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The Zero Divisor Graphs of Finite Rings of Cubefree Order

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Abstract. The aim of this paper is to classify the zero divisor graph of finite rings of cubefree order. It is proved that all zero divisor graphs can be interpreted as the extended join over well-known graphs.

1. Introduction

The notion of a zero divisor graph was introduced by Beck in [3] when he studied the coloring problem of a commutative ring. In order to define this graph, we assume that R is a ring and $G(R)$ is a simple graph such that $V(G(R)) = R$ and two distinct vertices x and y are adjacent provided that $xy = 0$. It is easy to prove that $G(R)$ is a connected graph of diameter at most 2. Anderson and Livingston [1], for simplification of the concept of Beck's zero divisor graph considered the set of all non-zero zero divisors as the vertex set. The edges can be defined in a similar way as Beck's seminal paper. This studied the interplay between the ring and graph theoretical properties of this structure. Throughout this paper we use the Anderson−Livingston's definition of zero devisor graph and so all rings considered here is not integral. We encourage to the interested readers to consult [5] for more information on this topic.

In [2, 6], a classification of finite rings of order p^2 and p^3 are presented. It is not so difficult to continue the lines of [6] for a classification of finite ring of square free orders. The aim of this paper is determining the zero divisor graphs of finite rings of order p^2, p is prime, and the zero divisor graphs of finite rings of cubefree orders.

We denote by K_n and ϕ_n the complete and empty graphs on n vertices, respectively. The join $G + H$ of graphs G and H with disjoint vertex sets $V(G)$ and $V(H)$ and edge sets $E(G)$ and $E(H)$ is the graph union $G \cup H$ together with all the edges joining $V(G)$ and $V(H)$. The complete bipartite and complete tripartite graphs $K_{m,n}$ and $K_{m,n,k}$ are defined by $K_{m,n} = \phi_m + \phi_n$ and $K_{m,n,k} = K_{m,n} + \phi_k$. Suppose $G_1, G_2, \cdots, G_k$ are graphs with disjoint vertices. The sequential join $G_1 + G_2 + \cdots + G_k$ is defined as the graph union $(G_1 + G_2) \cup (G_2 + G_3) \cup \cdots \cup (G_{n-1} + G_n)$.

The ring of integers modulo n is denoted by Z_n and $C_n(0)$ is another ring with the same elements and addition operation, but with the trivial multiplication. The opposite of a ring $(R, +, \cdot)$ is the ring $(R, +, *)$, whose multiplication " $*$ " is defined by $a * b = ba$. If Γ is a graph and $\Pi = \{P_1, P_2, \cdots, P_r\}$ is a partition of $V(\Gamma)$ then the quotient graph $\frac{\Gamma}{\Pi}$ is defined as follows:

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$$V(\tfrac{\Gamma}{\Pi}) = \Pi \quad \text{and} \quad E(\tfrac{\Gamma}{\Pi}) = \{P_iP_j \mid \exists v \in P_i \ \exists v^\star \in P_j \ s.t. \ vv^\star \in E(\Gamma)\}.$$

Suppose G is a labeled graph with $V(G) = \{x_1, ..., x_n\}$ and $\Gamma_1, \cdots, \Gamma_n$ are arbitrary graphs with disjoint vertex sets. An extended join of $\Gamma_1, \cdots, \Gamma_n$ by G is defined as follows:

$$\left(\biguplus_{i=1}^{n} \Gamma_i\right)_G = \bigcup_{x_rx_s \in E(G)} \Gamma_r + \Gamma_s.$$

It is clear that when $G = K_2$ the extended joint of graphs Γ_1 and Γ_2 by G is the ordinary join of graphs. If we use the $(n+1)-$vertex path P_n as G then the extended join of graphs $\Gamma_1, \cdots, \Gamma_{n+1}$ by P_n is called the sequential join of these graphs. The corona product of two graphs G and H is the disjoint union of one copy of G and $|V(G)|$ copies of H in such a way that each vertex of the copy of G is connected to all vertices of its corresponding copy of H [7]. Finally, for a subset A of a ring R, $A^\star$ denotes the set of nonzero elements of A. For concepts and notations not presented here, we refer to [8, 10].

