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Remarks and Comments on Some Recent Results

Vladimir Pavlović[a]

[a]*Faculty of Sciences and Mathematics, University of Niš, P.O. Box 224, 18000 Niš, Serbia*

Abstract. In this note we give shorter proofs of some recent results on star and left star orders on $\mathcal{B}(\mathcal{H})$ and correct a proof of one that was incomplete.

1. Remarks and Corrections

On $\mathbb{C}^{n\times n}$ many partial orders are defined. One such order is the *rank subtractivity order* (also known as the *minus order*) which was introduced by Hartwig [5] in the following way:

$$A \leq^{-} B \Leftrightarrow \mathrm{r}(A - B) = \mathrm{r}(A) - \mathrm{r}(B). \tag{1}$$

In [7] Šemrl considered the question of generalizing this order to $\mathcal{B}(\mathcal{H})$ and succeeded in finding an equivalent definition of the rank subtractivity partial order on $\mathbb{C}^{n\times n}$ that makes sense for elements of $\mathcal{B}(\mathcal{H})$:

Definition 1.1. *[7] Let $A, B \in \mathcal{B}(\mathcal{H})$. Then $A \leq^{-} B$ if and only if there exist projections $P, Q \in \mathcal{B}(\mathcal{H})$ such that*

(i) $\mathcal{R}(P) = \overline{\mathcal{R}(A)}$,

(ii) $\mathcal{N}(Q) = \mathcal{N}(A)$,

(iii) $PA = PB$,

(iv) $AQ = BQ$.

It was proved in [7] that the orders given by Definition 1.1 and by (1) coincide. This motivated Dolinar et al. [3] and Dolinar et al. [4] to, using the same approach as in [7], define partial orders on $\mathcal{B}(\mathcal{H})$ by modifying Definition 1.1.
More precisely, in [3] they introduced the following order

Definition 1.2. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then $A \overset{*}{\leq} B$ if and only if the following two conditions are satisfied:*

(1) $PB = A$ where P is the orthogonal projection onto $\overline{\mathcal{R}(A)}$,

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Email address: vlada@pmf.ni.ac.rs (Vladimir Pavlović)

(2) $BQ = A$ where Q is the orthogonal projection onto $\overline{\mathcal{R}(A^*)}$.

In the same paper they showed that this definition gives the usual *star order* on $\mathcal{B}(\mathcal{H})$ previously introduced by Drazin [2] as

$$A \leq^* B \Leftrightarrow A^*A = A^*B \text{ and } AA^* = BA^*. \tag{2}$$

Now, we will give a very short proof of this fact (Theorem 5 [3]) without using the polar decompositions of operators, which is the case in Theorem 5 in [3].

Theorem 1.3. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then $A \overset{*}{\leq} B$ if and only if $A^*A = A^*B$ and $AA^* = BA^*$.*

Proof. We have $A^*(B-A) = 0 \Leftrightarrow \mathcal{R}(B-A) \subseteq \mathcal{N}(A^*) = \mathcal{R}(A)^\perp \Leftrightarrow \mathcal{R}(A) \perp \mathcal{R}(B-A)$. Similarly $(B-A)A^* = 0 \Leftrightarrow \mathcal{R}(B^*-A^*) \subseteq \mathcal{N}(A) = \mathcal{R}(A^*)^\perp \Leftrightarrow \mathcal{R}(A^*) \perp \mathcal{R}(B^*-A^*)$. By Lemma 3 from [3] theorem follows.□

In [4] Dolinar et al. further introduced the following order:

Definition 1.4. *[7] For $A, B \in \mathcal{B}(\mathcal{H})$ we define $A * \leq B$ if and only the following two conditions are satisfied:*

(1) $PB = A$ where P is the orthogonal projection onto $\overline{\mathcal{R}(A)}$,

(2) $BQ = A$ for some projection $Q \in \mathcal{B}(\mathcal{H})$ such that $\mathcal{N}(Q) = \mathcal{N}(A)$.

