$; then $\partial_k g(0) = 0$, $\partial_{ij}^2 g(0) = 0$ and $g^1(X) = x^3 + xy^2 + xz^2$.

(i.4) For $|a| \geq 1$, f_a is co-Lipschitz on $\mathbb{B} \setminus F$, where F is a compact subset of \mathbb{B} .

Set $x' = f(x)$, $x = (x^1, x^2, x^3)$; then $|x'| = |x|^a$ and by the cosine formula, we find $|x|^{2a} + |y|^{2a} - |x' - y'|^2 = |x|^{a-1}|y|^{a-1}(|x|^2 + |y|^2 - |x - y|^2)$.

$$|x' - y'|^2 = |x|^{a-1}|y|^{a-1}|x - y|^2 + R(x, y), \tag{4}$$

where $R(x, y) = |x|^{2a} + |y|^{2a} - |x|^{a+1}|y|^{a-1} - |x|^{a-1}|y|^{a+1}$.

Without loss of generality we can suppose that $|y| = \lambda|x|$, $0 \leq \lambda \leq 1$. Then $R(x, y) = (1 - \lambda^{a-1})(1 - \lambda^{a+1})|x|^{2a} \geq 0$. Therefore $|x' - y'|^2 \geq |x|^{a-1}|y|^{a-1}|x - y|^2$ and thus

$$|x' - y'| \geq \lambda^{(a-1)/2}|x|^{a-1}|x - y|. \tag{5}$$

We need the following Proposition concerning the distortion property of qr mappings.

Proposition 2.12 (Corollary 11.3 [?]). Let $f : \mathbb{B}^n \rightarrow \mathbb{B}^n$ be K-qr, $f(0) = 0$ and $\alpha = K_I(f)^{1/(1-n)}$. Then

(i.5) $|f(x)| \leq \varphi_{K,n}(|x|) \leq \lambda_n^{1-\alpha}|x|^\alpha$.

(i.6) If $g : \mathbb{B}^n \rightarrow \mathbb{B}^n$ is K-qc, $g(0) = 0$ and $1/\alpha = K_I(g^{-1})^{1/(n-1)}$, then $m|x|^{1/\alpha} \leq |g(x)|$.

Set $f = g^{-1}$, $y = g(x)$, $x = f(y)$ and $\alpha = K_I(g^{-1})^{1/(1-n)}$. Then by (i.5), $|x| \leq \lambda_n^{1-\alpha}|y|^\alpha$ and hence (i.6).

Suppose that g is analytic (more generally $C^{(3)}$ at 0), $g(0) = 0$, $\partial_k g(0) = 0$ and $\partial_{ij}^2 g(0) = 0$. Then $|\partial_k g(x)| \leq M|x|^2$ and therefore $|g(x) - g(y)| \leq M|x|^2|x - y|$ if $|y| \leq |x|$. In particular, $|g(x)| \leq M|x|^3$.

Note that, if in addition, f is $C^{(4)}$ at 0 and $\partial_{ijk}^3 g(0) = 0$, then $|g(x) - g(y)| \leq M|x|^3|x - y|$ if $|y| \leq |x|$ and $f_a + g$ is K-qc for $a < 4$. In particular, $|g(x)| \leq M|x|^4$.

Proposition 2.13. Suppose that f has partial derivatives up to the order 3 at a point $x_0 \in \mathbb{R}^n$ and that $f : U(x_0) \rightarrow \mathbb{R}^n$ is K-qc, where $U(x_0)$ is a neighborhood of x_0 in \mathbb{R}^n . If $\partial_k f(x_0) = 0$ and $\partial_{ij}^2 f(x_0) = 0$, then $K_O(f) \geq 3^{n-1}$ on $U(x_0)$.

Example 2.11 shows that the result is optimal.

Proof. By the Taylor formula, there is M such $|f(x)| \leq M|x|^3$. Set $g = f^{-1}$ and $\alpha = K_I(g)^{1/(1-n)}$ so that $1/\alpha = K_I(f^{-1})^{1/(n-1)}$.

Then by Proposition 2.12, $m|x|^{1/\alpha} \leq |f(x)| \leq M|x|^3$. Hence $1/\alpha \geq 3$, ie. $K_I(f^{-1}) \geq 3^{n-1}$ and therefore $K_O(f) \geq 3^{n-1}$, because $K_O(f) = K_I(f^{-1})$. \square

Proposition 2.14. Suppose that f has continuous partial derivatives up to the order 3 at the origin 0 and that $f : U(0) \rightarrow \mathbb{R}^n$ is K-qc, where $U(0)$ is a neighborhood of 0 in \mathbb{R}^n . If $K_O(f) < 3^{n-1}$, then $J(f, 0) \neq 0$.

In particular, if g is analytic (more generally $C^{(3)}$ ($U(0)$) or g only has partial derivatives up to the order 3), and if g is K-qc with $K_O(g) < 3^{n-1}$, then $J(g, 0) \neq 0$.

Proof. Contrary suppose that $J(f, 0) = 0$. Since f is K-qc, $\partial_k f(x_0) = 0$, hence by Proposition 2.10 we find $\partial_{ij}^2 f(x_0) = 0$. Now, by Proposition 2.13, $K_O(f) \geq 3^{n-1}$ and this yields a contradiction. \square

Theorem 2.15. If in addition to hypothesis (a) of Theorem 2.9 we suppose that h is qc with $K_O(h) < 3^{n-1}$, then h is co-Lipschitz on \mathbb{B} .

Proof. The proof follows from Theorem 2.9 and Proposition 2.14. We leave the details to the interested reader. \square

3. The Lower Bound of the Jacobian in \mathbb{R}^3

It seems a natural project to generalize and develop the arguments used in planar theory of harmonic mappings to harmonic functions in domains of $\mathbb{R}^n, n \geq 3$. In Section 1 we used the fact that every complex harmonic function h on simple connected planar domain, can be written in the form $h = f + \bar{g}$, where f and g are holomorphic that $|f'|$ satisfies minimum principle. There is no appropriate analogy of this result in space. The next example shows that the minimum principle does not hold for modulus of vector valued harmonic mapping.