2. Main Results

The aim of this section is to present a complete classification of graphs, which can be represented as zero divisor graphs of finite rings of cubefree order. For the sake of completeness, we mention here [6, Theorem 2], [6, Corollary 3] and a characterization theorem on finite rings [9] which are crucial throughout this paper.

Theorem 1. (See [6, Theorem 2]) For any prime p there are, up to isomorphism, exactly 11 rings of order p^2 with the following presentations:

1. $A = \langle a \mid p^2a = 0, a^2 = a\rangle$,
2. $B = \langle a \mid p^2a = 0, a^2 = pa\rangle$,
3. $C = \langle a \mid p^2a = 0, a^2 = 0\rangle$,
4. $D = \langle a, b \mid pa = pb = 0, a^2 = a, b^2 = b, ab = ba = 0\rangle$,
5. $E = \langle a, b \mid pa = pb = 0, a^2 = a, b^2 = b, ab = a, ba = b\rangle$,
6. $F = \langle a, b \mid pa = pb = 0, a^2 = a, b^2 = b, ab = b, ba = a\rangle$,
7. $G = \langle a, b \mid pa = pb = 0, a^2 = 0, b^2 = b, ab = a, ba = a\rangle$,
8. $H = \langle a, b \mid pa = pb = 0, a^2 = 0, b^2 = b, ab = ba = 0\rangle$,
9. $I = \langle a, b \mid pa = pb = 0, a^2 = b, ab = 0\rangle$,
10. $J = \langle a, b \mid pa = pb = 0, a^2 = b^2 = 0\rangle$,
11. $K = GF(p^2)$ = The finite field of order p^2.

Theorem 2. (See [6, Corollary 3]) If $n = p_1 \cdots p_k$ is a square$-$free positive integer then up to isomorphism, there are exactly 2^k rings of order n. These are product rings in the form $R_1 \times R_2 \times \cdots \times R_k$ such that R_i is a ring of order p_i, its additive group is isomorphic to Z_{p_i} and its multiplication is either trivial or isomorphic to the integers modulo p_i.

Theorem 3. (See [9, Hilfssatz 1]) Every finite ring is isomorphic to a Cartesian product of rings of prime power order.

In the following theorem $Z(R)$ denotes the set of all zero devisors of R.

Theorem 4. Suppose R is a finite ring of order p^2. Then $\Gamma(R)$ is isomorphic to K_{p-1}, $K_{p-1} + \phi_{p^2-p}$, K_{p^2-1} or $K_{p-1,p-1}$.

Proof. Suppose R is a ring of order p^2. By Theorem 1, $R \cong A, B, C, D, E, F, G, H, I$ or J. Our main proof proceeds case by case as follows:

Case 1. $R \cong A$ *or* G. If $R \cong A$ then $Z(R) = \{0, p, 2p, \cdots, (p-1)p\}$ and so $\Gamma(R) \cong K_{p-1}$, as desired. Suppose $R \cong G$. Then by choosing $a = x + \langle x^2\rangle$ and $b = 1 + \langle x^2\rangle$ in the ring $\frac{Z_p[x]}{\langle x^2\rangle}$, one can see that

$$\frac{Z_p[x]}{\langle x^2\rangle} = \langle a, b \mid pa = pb = 0, a^2 = 0, b^2 = b, ab = a, ba = a\rangle.$$

This shows that $G \cong \frac{Z_p[x]}{\langle x^2\rangle}$. On the other hand, if $I = \langle x^2\rangle$ then

$$Z\left(\frac{Z_p[x]}{I}\right) = \{I, x+I, 2x+I, \cdots, (p-1)x+I\}.$$

Since $Z\left(\frac{Z_p[x]}{I}\right)$ is a commutative set with respect to multiplication, $G \cong K_{p-1}$.

