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A General Rational Sum Identity

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Abstract. In this paper, by means of divided differences and an inverse pair formula we present a general rational sum identity which generalizes some identities of Chu-Yan, Prodinger, Mansour-Shattuck-Song and Ismail-Stanton.

1. Introduction

In the article [8], Díaz-Barrero et al. obtained two identities involving rational sums:

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\binom{x+k}{k}^{-1}\sum_{1\le i\le j\le k}\frac{1}{x^2+(i+j)x+ij}=\frac{n}{(x+n)^3}, \tag{1}$$

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\binom{x+k}{k}^{-1}\left\{\sum_{j=1}^{k}\frac{1}{(x+j)^3}+\sum_{1\le i\le j\le k}\frac{1}{(x+i)(x+j)(2x+i+j)}+\sum_{1\le i<j<l\le k}\frac{1}{(x+i)(x+j)(x+l)}\right\}=\frac{n}{(x+n)^4}. \tag{2}$$

Eq. (1.1) includes Díaz-Barrero's result in [7] as a special case $x=0$ which states that

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\sum_{1\le i\le j\le k}\frac{1}{ij}=\frac{1}{n^2}. \tag{3}$$

Recently, Prodinger [13] made use of partial fraction decomposition [4] and inverse pairs and presented a more general formula:

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\binom{x+k}{k}^{-1}\sum_{l_1+2l_2+\cdots=l}\prod_{i\ge1}\frac{s_{k,i}^{l_i}}{l_i!i^{l_i}}=\frac{n}{(x+n)^{l+1}}, \tag{4}$$

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where $s_{k,i}=\sum_{j=1}^{k}(x+j)^{-i}$. Almost at the same time, Chu and Yan [3] employed binomial inversions to gave a more general identities of (1.3) with multiple l-fold sum:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x+k}{k}^{-1}\sum_{0\le j_1\le\cdots\le j_l\le k}\prod_{i=1}^{l}\frac{1}{x+j_i}=\frac{x}{(x+n)^{l+1}}. \tag{5}$$

A direct proof of (1.5) was also given by Chu [2]. For other generalizations of Díaz-Barrero's result by using integral method, one is referred to [15]. More recently, Mansour et al. [12] provided a q-analog for the rational sum identity (1.4):

$$\sum_{k=1}^{n}(-1)^{k-1}q^{\binom{k}{2}-k(n-1)}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}x+k\\k\end{bmatrix}_q^{-1}\sum_{l_1+2l_2+\cdots=l}\prod_{i\ge1}\frac{s_{k,i}(q)^{l_i}}{l_i!i^{l_i}}=\frac{q^{nl}[n]_q}{[x+n]_q^{l+1}}, \tag{6}$$

where $s_{k,i}(q)=\sum_{j=1}^{k}q^{ij}[x+j]_q^{-i}$. In particular, they gave a very nice bijective proof for the case $l=1$. For more generalizations of (1.1)-(1.3), one is referred to [18, 19]. By means of the technique of summations theorems for hypergeometric series [10, 14, 16, 17], Eqs. (1.1)-(1.3) were derived systematically.

Motivated by these interesting work, this paper will be devoted to a more general rational sum identity that includes all of the identities presented above as a special case. Our main tools are divided differences and inverse pairs.

Throughout this paper , we will use the standard notation

$$[n]_q=1+q+\cdots+q^{n-1}=\frac{1-q^n}{1-q},\ [n]_q!=[1]_q[2]_q\cdots[n]_q,$$
$$\begin{bmatrix}n\\k\end{bmatrix}_q=\frac{[n]_q!}{[k]_q![n-k]_q!},\ (x;q)_n=\prod_{i=0}^{n-1}(1-xq^i),$$

and by convention empty products take the value 1 and empty sums take the value 0.

2. Main Results

In this section, let us first recall that divided differences as the coefficients of the Newton interpolating polynomial have played an important role in numerical analysis, especially in interpolation and approximation by polynomials and in spline theory, see [6] for a recent survey. They also have many applications in combinatorics [1, 21–23].

Let $\Delta(a_0,a_1,\ldots,a_n)f(\cdot)$ denote the n-th divided difference of a function $f(x)$ at the points $a_0,a_1,\ldots,a_n$. It is well known that for the distinct points $a_0,a_1,\ldots,a_n$, the divided differences of the function f are defined recursively by the following formula:

$$\Delta(a_0)f(\cdot)=f(a_0),$$
$$\Delta(a_0,a_1,\ldots,a_n)f(\cdot)=\frac{\Delta(a_0,a_1,\ldots,a_{n-1})f(\cdot)-\Delta(a_1,a_2,\ldots,a_n)f(\cdot)}{a_0-a_n},\quad n=1,2,\ldots. \tag{7}$$