Example 3.1. Define $h(M) = (x, y, x^2 + y^2 - 1 - 2z^2)$, where $M = (x, y, z)$, $f_0(x, y) = (x, y, x^2 + y^2 - 1)$ and $G = \{(x, y, z) : z < x^2 + y^2 - 1\}$. Then $J(h) = -4z$, the restriction of h on xy -plane is f_0 , $\Gamma_{f_0} = \partial G$ and $h(\mathbb{R}^3) = G$. Let $d(M) = |OM|$. Then d attains minimum on Γ_{f_0} at some point M_1 and there is a point $M_0 = (x_0, y_0, 0)$ such that $f_0(M_0) = M_1 = h(M_0)$, h maps \mathbb{R}^3 onto G and therefore $|h|$ attains minimum at M_0 .

If h is a harmonic mapping from a domain in \mathbb{R}^n to \mathbb{R}^n , then $|h'_{x_k}|$ is subharmonic, but it does not satisfy minimum principle in general (adapt the above example to the dimension $n \geq 3$).

In fact, Lewy's theorem is false in dimensions higher than two (see [9] p. 25-27 for Wood's counterexample).

Consider the polynomial map from \mathbb{R}^3 to \mathbb{R}^3 defined by $h(x, y, z) = (u, v, w)$, where

$$u = x^3 - 3xz^2 + yz, \quad v = y - 3xz, \quad w = z.$$

Since $z = w, u = x^3 + zv = x^3 + vw$, we find $x = \sqrt[3]{u - vw}$ and therefore $y = v + 3xz = v + 3xw = v + 3w\sqrt[3]{u - vw}$, we conclude that h is injective. A calculation shows that h has the Jacobian

$$J_h(x, y, z) = 3x^2,$$

which vanishes on the plane $x = 0$. Jacobian of C^1 orientation preserving mapping f is nonnegative. Iwaniec [14] suggest a project (for students): Generalize and develop the arguments used in planar theory of harmonic mappings to gradient mappings of harmonic functions in domains of \mathbb{R}^3 . Recall that, in planar theory of harmonic mappings we used

(I.0) version of Hall and

(II.0) If analytic function does not vanish then its modulus satisfies minimum principle.

For harmonic gradient mapping in 3-space Proposition 3.4 and Theorem 3.2 are analogy of (I.0) and (II.0) respectively.

For $n = 3$, Lewy proved that the Hessian of a harmonic function (the determinant of its matrix of second derivatives) cannot vanish at an interior point of its domain without changing sign, unless it vanishes identically. More precisely, if the Hessian vanishes at some interior point x_0 without vanishing identically, then in each neighborhood of x_0 it must take both positive and negative values. But the Jacobian of a harmonic mapping $f = \text{grad } u$ is the Hessian of u .

As a consequence, the Jacobian of a locally univalent harmonic gradient mapping from \mathbb{R}^3 to \mathbb{R}^3 cannot vanish at any interior point of its domain. Gleason and Wolff [13] generalized this result to \mathbb{R}^n .

Throughout this text the subscripts with variables x, y and z designate partial derivatives.

A vector field $F = (f^1, f^2, f^3)$ is said to satisfy the Cauchy-Riemann equations (CR-equations, for short) if its coordinates, (conjugate harmonic functions) satisfy:

$$f_y^1 = f_x^2, f_z^1 = f_x^3, f_z^2 = f_y^3, f_x^1 + f_y^2 + f_z^3 = 0, \text{ locally } F = \text{grad } \phi \text{ and } \Delta \phi = 0.$$

Equivalently, the Jacobian matrix of F is symmetric and has trace 0.

Theorem 3.2 (Lewy-Gleason-Wolff,[13]). *Logarithm of modulus of the Hessian of a harmonic function in a domain $\Omega \subset \mathbb{R}^3$ is superharmonic outside its zeros. Precisely, $\Delta \ln |H| \leq 0$, wherever $H \neq 0$.*

Remark. This inequality fails in dimensions greater than 3. Obviously, it holds (as equality) for planar harmonic functions.

Proposition 3.3. *Suppose Hessian determinant H of a harmonic function in a domain $\Omega \subset \mathbb{R}^3$ is positive. Then for every compact $F \subset \Omega$ we have $\inf_F H \geq \inf_{\partial F} H$.*

In particular, if f is injective harmonic gradient mapping, we have $\inf_F J(f) \geq \inf_{\partial F} J(f)$.

Using a normal family argument one can prove (see subsection 3.1 for details):

Proposition 3.4. [35] *Suppose that G and G' are domains in \mathbb{R}^3 : If $f : G \rightarrow G'$ is injective harmonic K -qc gradient mapping, then f has weak H -property.*

A more general result will appear in a forthcoming paper.

Proposition 3.5. *Suppose that*

- (a1): *f is injective harmonic gradient mapping from \mathbb{B} onto $D \subset \mathbb{R}^3$ and*
- (b1): *partial derivatives of f have continuous extension to $\overline{\mathbb{B}}$*
- (i.1) *If $J_h \geq j_0 > 0$ on \mathbb{S}^2 , then $J_h \geq j_0$ on $\overline{\mathbb{B}}$.*
- (i.2) *If in addition f is K -qc, then $f : \mathbb{B} \rightarrow D$ is bi-Lipschitz.*

Theorem 3.6. *Suppose that*

- (a1) *f is continuous on $\overline{\mathbb{B}}$, and univalent harmonic K -qc gradient mapping from \mathbb{B} onto convex domain $D \subset \mathbb{R}^3$ and*
- (b1) *partial derivatives of f have continuous extension to $\overline{\mathbb{B}}$.*
- (ii.1) *Then $f : \mathbb{B} \rightarrow D$ is bi-Lipschitz and in particular $f^{-1} : D \rightarrow \mathbb{B}$ is L -Lipschitz.*
- (ii.2) *If f is injective harmonic K -qc gradient mapping from \mathbb{B} onto convex bounded domain $D \subset \mathbb{R}^3$, then f is co-Lipschitz.*

Proof. (i.1), (i.2) and (ii.1) are corollary of Proposition 3.3.