Case 2. $R \cong B, E, F, H$ *or* I. We first assume that $R \cong B$. It is clear that $B \cong \langle p\rangle \triangleleft Z_{p^3}$. Notice that "$\triangleleft$" is a notation which denotes the ideals. Set $B_1 = \{p^2, 2p^2, \cdots, (p-1)p^2\}$ and $B_2 = \langle p\rangle \setminus B_1$. Suppose x and y are arbitrary elements of $Z(R)^\star = B_1 \cup B_2$. If $x, y \in B_1$ or $x \in B_1$ and $y \in B_2$ then $xy = 0$. Otherwise, $xy \neq 0$. Thus, $\Gamma(R) \cong K_{p-1} + \phi_{p^2-p}$.

We now assume that $R \cong F$. Define:

$$S = \left\{\begin{bmatrix} x & y \\ x & y \end{bmatrix} \mid x, y \in Z_p\right\}, a = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, b = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, F_1 = \left\{\begin{bmatrix} k & -k \\ k & -k \end{bmatrix} \mid k \in Z_p^\star\right\}.$$

One can prove that $F = \langle a, b \mid pa = pb = 0, a^2 = a, b^2 = b, ab = b, ba = a\rangle$ and for each element $a, b \in F$, $ab = 0$ if and only if $a, b \in F_1$ or $a \in F_1$ and $b \in F_2 = F \setminus F_1$, This shows that $\Gamma(R) \cong K_{p-1} + \phi_{p^2-p}$. On the other hand, $E \cong F^{op}$ and so $\Gamma(E) \cong \Gamma(F) \cong K_{p-1} + \phi_{p^2-p}$.

Next we assume that $R \cong H$. Notice that $H \cong Z_p \times C_p(0)$. Set $H_1 = \{(0, b) \mid b \in C_p(0), b \neq 0\}$ and $H_2 = \{(a, b) \mid a \in Z_p^\star, b \in C_p(0)\}$. Again, one can see that $ab = 0$ if and only if $a, b \in H_1$ or $a \in H_1$ and $b \in H_2$. Therefore, $\Gamma(R) \cong K_{p-1} + \phi_{p^2-p}$, as desired. Finally, suppose that $R \cong I$. Since $L = \{cx + dx^2 + \langle x^3\rangle \mid c, d \in Z_p\} \trianglelefteq \frac{Z_p[x]}{\langle x^3\rangle}$, by choosing $a = x + \langle x^3\rangle$ and $b = x^2 + \langle x^3\rangle$, we can see that $I \cong L$. Set $L_1 = \{kx^2 + \langle x^3\rangle \mid k \in Z_p^\star\}$ and $L_2 = L \setminus L_1$. Again, it is not so difficult to prove $\Gamma(I) \cong K_{p-1} + \phi_{p^2-p}$.

Case 3. $R \cong C$ *or* J. Suppose $R \cong C$. Then one can easily see that $R \cong C_{p^2}(0)$ and all distinct elements of R are adjacent in its zero divisor graph. Therefore, $\Gamma(R) \cong K_{p^2-1}$. If $R \cong J$ then $R \cong C_p(0) \times C_p(0)$ and by definition $\Gamma(R) \cong K_{p^2-1}$.

Case 4. $R \cong D$. Suppose $R \cong D \cong Z_p \times Z_p$. Define $D_1 = \{(r, 0) \mid r \in Z_p^\star\}$ and $D_2 = \{(0, s) \mid s \in Z_p^\star\}$. Then for each element $x, y \in D$, $xy = 0$ if and only if $x \in D_1$ and $y \in D_2$. Therefore, $\Gamma(R) \cong K_{p-1,p-1}$.