From (2.1) the divided differences can be expressed by the explicit formula

$$\Delta(a_0,a_1,\ldots,a_n)f(\cdot)=\sum_{i=0}^{n}\frac{f(a_i)}{\prod_{j=0,\ne i}^{n}(a_i-a_j)}, \tag{8}$$

which can be shown by induction. From the above expression one sees that the divided differences are symmetric functions of their arguments. If $f(x)=x^j$ for $0\le j\le n$, then

$$\Delta(a_0,a_1,\ldots,a_n)(\cdot)^j=\delta_{n,j},$$

where $\delta_{p,q}$ is defined as

$$\delta_{p,q} = \begin{cases} 1, & p = q, \\ 0, & p \neq q. \end{cases}$$

Let $h(x) = f(x)g(x)$. If f and g are sufficiently smooth functions, then for arbitrary points $a_0, a_1, \ldots, a_n$, we have

$$\Delta(a_0, a_1, \ldots, a_n)h(\cdot) = \sum_{i=0}^{n} \Delta(a_0, a_1, \ldots, a_i)f(\cdot)\Delta(a_i, a_{i+1}, \ldots, a_n)g(\cdot). \tag{9}$$

This is called the Steffensen formula [20] (see also [22]). Furthermore, considering the multiplication of the m functions $\varphi_1, \varphi_2, \ldots, \varphi_m$, the Steffensen formula can be generalized. If φ_i ($i = 1, 2, \ldots, m$) are sufficiently smooth functions, then for arbitrary points $a_0, a_1, \ldots, a_n$, we have

$$\Delta(a_0, a_1, \ldots, a_n)h(\cdot) = \sum_{0=i_0 \le i_1 \le \cdots \le i_m = n} \prod_{k=0}^{m-1} \Delta(a_{i_k}, a_{i_k+1} \ldots, a_{i_{k+1}})\varphi_{k+1}(\cdot), \tag{10}$$

where $h(x) = \prod_{i=1}^{m} \varphi_i(x)$.

Now, let us consider the following lemma.

Lemma 2.1. *If the sequence $\{a_k\}_{k\ge 0}$ are distinct, then for $n \ge 0$ there holds*

$$f_n = \sum_{k=0}^{n} g_k \frac{\prod_{i=0}^{n-1}(a_n - a_i)}{\prod_{i=0,\neq k}^{n}(a_k - a_i)} \Leftrightarrow g_n = \sum_{k=0}^{n} f_k \prod_{i=0}^{k-1} \frac{a_n - a_i}{a_k - a_i}. \tag{11}$$

Proof. First we will prove an equivalent form of (2.5) as follows

$$f_n = \sum_{k=0}^{n} g_k \frac{1}{\prod_{i=0,\neq k}^{n}(a_k - a_i)} \Leftrightarrow g_n = \sum_{k=0}^{n} f_k \prod_{i=0}^{k-1}(a_n - a_i). \tag{12}$$

Substituting the second equality into the right hand side of the first equality yields

$$\begin{aligned}
\sum_{k=0}^{n} g_k \frac{1}{\prod_{i=0,\neq k}^{n}(a_k - a_i)} &= \sum_{k=0}^{n} \frac{1}{\prod_{i=0,\neq k}^{n}(a_k - a_i)} \sum_{j=0}^{k} f_j \prod_{i=0}^{j-1}(a_k - a_i) \\
&= \sum_{j=0}^{n} f_j \sum_{k=j}^{n} \frac{\prod_{i=0}^{j-1}(a_k - a_i)}{\prod_{i=0,\neq k}^{n}(a_k - a_i)} = \sum_{j=0}^{n} f_j \sum_{k=j}^{n} \frac{1}{\prod_{i=j,\neq k}^{n}(a_k - a_i)} \\
&= f_n.
\end{aligned}$$

The last equality holds because

$$\sum_{k=j}^{n} \frac{1}{\prod_{i=j,\neq k}^{n}(a_k - a_i)} = \Delta(a_j, a_{j+1}, \ldots, a_n)e(\cdot) = \delta_{n,j},$$

where the function $e(x) \equiv 1$.

On the other hand, substituting the first equality into the right hand side of the second equality yields

$$\begin{aligned}
\sum_{k=0}^{n} f_k \prod_{i=0}^{k-1}(a_n - a_i) &= \sum_{k=0}^{n} \prod_{i=0}^{k-1}(a_n - a_i) \sum_{j=0}^{k} g_j \frac{1}{\prod_{i=0,\neq j}^{k}(a_j - a_i)} \\
&= \sum_{j=0}^{n} g_j \sum_{k=j}^{n} \frac{\prod_{i=0}^{k-1}(a_n - a_i)}{\prod_{i=0,\neq j}^{k}(a_j - a_i)}.
\end{aligned}$$