(ii.2) is a corollary of Proposition 3.4 and Theorem 2.4 (i.3). \square

Note that the above outline of proof of (ii.2) is not based on Theorem 3.2 (see subsection 3.1 for more details). Astala-Manojlović first made publicly available proof of (ii.2) in Math.ArXiv, [6].

In particular, (ii.2) yields:

Proposition 3.7. (b) *Suppose that f is univalent harmonic gradient mapping from \mathbb{B}^3 onto itself. Then f is co-Lipschitz.*

Note that this result also follows directly from Lewy-Gleason-Wolff Theorem 3.2, Astala-Gehring Lemma 2.5 and Theorem 2.4.

In communication between V. Zorich and the author, the question was asked to find examples of functions that satisfy the condition (c). For example, if $u = x^2 + y^2 - 2z^2$, then $f = \nabla u = (2x, 2y, -4z)$ is injective harmonic gradient mapping from \mathbb{B}^3 onto the ellipsoid.

If u is real-valued function such that $f = \nabla u = (x, y, z)$, then $u = x^2/2 + y^2/2 + z^2/2 + c$.

In particular, Id is not harmonic gradient mapping.

In complex plane, if u is real-valued harmonic function, then $u_z = \frac{1}{2}(u'_x - u'_y)$ is analytic function and therefore $\nabla u = \overline{F}$, where $F = 2u_z$ is analytic function.

3.1. Hall lemma and co-Lipschitz property of qc gradient harmonic mappings

Here, we outline a proof of Theorem 3.6, the part (ii.2), stated here as:

(A0) A euclidean gradient harmonic mapping from the unit ball $\mathbb{B} \subset \mathbb{R}^3$ onto a bounded convex domain is co-Lipschitz.

Note that our proof of (A0) is based on the Hall lemma and a normal family argument and our approach is different from that in [6]; see also [8].

We first prove Hall lemma for harmonic injective mappings in n -dimensional space.

If G and G' are domains in \mathbb{R}^n , by $QCH(G, G')$ (respectively $QCH_K(G, G')$) we denote the set of Euclidean harmonic quasiconformal mappings (respectively K -qc) of G onto G' .

Lemma 3.8. Let F_K be a family of harmonic K -qc mapping $f : \mathbb{B} \rightarrow \mathbb{R}^n$ such that for $f \in F_K$, $J(f)$ has no zeros, F_K is closed with respect to uniform convergence, $f(\mathbb{B}) \supset \mathbb{B}$ and $f(0) = 0$. Then there is a constant $c > 0$ such that if $f \in F_K$ is harmonic K -qc mapping $f(\mathbb{B}) \supset \mathbb{B}$, $f(0) = 0$, then $\Lambda_f(0) \geq c$.

Proof. Suppose, on the contrary, that there is a sequence $f_n \in F_K$ such that $\Lambda_{f_n}(0) \rightarrow 0$. The sequence (f_n) forms a normal family and there is a subsequence of f_n which converges uniformly to a limit $f_0 \in F_K$; this is a contradiction. \square

Our further considerations are related to weak H -property (see Definition 2.2).

By Hall lemma planar euclidean harmonic mappings have H -property. Let $QCH_K^0 = QCH_K^0(\mathbb{B}, D)$ be family of gradient harmonic mappings which maps \mathbb{B} onto D .

Definition 3.9. We say that $\underline{F} \subset QCH_K(G, G')$ has J -property if F is closed with respect to uniform convergence, and for $f \in \underline{F}$, $J(f)$ has no zeros in G .

Now we sketch a proof of the statement (A0) in few steps (A1-A5):

(A1) If $\underline{F} \subset QCH_K(G, G')$ has J -property, then $f \in \underline{F}$ has weak H -property.

Outline of proof. By Lemma 3.8, there is a constant $c > 0$ such that $d(x)\Lambda_f(x) \geq cd(fx)$, $x \in G$.

(A2) Suppose that h is a euclidean harmonic mapping from the unit ball $\mathbb{B} \subset \mathbb{R}^n$ onto a bounded convex domain $D = h(\mathbb{B})$. Then there is a constant $c > 0$ such that $d^*(f(x)) \geq cd(x)$, $x \in \mathbb{B}$.

(A3) If f is injective harmonic gradient mapping, which maps $\mathbb{B} \subset \mathbb{R}^n$ onto a bounded convex domain $D = h(\mathbb{B})$, then any associated limit of f is harmonic gradient mapping.

In dimension $n = 3$, then it has Jacobian non zero normal family-property.

(A4) If $\underline{F} \subset QCH_K(B, D)$ has H -property, then every $f \in \underline{F}$ is co-Lipschitz.

By (A1) there is a constant $c > 0$ such that $d(x)\Lambda_f(x) \geq cd^*(f(x))$, $x \in \mathbb{B}$, and by (A2), a constant $c_1 > 0$ such that $d^*(f(x)) \geq c_1d(x)$. Hence $\lambda_f(x) \geq c_2$, where $c_2 = cc_1/K$.

(A5) In 3-dimensional space, $QCH_K^0 = QCH_K^0(\mathbb{B}, D)$ has H -property.

From (A1-A5), it follows (A0).

4. Appendix 1

In this review section we follow [32, 33]. First we recall some results from Section 1 (Theorem 1.2) and prove (I0) version of Hall lemma.

Theorem 4.1. Let h be an euclidean harmonic orientation preserving univalent mapping of the unit disc onto convex domain Ω . If Ω contains a disc $B(a; R)$ and $h(0) = a$ then

$$|\partial h(z)| \geq \frac{R}{4}, \quad z \in \mathbb{D}.$$

As a corollary of the previous Theorem we obtain

Theorem 4.2. Let h be an euclidean harmonic orientation preserving K -qc mapping of the unit disc onto convex domain Ω . If Ω contains a disc $B(a; R)$ and $h(0) = a$ then

$$|\partial h(z)| \geq \frac{R}{4}, \quad z \in \mathbb{D},$$

$$l_h(z) \geq \frac{1-k}{4}R.$$

Let $c = \frac{1-k}{4}R$. Since

$$|\bar{D}h(z)| \leq k |Dh(z)|,$$

it follows that $l_h(z) \geq c = \frac{1-k}{4}R$ and therefore $|h(z_2) - h(z_1)| \geq c |z_2 - z_1|$.