This completes the proof. □

Suppose R is a cubefree finite ring. Then by Theorem 3, R is isomorphic to a Cartesian product of rings of prime power order. Among rings of order p^2, $Z_p \times Z_p$, $Z_p \times C_p(0)$ and $C_p(0) \times C_p(0)$ are the only rings which are product of rings of order p. So, we can write $R \cong \prod_{i=1}^n R_i$, where for each i, $1 \leq i \leq n$, R_i is not isomorphic to three mentioned rings. Define:

$$\begin{aligned} N_1 &= \{1, 2, \ldots, n\}, \\ N_2 &= \{i \in N_1 \mid R_i \ncong Z_p, C_p(0)\}, \\ N_3 &= \{i \in N_1 \mid R_i \cong C\}, \\ N_4 &= \{i \in N_1 \mid \text{R}_\text{i} \text{ is not a field}\}, \\ N_5 &= \{i \in N_2 \mid R_i \cong B \text{ or } C \text{ or } E \text{ or } F \text{ or } I\}. \end{aligned}$$

The eccentricity of a vertex v, $\varepsilon(v)$, is the greatest distance between v and any other vertex and the minimum eccentricity among vertices of the graph is called its radius. A central vertex in a graph of radius r is one whose eccentricity is r. The center of the graph is defined as the set of all central vertices. We denote the center of a graph G, by $C(G)$. For each i, $1 \le i \le n$, we define three subsets T_i, $T_i^\star$ and S_i from R_i, as follows:

$$T_i^\star = \begin{cases} \Omega & R_i \cong GF(p^2) \text{ or } C \\ Z(R_i)^\star & R_i \cong A \text{ or } G \\ C(\Gamma(R_i)) & R_i \cong B \text{ or } E \text{ or } F \text{ or } I \\ R_i^\star & R_i \cong Z_p \text{ or } C_p(0). \end{cases},$$

where Ω is a fixed subset of $R_i^\star$ of cardinality $p_i - 1$, $T_i = T_i^\star \cup \{0_{R_i}\}$ and $S_i = R_i \setminus T_i$. Here we can easily prove that $C(\Gamma(R_i)) = \mathrm{Nil}(R_i)^\star$ in which $\mathrm{Nil}(R_i)$ is the nil radical of R_i [8, p. 379]. On the other hand, for each $x = (x_1, \ldots, x_n) \in R$, $\mu_x = \{i \in N_2 \mid x_i \in S_i\}$. Define $x \sim y$ if and only if $\mu_x = \mu_y$, where $x, y \in R$. It is easy to see that $\sim$ is an equivalence relation. Moreover, we assume that $[x]$ denotes the equivalence class of x under $\sim$ and X is a set of representatives of the equivalence relation $\sim$.

Suppose that $x \in X$ and $\emptyset \neq \nu \subseteq N_1$. Set

$$[x]_\nu = \{y = (y_1, \ldots, y_n) \mid \mu_x = \mu_y \ \&\ y_i = 0 \text{ if and only if } i \notin \nu\}.$$

The induced subgraph of $\Gamma(R)$ generated by $[x]_\nu$ is denoted by $\Gamma([x]_\nu)$. For each $x_1, x_2 \in X$, $x_1 \neq x_2$, and for each ν_1, ν_2 such that $\nu_1, \nu_2 \subseteq N_1$, $\nu_1 \neq \nu_2$ and $\nu_1, \nu_2 \neq \emptyset$, we say $\{(x_1, \nu_1), (x_2, \nu_2)\}$ satisfies condition (P) if and only if

i) $\mu_{x_1} \cap \mu_{x_2} \subseteq N_3$;

ii) $(\nu_1 \setminus \mu_{x_1}) \cap (\nu_2 \setminus \mu_{x_2}) \subseteq N_4$;

iii) $\mu_{x_1} \cap (\nu_2 \setminus \mu_{x_2}), \mu_{x_2} \cap (\nu_1 \setminus \mu_{x_1}) \subseteq N_5$.

Finally, for each $x \in X$ and $\emptyset \neq \nu \subseteq N_1$, we say that the pair (x, ν) satisfies $Q_{x,\nu}$ (or $(x, \nu) \in E_{x,\nu}$) if and only if $[\nu \subset N_1$ and $\mu_x \subseteq \nu]$ or $[\nu = N_1$ and $(\mu_x \cap N_5) \cup ((\nu \setminus \mu_x) \cap N_4) \neq \emptyset]$. For simplicity of our argument,

Lemma 5. $V(\Gamma(R)) = \bigcup_{x \in X,\ \emptyset \neq \nu \subseteq N_1,\ (x,\nu) \in E_{x,\nu}} V(\Gamma([x]_\nu))$.