In the same paper, they note in Theorem 5 that the order given by Definition 1.4 is the same as the left star order in the sense of Baksalary and Mitra. When showing that the conditions $A^*A = A^*B$ and $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ imply that $A * \leq B$, the authors observe that "*the left-star partial order implies minus partial order*", meaning that the left star partial order as given by Definition 1.4 implies the minus partial order, which is indeed a trivial fact, but to prove that $A * \leq B$ as defined in Definition 1.4 is the goal there, not an assumption. Here, we will give a complete proof of this result:

Theorem 1.5. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then $A * \leq B$ if and only if $A^*A = A^*B$ and $\mathcal{R}(A) \subseteq \mathcal{R}(B)$.*

Proof. ($\Leftarrow$) : Let

$$B = \begin{bmatrix} B_0 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \overline{\mathcal{R}(B^*)} \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \overline{\mathcal{R}(B)} \\ \mathcal{N}(B^*) \end{bmatrix}$$

where $B_0 \in \mathcal{B}\left(\overline{\mathcal{R}(B^*)}, \overline{\mathcal{R}(B)}\right)$ is injective. Since $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ we have that

$$A = \begin{bmatrix} A_0 & A_{00} \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \overline{\mathcal{R}(B^*)} \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \overline{\mathcal{R}(B)} \\ \mathcal{N}(B^*) \end{bmatrix}$$

for some $A_0 \in \mathcal{B}\left(\overline{\mathcal{R}(B^*)}, \overline{\mathcal{R}(B)}\right)$. From $A^*A = A^*B$ it follows that $A_{00} = 0$ and $A_0^*A_0 = A_0^*B_0$. If $A_1 \in \mathcal{B}\left(\overline{\mathcal{R}(A_0^*)}, \overline{\mathcal{R}(A_0)}\right)$ is (the injective operator) such that

$$A_0 = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \overline{\mathcal{R}(A_0^*)} \\ \mathcal{N}(A_0) \end{bmatrix} \to \begin{bmatrix} \overline{\mathcal{R}(A_0)} \\ \mathcal{N}(A_0^*) \end{bmatrix},$$

then $A_0^*A_0 = B_0^*A_0$ implies that

$$B_0^* = \begin{bmatrix} A_1^* & B_1 \\ 0 & B_2 \end{bmatrix} : \begin{bmatrix} \overline{\mathcal{R}(A_0)} \\ \mathcal{N}(A_0^*) \end{bmatrix} \to \begin{bmatrix} \overline{\mathcal{R}(A_0^*)} \\ \mathcal{N}(A_0) \end{bmatrix}$$

for some $B_1 \in \mathcal{B}\left(\mathcal{N}(A_0^*), \overline{\mathcal{R}(A_0^*)}\right)$, $B_2 \in \mathcal{B}\left(\mathcal{N}(A_0^*), \mathcal{N}(A_0)\right)$. The inclusion $\mathcal{R}(A_0) = \mathcal{R}(A) \subseteq \mathcal{R}(B) = \mathcal{R}(B_0)$ means that for every $x \in \overline{\mathcal{R}(A_0^*)}$ there are $x' \in \overline{\mathcal{R}(A_0^*)}$ and $y \in \mathcal{N}(A_0)$ such that

$$\begin{bmatrix} A_1x \\ 0 \end{bmatrix} = \begin{bmatrix} A_1x' \\ B_1^*x' + B_2^*y \end{bmatrix}.$$

The operator A_1 being injective, this further implies $\mathcal{R}(B_1^*) \subseteq \mathcal{R}(B_2^*)$, which gives us an operator $S \in \mathcal{B}\left(\mathcal{N}(A_0), \overline{\mathcal{R}(A_0^*)}\right)$ such that $B_1 = SB_2$.