As a corollary of Theorem 4.1 we obtain

Proposition 4.3. *Let h be an euclidean harmonic orientation preserving univalent mapping of the unit disc into \mathbb{C} such that $f(\mathbb{D})$ contains a disc $B_R = B(a; R)$ and $h(0) = a$. Then*

$$|\partial h(0)| \geq \frac{R}{4}. \tag{6}$$

Proof: Let $V = V_R = h^{-1}(B_R)$ and φ be a conformal mapping of the unit disc U onto V such that $\varphi(0) = 0$ and let $h_R = h \circ \varphi$. By Schwarz lemma

$$|\varphi'(0)| \leq 1. \tag{7}$$

Since $\partial h_R(0) = \partial h(0)\varphi'(0)$, by Proposition 4.3 we get $|\partial h_R(0)| = |\partial h(0)||\varphi'(0)| \geq \frac{R}{4}$. Hence, using (0.2) we get (0.1).

Also as an immediate corollary of Theorem 4.1 we obtain

Theorem 4.4 ([20, 22]). *Let h be an euclidean harmonic diffeomorphism of the unit disc onto convex domain Ω . If Ω contains a disc $B(a; R)$ and $h(0) = a$ then*

$$D(h)(z) \geq \frac{1}{16}R^2, z \in \mathbb{D},$$

where $D(h)(z) = |\partial h(z)|^2 + |\bar{\partial}h(z)|^2$.

The following example shows that previous results are not true if we omit the condition $h(0) = a$.

Example. The mapping

$$\varphi_b(z) = \frac{z - b}{1 - \bar{b}z}, \quad |b| < 1,$$

is a conformal automorphism of the unit disc onto itself and

$$|\varphi'_b(z)| = \frac{1 - |b|^2}{|1 - \bar{b}z|^2}, \quad z \in \mathbb{D}.$$

In particular $\varphi'_b(0) = 1 - |b|^2$.

Heinz proved (see [16]) that if h is a harmonic diffeomorphism of the unit disc onto itself such that $h(0) = 0$, then

$$D(h)(z) \geq \frac{1}{\pi^2}, \quad z \in \mathbb{D}.$$

Using Proposition 4.3 we can prove Heinz theorem:

Theorem 4.5 (Heinz). *There exists no euclidean harmonic diffeomorphism from the unit disc \mathbb{D} onto \mathbb{C} .*

Note that this result was a key step in his proof of the Bernstein theorem for minimal surfaces in \mathbb{R}^3 .

Schoen obtained a nonlinear generalization of Proposition 4.3 by replacing the target by complete surface of nonnegative curvature (see Proposition 2.4 [46]) and using this result he proved

Theorem 4.6 (Schoen). *There exists no harmonic diffeomorphism from the unit disc onto a complete surface (S, ρ) of nonnegative curvature $K_\rho \geq 0$.*

Suppose f is a harmonic diffeomorphism from B_r to (S, ρ) and $\text{dist}(f(0), \partial(f(B_r))) \geq R$. Then it suffices to show that

$$|df|^2(0) \geq C \frac{R^2}{r^2},$$

where C is a universal constant. By hypothesis, we have $|\partial f| > |\bar{\partial} f| \geq 0$ and

$$\Delta \ln |\partial f| = -K_\rho J_f \leq 0. \tag{8}$$

If we define a Riemannian metric λ on B_r by $\lambda = |\partial f|^2 |dz|^2$, then (8) implies $K_\lambda \geq 0$. Therefore $\text{dist}(0, \partial(B_r)) \geq \frac{1}{2} \text{dist}(f(0), \partial(f(B_r))) \geq \frac{1}{2}R$.

Lemma 4.7. *If σ is a metric density of nonnegative curvature $K_\sigma \geq 0$ on B_r and $d = \text{dist}_\sigma(0, T_r)$, then $\sigma(0) \geq C \frac{d^2}{r^2}$, where C is a universal constant.*

A proof can be given by means the estimate of harmonic function in terms of curvature (Cheng-Yau, CPAM 28, 333-354 (1975)). We apply this lemma to metric density $\lambda = |\partial f|^2$. By the above estimate,

$$|\partial f|^2(0) \geq C \frac{R^2}{r^2}.$$

This proves the theorem.

Question 2. Can we prove Lemma 4.7 elementary? Note that $\ln \sigma$ is superharmonic function. Therefore $\ln \frac{1}{\sigma}$ and $\frac{1}{\sigma}$ are subharmonic functions.

4.1. Distortion of conformal mappings

The following form of Koebe’s One-Quarter Theorem applies in fact to all conformal mappings.

Theorem 4.8. *Suppose that f is bijective conformal in D and $f(D) = D'$, $z_0 \in D$. Then*

$$\frac{1}{4} |f'(z_0)| \text{dist}(z_0, \partial D) \leq \text{dist}(f(z_0), \partial D') \leq 4 |f'(z_0)| \text{dist}(z_0, \partial D).$$

If we set $d = d(z_0) = \text{dist}(z_0, \partial D)$ and $d' = d'(f(z_0)) = \text{dist}(f(z_0), \partial D')$ then (iv.1): $d' \asymp d$.

Proof. Let $d = d(z_0) = d_D(z_0) = \text{dist}(z_0, \partial D)$, $d' = d(f(z_0)) = d_{D'}(f(z_0)) = \text{dist}(f(z_0), \partial D')$;

$$g(z) = \frac{f(z) - f(z_0)}{d \cdot f'(z_0)} \text{ and } f_0(z) = g(z_0 + zd).$$

Set $D_0 = g(\mathbb{D}(z_0; d))$ and $d_0 = \text{dist}(g(z_0), \partial D_0)$. Note that $d_1 = \text{dist}(g(z_0), \partial D) = \frac{d'}{|f'(z_0)|}$ and $d_1 \geq d_0$. Since $f'_0(0) = 1$, it follows from Koebe’s One-Quarter Theorem, applied to f_0 that $d_0 \geq 1/4$. Hence, since $\frac{d'}{|f'(z_0)|} = d_1 \geq d_0 \geq 1/4$, we get the left inequality.