Proof. To simplify our argument, we define $W = \bigcup_{x \in X,\ \emptyset \neq \nu \subseteq N_1,\ (x,\nu) \in E_{x,\nu}} V(\Gamma([x]_\nu))$. Suppose $a \in V(\Gamma(R))$. Then there are $x \in X$ and $\emptyset \neq \nu \subseteq N_1$ such that $a \in [x]_\nu$. If $\nu \neq N_1$ then (x, ν) satisfies $Q_{x,\nu}$ and so $a \in W$. Assume that $\nu = N_1$. Since $a = (a_1, \ldots, a_n)$, for each i, $i \in \nu = N_1$, $a_i \neq 0$. On the other hand, $a \in V(\Gamma(R))$ implies that there exists $j \in N_1$ such that a_j is not unit. We claim that $j \in (\mu_x \cap N_5) \cup ((\nu \setminus \mu_x) \cap N_4)$. Suppose $j \notin \mu_x$. Since a_j is not unit, $j \in N_4$ and so $j \in (\nu \setminus \mu_x) \cap N_4$, as desired. If $j \in \mu_x$ and $j \notin \mu_x \cap N_5$ then $a_j \in S_j$. Since $j \notin N_5$, a_j is unit which is impossible.

Conversely, we assume that $a \in W$. Then there are $x \in X$ and $\emptyset \neq \nu \subseteq N_1$ such that $a \in V(\Gamma([x]_\nu))$. If $\nu \neq N_1$ then $(0, \ldots, 0, t, 0, \ldots, 0)$ is a non-zero zero divisor for a, where $j \in N_1 \setminus \nu$ and $0 \neq t \in R_j$. This shows that $a \in V(\Gamma(R))$. Next we assume that $\nu = N_1$. Since (x, ν) satisfies $Q_{x,\nu}$, $j \in (\mu_x \cap N_5) \cup ((\nu \setminus \mu_x) \cap N_4)$ exists. Since for each i, $i \in N_5$, the elements of S_i and T_i are zero divisors of each other, $j \in \mu_x \cap N_5$ implies that $(0, \ldots, 0, t, 0, \ldots, 0)$ is a non-zero zero divisor for a, where $0 \neq t \in T_j$. Since for $i \in N_4$, the elements of T_i are zero divisors of each other, $j \in (\nu \setminus \mu_x) \cap N_4$ implies that $(0, \ldots, 0, t, 0, \ldots, 0)$ is a non-zero zero divisor for a, where $t \in T_j$. This completes the proof. □

Lemma 6. There is a partition $\mathcal{P}$ such that $\frac{\Gamma(R)}{\mathcal{P}}$ is isomorphic to a graph Λ such that

$$\begin{aligned} V(\Lambda) &= \{(x, \nu) \mid x \in X, \emptyset \neq \nu \subset N_1, \mu_x \subseteq \nu\} \cup \{(x, N_1) \mid x \in X, (\mu_x \cap N_5) \cup ((N_1 \setminus \mu_x) \cap N_4) \neq \emptyset\}, \\ E(\Lambda) &= \{(x_1, \nu_1)(x_2, \nu_2) \mid (x_1, \nu_1), (x_2, \nu_2) \in V(G), P \text{ is satisfied}\}, \\ \mathcal{P} &= \{[x]_\nu \mid x \in X, \emptyset \neq \nu \subseteq N_1, Q_{x,\nu} \text{ is satisfied}\}. \end{aligned}$$

Proof. By Lemma 5, the mapping $f : \frac{\Gamma(R)}{\mathcal{P}} \longrightarrow \Lambda$ which sends $[x]_\nu$ to (x, ν) is an isomorphism. So, $\Lambda \cong \frac{\Gamma(R)}{\mathcal{P}}$ which proves the theorem. □