We will show that $\overline{\mathcal{R}(B^*)} = \overline{\mathcal{R}(A^*)} \oplus \overline{\mathcal{R}(B^* - A^*)}$. Note that $\mathcal{R}(B^*) = \mathcal{R}(B_0^*)$, $\mathcal{R}(A^*) = \mathcal{R}(A_0^*) = \mathcal{R}(A_1^*)$ and $\mathcal{R}(B^* - A^*) = \mathcal{R}(B_0^* - A_0^*) = \mathcal{R}(\begin{bmatrix} B_1 \\ B_2 \end{bmatrix})$.

Suppose that $\begin{bmatrix} A_1^* x_n + B_1 y_n \\ B_2 y_n \end{bmatrix} \to \begin{bmatrix} u \\ v \end{bmatrix}$ for some $x_n \in \overline{\mathcal{R}(A_0)}$, $y_n \in \mathcal{N}(A_0^*)$ for $n \in \mathbb{N}$. Then $B_1 y_n = SB_2 y_n \to Sv$ so $A_1^* x_n \to u - Sv \in \overline{\mathcal{R}(A_1^*)}$. Hence $\begin{bmatrix} B_1 y_n \\ B_2 y_n \end{bmatrix} \to \begin{bmatrix} Sv \\ v \end{bmatrix}$ so

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u - Sv \\ 0 \end{bmatrix} + \begin{bmatrix} Sv \\ v \end{bmatrix}$$

finally implies $\begin{bmatrix} u \\ v \end{bmatrix} \in \overline{\mathcal{R}(A^*)} + \overline{\mathcal{R}(B^* - A^*)}$.

To see that the sum is direct let $u \in \overline{\mathcal{R}(A_0^*)}$, $v \in \mathcal{N}(A_0)$ be such that $\begin{bmatrix} A_1^* x_n \\ 0 \end{bmatrix} \to \begin{bmatrix} u \\ v \end{bmatrix}$, $\begin{bmatrix} B_1 y_n \\ B_2 y_n \end{bmatrix} \to \begin{bmatrix} u \\ v \end{bmatrix}$ for some $x_n \in \overline{\mathcal{R}(A_0)}$, $y_n \in \mathcal{N}(A_0^*)$ for $n \in \mathbb{N}$. From $v = 0$ it follows $B_1 y_n = SB_2 y_n \to Sv = 0$. Thus $u = 0$ and we are done.

From $A^*(B - A) = 0$ we have $\mathcal{R}(B - A) \subseteq \mathcal{N}(A^*) = \mathcal{R}(A)^\perp$ so $\mathcal{R}(A) \perp \mathcal{R}(B - A)$. By Lemma 2 from [4] we conclude that $A * \leq B$.

$(\Rightarrow)$: Suppose that $A * \leq B$. From Lemma 2 [4] it immediately follows that $\mathcal{R}(A) \subseteq \mathcal{R}(B)$, and also that $\mathcal{R}(A) \perp \mathcal{R}(B - A)$. Now, for every $x \in \mathcal{H}$ we have that $\langle (B - A)x, Ax \rangle = 0$, implying that $A^*(B - A) = 0$.□

We end the note by a remark about the proof of Theorem 15 [4] in which the authors presented a very interesting result in which they characterized all the bijective additive maps on $\mathcal{B}(\mathcal{H})$ which preserve the left (right) star order in both directions. Taking into account that ϕ is additive and using the fact that a bijective map $\phi : \mathcal{P}(\mathcal{H}) \to \mathcal{P}(\mathcal{H})$, where $\mathcal{P}(\mathcal{H})$ is the set of all orthogonal projections, preserves the usual order $P \leq Q \Leftrightarrow PQ = QP = P$ in both directions and satisfies $\phi(I - P) = I - \phi(P)$, if and only if there is an operator $U : \mathcal{H} \to \mathcal{H}$ either unitary or antiunitary, such that $\phi(P) = UPU^*$ for all $P \in \mathcal{P}(\mathcal{H})$ (see [6], page 13), we can eliminate the items 10 and 11 of the proof and skip directly to the conclusion reached in item 12.

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