Koebe’s Theorem applied to f^{-1} at $w_0 = f(z_0)$ gives $\frac{1}{4} |(f^{-1})'(w_0)| d' \leq d$ and the right inequality follows. \square

If we define $D_f^*(z_0) = \frac{d|f'(z_0)|}{d'}$, we can reformulate the above theorem as $1/4 \leq D_f^*(z_0) \leq 4$. The interested reader can check that

$$D_{f^{-1}}^*(w_0) = D_f^*(z_0).$$

The following two basic theorems are important for our research.

Theorem 4.9 (Kellogg, see for example [11, 12]). *If a domain $D = \text{Int}(\Gamma)$ is $C^{1,\alpha}$, $0 < \alpha < 1$, and ω is a conformal mapping of \mathbb{D} onto D , then ω' and $\ln \omega'$ are in Lip_α . In particular, $|\omega'|$ is bounded from above and below on \mathbb{D} .*

Theorem 4.10 (Kellogg and Warschawski, see [49], Theorem 3.6). *If a domain $D = \text{Int}(\Gamma)$ is $C^{2,\alpha}$ and ω is a conformal mapping of \mathbb{D} onto D , then $|\omega''|$ has a continuous extension to the boundary. In particular it is bounded from above on \mathbb{D} .*

4.2. The uniformization theorem

The uniformization theorem says that every simply connected Riemann surface is conformally equivalent to one of the three domains: the open unit disk, the complex plane, or the Riemann sphere. In particular it admits a Riemannian metric of constant curvature. This classifies Riemannian surfaces as elliptic (positively curved rather, admitting a constant positively curved metric), parabolic (flat), and hyperbolic (negatively curved) according to their universal cover.

The uniformization theorem is a generalization of the Riemann mapping theorem from proper simply connected open subsets of the plane to arbitrary simply connected Riemann surfaces. The uniformization theorem implies a similar result for arbitrary connected second countable surfaces: they can be given Riemannian metrics of constant curvature. Every Riemann surface is the quotient of a free, proper and holomorphic action of a discrete group on its universal covering and this universal covering is holomorphically isomorphic (one also says: “conformally equivalent”) to one of the following: the Riemann sphere, the complex plane or the unit disk in the complex plane. Koebe proved the general uniformization theorem that if a Riemann surface is homeomorphic to an open subset of the complex sphere (or equivalently if every Jordan curve separates it), then it is conformally equivalent to an open subset of the complex sphere. In 3 dimensions, there are 8 geometries, called the eight Thurston geometries. Not every 3-manifold admits a geometry, but Thurston’s geometrization conjecture proved by Grigori Perelman states that every 3-manifold can be cut into pieces that are geometrizable. The simultaneous uniformization theorem of Lipman Bers shows that it is possible to simultaneously uniformize two compact Riemann surfaces of the same genus > 1 with the same quasi-Fuchsian group. The measurable Riemann mapping theorem shows more generally that the map to an open subset of the complex sphere in the uniformization theorem can be chosen to be a quasiconformal map with any given bounded measurable Beltrami coefficient.

5. Appendix 2, Harmonic Maps Between surfaces

In [17] it is given self-contained account of the results on harmonic maps between surfaces. This treatment contains several simplifications and unifications compared to the presentations available in the existing literature. Upper and lower bounds for the sectional curvature K of a manifold are often denoted by k^2 and $-\omega^2$, i.e. $-\omega^2 \leq K \leq k^2$. This notation avoids square roots. It differs, however, from the terminology in some of the papers frequently referred to in the book [17].

Here we give short review of results related to our consideration in section 1. First the lower bound for Jacobian are considered.

Theorem 5.1. *Suppose $u : \mathbb{D} \rightarrow \Sigma$ is harmonic, and $u(D) \subset B(p, M)$, where $B(p, M)$ again is a disc with radius $M < \pi/2k$. Suppose that $\partial u(\mathbb{D}) = u(\mathbb{T})$ and that $g := u|_{\mathbb{T}} : \mathbb{T} \rightarrow \partial u(D)$ is a C^2 - diffeomorphism with (d1) $0 < b \leq |g'(t)|$ for all $t \in [0, 2\pi]$.*

Assume furthermore that $g(\mathbb{T})$ is strictly convex w.r.t. $u(\mathbb{D})$, and that we have the following estimates for the geodesic curvature of $g(\mathbb{T})$

(d2) $0 < a_1 \leq \kappa_g(g(t)) \leq a_2$ for all $t \in [0, 2\pi]$.

Then (iv.1):

$J \geq \delta_1^{-1}$, where $\delta_1 = \delta_1(\omega, k, M, \tau, a_1, a_2, b, |g|_{1,\alpha})$ (τ is given in Thm. 6.2 [17]).

Note that it is not assumed in the theorem that u is univalent; we needed only that u maps \mathbb{D} onto the convex side of $u(\mathbb{T})$. If u is an injective harmonic map, then $J \geq \delta_2^{-1}$ on \mathbb{D} .

Proposition 5.2. *Assume $u : \mathbb{D} \rightarrow \Sigma$ is an injective harmonic map, where $u(D) \subset B(p, M)$, and $B(p, M)$ is a disc with radius $M < \pi/2k$. Suppose that $g := u|_{\mathbb{T}} \in C^{1,\alpha}$, and that (d1) and (d2) hold. Then (iii.2): for all $z \in \mathbb{D}$, $J \geq \delta_2^{-1}$, where $\delta_2 = \delta_2(\omega, k, M, \tau, a_1, a_2, b, |g|_{1,\alpha})$.*

We define $d_*(q) := -\text{dist}(q, \partial u(D))$ for $q \in u(D)$. Since $\Delta(d_* \circ u) \geq a_1 b^2$, $d_* \circ u$ is subharmonic function. This will enable us to get a lower bound for the radial derivative of $d_* \circ u$ at boundary points with the argument of the boundary lemma of E. Hopf.