Lemma 7. For each $x \in X$ and $\emptyset \neq \nu \subseteq N_1$ which satisfy the condition $Q_{x,\nu}$, we have:

$$\Gamma([x]_\nu) = \begin{cases} K_{(\prod_{i\in\mu_x} p_i \times \prod_{i\in\nu}(p_i-1))} & \emptyset \neq \mu_x \subseteq N_3 \text{ and } \nu \subseteq N_4 \\ K_{\prod_{i\in\nu}(p_i-1)} & \mu_x = \emptyset \text{ and } \nu \subseteq N_4 \\ \phi_{(\prod_{i\in\mu_x} p_i \times \prod_{i\in\nu}(p_i-1))} & \mu_x \neq \emptyset \text{ and } [\mu_x \nsubseteq N_3 \text{ or } \nu \nsubseteq N_4] \\ \phi_{\prod_{i\in\nu}(p_i-1)} & \text{Otherwise.} \end{cases}$$

Proof. Consider two arbitrary elements $y = (y_1, \dots, y_n), y' = (y'_1, \dots, y'_n) \in [x]_\nu$. Obviously, $\mu_y = \mu_{y'} = \mu_x$. Suppose that $\mu_x \subseteq N_3, \nu \subseteq N_4$. Thus, for every $i \in \mu_x$, $y_i, y'_i \in C$ and so $y_i y'_i = 0$. If $i \in \nu \setminus \mu_x$ then $y_i, y'_i \in T_i$ and $\Gamma(T^\star)$ is an induced subgraph of $\Gamma(R_i)$ isomorphic to K_{p_i-1}. So, again $y_i y'_i = 0$. Finally, if $i \in N_1 \setminus \nu$ then $y_i = y'_i = 0$ and so $y_i y'_i = 0$. Therefore, $y_i y'_i = 0$ and $\Gamma([x]_\nu)$ is a complete graph. If $\mu_x \neq \emptyset$ then and we have:

$$\begin{aligned} |[x]_\nu| &= \prod_{i\in\mu_x} |S_i| \times \prod_{i\in\nu\setminus\mu_x} |T_i^\star| \\ &= \prod_{i\in\mu_x} (p_i^2 - p_i) \times \prod_{i\in\nu\setminus\mu_x} (p_i - 1) \\ &= \prod_{i\in\mu_x} p_i \times \prod_{i\in\mu_x} (p_i - 1) \times \prod_{i\in\nu\setminus\mu_x} (p_i - 1) \\ &= \prod_{i\in\mu_x} p_i \times \prod_{i\in\nu} (p_i - 1). \end{aligned}$$

If $\mu_x = \emptyset$ then $|[x]_\nu| = \prod_{i\in\nu}(p_i - 1)$. If $\mu_x \nsubseteq N_3$ then there exists $i \in \mu_x$ such that $i \notin N_3$. Hence there are $y_i, y'_i \in S_i$ such that $y_i, y'_i \notin C$ and so $y_i y'_i \neq 0$. This shows that $yy' \neq 0$. If $\nu \nsubseteq N_4$ then there exists $i \in \nu$ such that $i \notin N_4$. By our notation, R_i is a field and $y_i, y'_i \in R_i$. So, $i \in \nu$ implies that $y_i y'_i \neq 0$. Again $yy' \neq 0$ and $\Gamma([x]_\nu) = \phi_{|[x]_\nu|}$, which completes the proof. □

Theorem 8. Suppose R is a cubefree order ring. Then,

$$\Gamma(R) = \biguplus_{\substack{x,x'\in X,\ \emptyset\neq\nu,\nu'\subseteq N_1,\ (x,\nu)(x',\nu')\in E(\Lambda), \\ Q_{x,\nu}, Q_{x',\nu'} \text{ and } (P) \text{ are satisfied}}} \Big(\Gamma([x]_\nu) + \Gamma([x']_{\nu'})\Big)_\Lambda. \quad (1)$$

Proof. Suppose L denotes the right hand side graph of the Equation 1. We first prove that $V(L) = V(\Gamma(R))$. Clearly, $V(L) \subseteq V(\Gamma(R))$ and so it is enough to show that $V(\Gamma(R)) \subseteq V(L)$. Suppose $y = (y_1, \dots, y_n) \in V(\Gamma(R))$. Then there exists $\emptyset \neq \nu \subseteq N_1$ such that $y_i \neq 0$ if and only if $i \in \nu$. We can also find a subset μ of N_2 such that $y_i \in S_i$ if and only if $i \in \mu$. Therefore, there exists $x \in X$ such that $\mu_x = \mu_y = \mu$, as desired.

We now prove that $E(\Gamma(R)) = E(L)$. Suppose $yy' \in E(\Gamma(R))$, $y = (y_1, \dots, y_n)$ and $y' = (y'_1 \dots, y'_n)$. By definition of $E(\Gamma(R))$, for each $i \in N_1$, $y_i y'_i = 0$. Since $y, y' \in V(\Gamma(R))$, there are $\emptyset \neq \nu, \nu' \subseteq N_1$ such that $i \in \nu$ if and only if $y_i \neq 0$, and $j \in \nu'$ if and only if $y'_j \neq 0$. We first assume that $\nu \neq \nu'$. By definition of $V(\Gamma(R))$, there are $x, x' \in X$ such that $\mu_y = \mu_x$ and $\mu_{y'} = \mu_{x'}$. This shows that $y \in [x]_\nu$ and $y' \in [x']_{\nu'}$. Since for each $i \in N_1$, $y_i y'_i = 0$, $\mu_x \cap \mu_{x'} \subseteq N_3$. If $(\nu \setminus \mu_x) \cap (\nu' \setminus \mu_{x'}) \nsubseteq N_4$ then there exists $j \in \nu \setminus \mu_x$ such that $j \notin N_4$. Therefore, by definition of N_4, R_j is a field. Now $y_j y'_j = 0$ implies that $y_j = 0$ or $y'_j = 0$, which is impossible. Thus $(\nu \setminus \mu_x) \cap (\nu' \setminus \mu_{x'}) \subseteq N_4$. Next we prove that $\mu_x \cap (\nu' \setminus \mu_{x'}) \subseteq N_5$. Suppose $i \in \mu_x \cap (\nu' \setminus \mu_{x'})$. Hence $y'_i \in T_i$. Again from the equation $y_i y'_i = 0$ we deduce that $i \in N_5$. In a similar way, $\mu_{x'} \cap (\nu \setminus \mu_x) \subseteq N_5$. Therefore, $yy' \in E(L)$. If $\nu = \nu'$ and $x \neq x'$ then a similar argument as above shows that $yy' \in E(L)$. Assume that $y, y' \in [x]_\nu$, for some $x \in X$ and $\emptyset \neq \nu \subseteq N_1$. Since $y_i y'_i = 0$, $i \in N_1$, we have $\mu_x \subseteq N_3$ and $\nu \subseteq N_4$. By Lemma 7, $\Gamma([x]_\nu)$ is a complete graph and so $yy' \in E(L)$. Conversely, we assume that $ab \in E(L)$. Put $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$. Then there are $x, x' \in X$ and $\emptyset \neq \nu, \nu' \subseteq N_1$, such that $a \in V(\Gamma([x]_\nu))$ and $b \in V(\Gamma([x']_{\nu'}))$. Our main proof will consider four cases as follows:

a. $x = x'$ *and* $\nu = \nu'$. Suppose $a, b \in V(\Gamma([x]_\nu))$. Then $\Gamma([x]_\nu)$ is a complete graph and so $\mu_x \subseteq N_3$, $\nu \subseteq N_4$. Since $\mu_x \subseteq N_3$, $a_i b_i = 0$, for each $i \in \mu_x$. If $i \in \nu \setminus \mu_x$ then $a_i, b_i \in T_i$. But T_i is not a subset of any field, so $a_ib_i = 0$, for each $i \in \nu \setminus \mu_x$. On the other hand, for any $i \in N_1 \setminus \nu$, we have $a_i = b_i = 0$ which implies that $a_ib_i = 0$. Hence $ab \in E(\Gamma(R))$, as desired.