Taking Cor. 6.2[17] into account, we can therefore find a neighborhood V_0 of \mathbb{T} in \mathbb{D} with the property that d_* is a C^2 function with strictly convex level curves on $u(V_0)$. Suppose $z_0 \in \mathbb{T}$; we can choose some disc $B_1 = B(z_1, r_1) \subset D$, $z_0 \in S(z_1, r_1)$, in such a way that $\Delta(d_* \circ u) \geq a_1 b^2/2$ for $z \in B_1$.

Defining the auxiliary function v via $v(z) = \frac{r_1^2}{8} a_1 b^2 (1 - \frac{|z-z_1|^2}{r_1^2})$, we find $\Delta v = -a_1 b^2/2$ and therefore $\Delta(d_* \circ u + v) \geq 0$. The maximum principle now controls the derivative of $d_* \circ u + v$ at z_0 in the direction of the outer normal.

Now, using the above estimates, the existence of harmonic diffeomorphisms which solve a Dirichlet problem is considered.

Theorem 5.3. *Suppose (e1): Ω is a compact domain with Lipschitz boundary $\partial\Omega$ on some surface, and that Σ is another surface. We assume (e2): that $f : \overline{\Omega} \rightarrow \Sigma$ maps $\overline{\Omega}$ homeomorphically onto its image, that $f(\partial\Omega)$ is contained in some disc $B(p, M)$ with radius $M < \pi/2k$ (where $k^2 \geq 0$ is an upper curvature bound on $B(p, M)$) and that the curves $f(\partial\Omega)$ are of Lipschitz class and convex w.r.t. $f(\Omega)$. Then (v.1): there exists a harmonic mapping $u : \Omega \rightarrow B(p, M)$ with the boundary values prescribed by f which is a homeomorphism between $\overline{\Omega}$ and its image, and a diffeomorphism in the interior.*

(v.2) *Moreover, if $f|\partial\Omega$ is even a C^2 -diffeomorphism between C^2 -curves, then u is a diffeomorphism up to the boundary.*

First of all, $\partial\Omega$ is connected. Otherwise, $f(\partial\Omega)$ would consist of at least two curves, both of them convex w.r.t. $f(\Omega)$. Since Ω is homeomorphic to $f(\Omega)$, we conclude that Ω is a disc, topologically. Since there is a conformal map $\phi : D \rightarrow g(\mathbb{D})$, one have to prove the theorem only for the case where Ω is the plane unit disc \mathbb{D} .

We first assume that $f : \mathbb{T} \rightarrow f(\mathbb{T})$ is C^2 -diffeomorphism between curves of class $C^{2,\alpha}$, that $f(\mathbb{T})$ is not only convex, but strictly convex, and that we have the following quantitative bounds

(e2) $|f'(t)| \geq b_2^{-1}$ and $|f''(t)| \leq b_1$.

Now let Γ be the parametrization of the boundary curve of $g(D)$ by arclength. If l is length of $\partial g(D)$, then Γ maps $[0, l]$ on $\partial g(D)$. We set $w(z, \lambda) = \lambda \Gamma^{-1}(\phi(z)) + (1 - \lambda) \Gamma^{-1}(g(z))$, $\omega(z, \lambda) = \Gamma(w(t, \lambda))$, $z \in T$, $\lambda \in [0, 1]$, and $\underline{\omega}(t, \lambda) = \omega(e^{it}, \lambda)$. Using (e2), one can check that

$$\underline{\omega}(t, \lambda), \frac{\partial \underline{\omega}(t, \lambda)}{\partial t} \text{ and } \frac{\partial^2 \underline{\omega}(t, \lambda)}{\partial t^2} \text{ are continuous functions of } \lambda.$$

Let now u_λ denote the harmonic map from D to $B(p, M)$ with boundary values $\omega(\cdot, \lambda)$. In particular, $m(\lambda) := \inf_{z \in D} |J(u_\lambda)(z)|$ depends continuously on λ . We define $L := \{\lambda \in [0, 1] : m(\lambda) > 0\}$. $0 \in L$; (u_0 is the conformal map ϕ), and therefore L is not empty.

By Proposition 5.2, $m(\lambda) \geq m_0 > 0$ for $\lambda \in L$. Since $m(\lambda)$ depends continuously on λ , (8.1.7) implies $L = [0, 1]$. Thus, u_1 is a local diffeomorphism and a diffeomorphism between the boundaries of D and $u_1(D)$, and consequently a global diffeomorphism by topology. Theorem 5.3 and the uniqueness theorem of Jager and Kaul (cf. Theorem 5.1 [17]) imply

Corollary 5.4. *Under the assumptions of Theorem 5.3, each harmonic map which solves the Dirichlet problem defined by g and which maps Ω into a geodesic disc $B(p, M)$ with radius $M < \pi/2k$, is a diffeomorphism in Ω .*

The above, we have assumed that the boundary of the image is strictly convex, and, in addition, that the boundary values are a diffeomorphism of class C^2 . The theorem also holds for the case that

(f1): the boundary is only supposed to be convex and that

(f2): the boundary values are only supposed to induce a homeomorphism of the boundaries.

The procedure to handle the case (f1) is called the first approximation argument. It is a modification of the corresponding one given by E. Heinz.

The case (f1) of a general boundary is handled by an approximation by smooth curves. More precisely it is supposed that the boundary of the image $f(\mathbb{D})$ is only convex, while the boundary values f are still assumed to be a diffeomorphism of class C^2 . For approximation arguments in planar case see also [14, 15, 24].