b. $x \neq x'$ *and* $\nu = \nu'$. If $i \in \mu_x \setminus \mu_{x'}$ then the inclusion $\mu_{x'} \cap (\nu \setminus \mu_x) \subseteq N_5$ shows that $a_i \in T_i$ and $b_i \in S_i$. Since the elements of S_i and T_i are zero divisors of each other, $a_ib_i = 0$. We now assume that $i \in \mu_x \cap \mu_{x'}$. Then $\mu_x \cap \mu_{x'} \subseteq N_3$ and so $a_i, b_i \in C$. Hence $a_ib_i = 0$. If $i \in \nu \setminus \mu_x$ then we have two cases that $i \in \mu_{x'}$ or $i \notin \mu_{x'}$. In the first case, the inclusion $\mu_{x'} \cap (\nu \setminus \mu_x) \subseteq N_5$ proves that $a_i \in T_i$ and $b_i \in S_i$. Thus $a_ib_i = 0$. In the later, the inclusion $(\nu \setminus \mu_x) \cap (\nu \setminus \mu_{x'}) \subseteq N_4$ proving that $a_ib_i = 0$. Finally, if $i \notin \nu$ then $a_i = b_i = 0$ and hence $a_ib_i = 0$ which completes this part.

c. $x = x'$ *and* $\nu \neq \nu'$. We consider four subcases that $i \in \mu_x$, $i \in (\nu \setminus \mu_x) \cap (N_1 \setminus \nu')$, $i \in (\nu \setminus \mu_x) \cap \nu'$ or $i \notin \nu$. In the first subcase, $\mu_x = \mu_{x'} \subseteq N_3$ and so $a_i, b_i \in C$ which implies that $a_ib_i = 0$. In the second and forth subcases, $b_i = 0$ and $a_i = 0$, respectively, and so $a_ib_i = 0$. Finally, in the third subcase, the inclusion $(\nu \setminus \mu_x) \cap (\nu' \setminus \mu_x) \subseteq N_4$ deduces $a_ib_i = 0$, which completes this part.

d. $x \neq x'$ *and* $\nu \neq \nu'$. By a similar argument as Cases a-c, we can conclude this part.
This completes our argument. □

We end this paper by determining the zero divisor graph of all finite rings of order p^2q, where p and q are distinct primes.

Corollary 9. Suppose R is a finite ring of order p^2q, where p and q are distinct primes. Then $\Gamma(R)$ is isomorphic to one of the following graphs:

1. K_{p^2q-1},
2. $K_{p^2-1,q-1}$,
3. $K_{pq-1} + \phi_{pq(p-1)}$,
4. $K_{q-1} + \phi_{q(p^2-1)}$,
5. $K_{p^2-1} + \phi_{p^2(q-1)}$,
6. $K_{q(p-1)} + K_{q-1} + \phi_{pq(p-1)}$,
7. $K_{q(p-1),q(p-1)} + K_{q-1} + \phi_{q(p-1)^2}$,
8. $K_{p(p-1),p(q-1)} + K_{p-1} + \phi_{p(p-1)(q-1)}$,
9. $\phi_{(p-1)(q-1)} + K_{p-1} + \phi_{q-1} + \phi_{p(p-1)}$,
10. $\left(\phi_{(p-1)(q-1)} \uplus \phi_{p-1} \uplus \phi_{q-1} \uplus \phi_{(p-1)^2} \uplus \phi_{p-1} \uplus \phi_{(p-1)(q-1)}\right)_{G_1}$, where G_1 is the corona product of a triangle by K_1 in such a way that two copies of $\phi_{(p-1)(q-1)}$ and a copy of $\phi_{(p-1)^2}$ are corresponding to vertices of three copies of K_1. Moreover, two copies of $\phi_{(p-1)(q-1)}$ are adjacent to ϕ_{p-1}.

Proof. Apply Theorems 1, 3 and 8. □

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