6. Appendix 3

During the visiting position at Wayne State University, Detroit, 1988/89, the author²⁾ started considering hqc mappings. In particular, the author observed that the following results hold (see Proposition 6.1 and 6.2 below). When I returned to Belgrade, I used to talk on the seminar permanently and asked several open questions related to the subject. Many research papers are based on these communications. Since I had not published all of them it happens that some researchers discovered them later. Here we only discuss a few results from Revue Roum. Math. Pures Appl. Vol. 51 (2006) 5–6, 711–722³⁾:

Proposition 6.1 (Proposition 5 [33]). *If h is a harmonic univalent orientation preserving K -qc mapping of domain D onto D' , then*

$$d(z)\lambda_h(z) \leq 16Kd_h(z), \text{ and } d(z)\lambda_h(z) \geq \frac{1-k}{4}d_h(z). \quad (9)$$

Proposition 6.2 (Corollary 1, Proposition 5 [33]). *Every e -harmonic quasi-conformal mapping of the unit disc (more generally of a strongly hyperbolic domain) is a quasi-isometry with respect to hyperbolic distances.*

From Proposition 6.1 directly follows next result (Proposition 6.3).⁴⁾

Proposition 6.3 ([30]). *Every e -harmonic quasi-conformal mapping of a domain different from \mathbb{C} is a quasi-isometry with respect to quasi-hyperbolic distances.*

The next theorem concerns harmonic maps onto a convex domain. For the planar version of Theorem 2.4 cf. [32, 33], also [38], pp. 152-153. The space version was communicated on International Conference on Complex Analysis and Related Topics (Xth Romanian-Finnish Seminar, August 14-19, 2005, Cluj-Napoca, Romania), by Mateljević and stated in [33], cf. also [40].

Theorem 6.4 (Theorem 1.3, [33]). *Suppose that h is an Euclidean harmonic mapping from the unit ball \mathbb{B}^n onto a bounded convex domain $D = h(\mathbb{B}^n)$, which contains the ball $h(0) + R_0\mathbb{B}^n$. Then for any $x \in \mathbb{B}^n$*

$$d(h(x), \partial D) \geq (1 - \|x\|)R_0/2^{n-1}.$$

Now we outline an example related to a question that arose during author's visit to Helsinki and Turku in October 2005 (at a Seminar in Helsinki-Turku) and the small progress related to discussion there. Let \mathbb{H}^n denote the half-space in \mathbb{R}^n . If D is a domain in \mathbb{R}^n , by $QCH(D)$ we denote the set of Euclidean harmonic quasiconformal mappings of D onto itself. In particular, if $x \in \mathbb{R}^3$, we use the notation $x = (x_1, x_2, x_3)$ and denote $f'_{x_k} = D_{x_k}f = f_{x_k}$ the partial derivative with respect to x_k . A fundamental solution of the Laplace equation in \mathbb{R}^3 is $\frac{1}{r}$, where $r = |x|$. Let $u_0 = \frac{1}{|x+e_3|}$, where $e_3 = (0, 0, 1)$. Define $h(x) = (x_1 + e_1u_0, x_2 + e_2u_0, x_3)$. We can verify that $h \in QCH(\mathbb{H}^3)$ (harmonic quasiconformal of \mathbb{H}^3 onto itself) for small e_1 and e_2 .

Now we shortly follow [36]. Using the Herglotz representation of a nonnegative harmonic function u (see Theorem 7.24 and Corollary 6.36 [7]), one can get:

Lemma A. *If u is a nonnegative harmonic function on a half space \mathbb{H}^n , continuous up to the boundary with $u = 0$ on \mathbb{H}^n , then u is (affine) linear.*

In [33], the author has outlined a proof of the following result:

Theorem A. *If h is a quasiconformal harmonic mapping of the upper half space \mathbb{H}^n onto itself and $h(\infty) = \infty$, then h is quasi-isometry with respect to both the Euclidean and the Poincaré distance.*

Note that the outline of proof in [33] can be justified by Lemma A.

We show that the analog statement of this result holds for p -harmonic vector functions (solutions of p -Laplacian equations) using the mentioned result obtained in the paper [19], stated here as:

Theorem B. *If u is a nonnegative p -harmonic function on a half space \mathbb{H}^n , continuous up to the boundary with $u = 0$ on \mathbb{H}^n , then u is (affine) linear.*

²⁾the author refers to me

³⁾It seems that there is some problems concerning visibility of this journal.

⁴⁾In that time, the author did not realized that quasi-hyperbolic metrics have important applications and did not state this version which due to V. Manojlovic.

Theorem 6.5 ([36]). *If h is a quasiconformal p -harmonic mapping of the upper half space \mathbb{H}^n onto itself and $h(\infty) = \infty$, then both $h : (\mathbb{H}^n, |\cdot|) \rightarrow (\mathbb{H}^n, |\cdot|)$ and $h : (\mathbb{H}^n, \rho_{\mathbb{H}^n}) \rightarrow (\mathbb{H}^n, \rho_{\mathbb{H}^n})$ are bi-Lipschitz where $\rho_{\mathbb{H}^n}$ is the Poincaré distance.*

Since 2-harmonic mapping are Euclidean harmonic this result includes Theorem A.

For a domain $G \subset \mathbb{R}^n$ let $\rho : G \rightarrow (0, \infty)$ be a function. We say that ρ is a weight function or a metric density if for every locally rectifiable curve γ in G , the integral

$$l_\rho(\gamma) = \int_\gamma \rho(x) ds$$

exists. In this case we call $l_\rho(\gamma)$ the ρ -length of γ . A metric density defines a metric $d_\rho : G \times G \rightarrow (0, \infty)$ as follows. For $a, b \in G$, let

$$d_\rho(a, b) = \inf_\gamma l_\rho(\gamma)$$

where the infimum is taken over all locally rectifiable curves in G joining a and b . It is an easy exercise to check that d_ρ satisfies the axioms of a metric. For instance, the hyperbolic (or Poincaré) metric of \mathbb{D} is defined in terms of the density $\rho(x) = c/(1 - |x|^2)$ where $c > 0$ is a constant.

The quasihyperbolic metric $k = k_G$ of G is a particular case of the metric d_ρ when $\rho(x) = \frac{1}{d(x, \partial G)}$ (see [45]). For a domain $G \subset \mathbb{R}^n, n \geq 2, x, y \in G$, let

$$r_G(x, y) = \frac{|x - y|}{\min\{d(x), d(y)\}} \text{ where } d(x) = d(x, \partial G) \equiv \inf\{|z - x| : z \in \partial G\}.$$

If the domain G is understood from the context, we write r instead r_G . This quantity is used, for instance, in the study of quasiconformal and quasiregular mappings, cf. [45]. It is a basic fact that [44, Theorem 18.1] for $n \geq 2, K \geq 1, c_2 > 0$ there exists $c_1 \in (0, 1)$ such that whenever $f : G \rightarrow fG$ is a quasiconformal mapping with $G, fG \subset \mathbb{R}^n$ then $x, y \in G$ and $r_G(x, y) \leq c_1$ imply $r_{fG}(f(x), f(y)) \leq c_2$. We call this property the local uniform boundedness of f with respect to r_G . Note that quasiconformal mappings satisfy the local uniform boundedness property and so do quasiregular mappings under appropriate conditions; it is known that one to one mappings satisfying the local uniform boundedness property may not be quasiconformal. We also consider a weaker form of this property and say that $f : G \rightarrow fG$ with $G, fG \subset \mathbb{R}^n$ satisfies the weak uniform boundedness property on G (with respect to r_G) if there is a constant $c > 0$ such that $r_G(x, y) \leq 1/2$ implies $r_{fG}(f(x), f(y)) \leq c$.

Theorem 6.6 ([36]). *Suppose that G is a proper subdomain of \mathbb{R}^n and $h : G \rightarrow \mathbb{R}^n$ is a harmonic mapping. Then the following conditions are equivalent*

- (1) h satisfies the weak uniform boundedness property.
- (2) $h : (G, k_G) \rightarrow (h(G), k_{h(G)})$ is Lipschitz.

Theorem 6.7 ([36]). *Suppose that $G \subset \mathbb{R}^n, f : G \rightarrow \mathbb{R}^n$ is K -qr and $G' = f(G)$. Let $\partial G'$ be a continuum containing at least two distinct points. If f is a harmonic mapping, then $f : (G, k_G) \rightarrow (G', k_{G'})$ is Lipschitz.*

Now, we give a sufficient condition for a qc mapping $f : G \rightarrow f(G)$ to be a pseudo-isometry w.r.t. quasihyperbolic metrics on G and $f(G)$. First we adopt the following notation.

If V is a subset of \mathbb{R}^n and $u : V \rightarrow \mathbb{R}^m$, we define

$$\text{osc}_V u = \sup\{|u(x) - u(y)| : x, y \in V\}.$$

Suppose that $G \subset \mathbb{R}^n$ and $B_x = B(x, d(x)/2)$. Let $OC^1(G)$ denote the class of $f \in C^1(G)$ such that

$$d(x)|f'(x)| \leq c_1 \text{osc}_{B_x} f \tag{10}$$

for every $x \in G$. Similarly, let $SC^1(G)$ be the class of functions $f \in C^1(G)$ such that

$$|f'(x)| \leq ar^{-1} \omega_f(x, r) \quad \text{for all } B^n(x, r) \subset G, \quad (11)$$

where $\omega_f(x, r) = \sup\{|f(y) - f(x)| : y \in B^n(x, r)\}$.

The proof of Theorem 6.6 gives the following more general result:

Theorem 6.8 ([36]). *Suppose that $G \subset \mathbb{R}^n$, $f : G \rightarrow G'$, $f \in OC^1(G)$ and it satisfies the weak property of uniform boundedness with a constant c on G . Then*

(e) $f : (G, k_G) \rightarrow (G', k_{G'})$ is Lipschitz.

(f) In addition, if f is K -qc, then f is pseudo-isometry w.r.t. quasihyperbolic metrics on G and $f(G)$.

In [25, 26, 50] Kalaj and the author study mappings in plane and space which satisfy

$$|\Delta u| \leq a|\nabla u|^2 + b. \quad (12)$$

Suppose that domains D and Ω are bounded domains in \mathbb{R}^n and its boundaries belong to class $C^{k,\alpha}$, $0 \leq \alpha \leq 1$, $k \geq 2$ (more generally C^2). Suppose further that g and g' are C^1 metric on \bar{D} and $\bar{\Omega}$ respectively. Using inner estimate (cf. Theorem 6.14 [10]) we can prove

Theorem 6.9. *If $u : D \rightarrow \Omega$ a qc (g, g')-harmonic map (or satisfies (12)), then u is Lipschitz on D .*

We discussed this result at Workshop on Harmonic Mappings and Hyperbolic Metrics, Chennai, India, Dec. 10-19, 2009, see Course-materials [50]. During the lectures we also pointed some differences between the theory in the plane and the space.

We now present a few open questions; first recall two questions cf. [33].

Q1. In connection with Theorem A, we ask the corresponding question for p -harmonic functions.

Q2. If f is a homeomorphism of \mathbb{R}^2 onto itself with $f(\infty) = \infty$, let h be the solution of the corresponding Dirichlet problem in \mathbb{H}^3 . Find a sufficient condition for f such that a) h is one-to-one, and b) $h \in QCH(\mathbb{H}^3)$. Consider also the corresponding question for dimension $n > 3$.

Note that Theorem 6.5 yields an affirmative answer to Q1.

Q3. Suppose that $G \subset \mathbb{R}^n$ is a proper subdomain, $f : G \rightarrow \mathbb{R}^n$ is harmonic K -qc and $G' = f(G)$. Determine whether f is a quasi-isometry (ie. bi-Lipschitz) w.r.t. quasihyperbolic metrics on G and G' .

Q4. Describe zero set of Jacobian of injective harmonic maps in space?

Since this question sounds more general, we can specify it. For example consider the following question:

1. Let \mathbb{B}^m be hyperbolic space and let $f : \mathbb{B}^m \rightarrow \mathbb{B}^m$, $m \geq 3$, be harmonic K -quasiconformal map. Whether f has critical points on \mathbb{B}^m ?

2. Whether injective euclidan-harmonic (or p -harmonic), $m \geq 3$, quasiconformal map has critical points?

Q5. Is there any appropriate version of the Radó-Kneser-Choquet theorem (RKC-Theorem) in space?

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⁵⁾In particular they gave some remarks concerning previous drafts, [40] v1 and [8].

⁶⁾In particular in the communication the question who has the priority of proof of Theorem 3.6(ii.2) has appeared and all the above mentioned colleagues have been involved with it.

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