Write

$$A(n,j)=\sum_{k=j}^{n}\prod_{i=0}^{k-1}(a_n-a_i)\prod_{i=k+1}^{n}(a_j-a_i).$$

If $j=n$, it is obvious that $A(n,j)=\prod_{i=0}^{n-1}(a_n-a_i)$. If $j<n$, then

$$\begin{aligned}A(n,j)&=\prod_{i=0}^{j-1}(a_n-a_i)\prod_{i=j+1}^{n}(a_j-a_i)+\sum_{k=j+1}^{n}\prod_{i=0}^{k-1}(a_n-a_i)\prod_{i=k+1}^{n}(a_j-a_i)\\&=\prod_{i=0,\neq j}^{j+1}(a_n-a_i)\prod_{i=j+2}^{n}(a_j-a_i)+\sum_{k=j+2}^{n}\prod_{i=0}^{k-1}(a_n-a_i)\prod_{i=k+1}^{n}(a_j-a_i)\\&=\prod_{i=0,\neq j}^{j+2}(a_n-a_i)\prod_{i=j+3}^{n}(a_j-a_i)+\sum_{k=j+3}^{n}\prod_{i=0}^{k-1}(a_n-a_i)\prod_{i=k+1}^{n}(a_j-a_i)\\&=\cdots\\&=\prod_{i=0,\neq j}^{n-1}(a_n-a_i)(a_j-a_n)+\prod_{i=0}^{n-1}(a_n-a_i)\\&=0.\end{aligned}$$

Thus, this implies that

$$\sum_{k=j}^{n}\frac{\prod_{i=0}^{k-1}(a_n-a_i)}{\prod_{i=0,\neq j}^{k}(a_j-a_i)}=\delta_{n,j}.$$

Replacing f_n by $\frac{f_n}{\prod_{i=0}^{n-1}(a_n-a_i)}$ we arrive at (2.5). □

Remark 2.2. *For $n\geq 1$, this inverse pair formula can be written alternatively as*

$$f_n=\sum_{k=1}^{n}g_k\frac{\prod_{i=0}^{n-2}(a_{n-1}-a_i)}{\prod_{i=0,\neq k-1}^{n}(a_{k-1}-a_i)}\Leftrightarrow g_n=\sum_{k=1}^{n}f_k\prod_{i=0}^{k-2}\frac{a_{n-1}-a_i}{a_{k-1}-a_i}.$$

Making use of Lemma 2.1, we can obtain the following theorem.

Theorem 2.3. *If the sequence $\{a_k\}_{k\geq 0}$ are distinct, then for $l\geq 1$ there holds*

$$\frac{1}{(x+a_n)^{l+1}}=\sum_{k=0}^{n}\frac{\prod_{i=0}^{k-1}(a_i-a_n)}{\prod_{i=0}^{k}(x+a_i)}\sum_{0\leq i_1\leq\cdots\leq i_l\leq k}\prod_{j=1}^{l}\frac{1}{x+a_{i_j}}. \qquad (13)$$

Proof. Let $g_k=\frac{1}{(x+a_k)^{l+1}}$ in Lemma 2.1. There holds

$$f_n=\sum_{k=0}^{n}\frac{1}{(x+a_k)^{l+1}}\frac{\prod_{i=0}^{n-1}(a_n-a_i)}{\prod_{i=0,\neq k}^{n}(a_k-a_i)}=\prod_{i=0}^{n-1}(a_n-a_i)\Delta(a_0,a_1,\ldots,a_n)\left(\frac{1}{x+\cdot}\right)^{l+1}.$$

By the recurrence of divided differences, it is easy to obtain

$$\Delta(a_0,a_1,\ldots,a_k)\left(\frac{1}{x+\cdot}\right)=\frac{(-1)^k}{\prod_{i=0}^{k}(x+a_i)}.$$

Applying (2.4), we have

$$\Delta(a_0, a_1, \ldots, a_n)\left(\frac{1}{x+\cdot}\right)^{l+1} = \frac{(-1)^n}{\prod_{i=0}^{n}(x+a_i)} \sum_{0\le i_1\le\cdots\le i_l\le n} \prod_{j=1}^{l} \frac{1}{x+a_{i_j}},$$

which leads to

$$f_n = \frac{\prod_{i=0}^{n-1}(a_i - a_n)}{\prod_{i=0}^{n}(x+a_i)} \sum_{0\le i_1\le\cdots\le i_l\le n} \prod_{j=1}^{l} \frac{1}{x+a_{i_j}}.$$

In view of Lemma 2.1, the desired result is obtained. □

Remark 2.4. *Eq.* (2.7) *is expressed with multiple l-fold sum. Let* $u_i = (x+a_i)^{-1}, i = 0, 1, \ldots, k$. *It is not hard to verify*

$$\prod_{i=0}^{k} \frac{1}{1-u_i t} = \prod_{i=0}^{k} \sum_{j\ge 0} (u_i t)^j = \sum_{l\ge 0} t^l \sum_{0\le i_1\le\cdots\le i_l\le k} \prod_{j=1}^{l} u_{i_j}. \tag{14}$$

Considering the l-th derivative of $\prod_{i=0}^{k} \frac{1}{1-u_i t}$ *at* $t = 0$, *we have*

$$\frac{d^l}{dt^l} \prod_{i=0}^{k} \frac{1}{1-u_i t}\bigg|_{t=0} = \frac{d^l}{dt^l} e^{-\sum_{i=0}^{k} \log(1-u_i t)}\bigg|_{t=0}.$$

Applying Faà di Bruno's formula [5] yields

$$\frac{d^l}{dt^l} \prod_{i=0}^{k} \frac{1}{1-u_i t}\bigg|_{t=0} = Y_l(U_{k,1}(\boldsymbol{a}), U_{k,2}(\boldsymbol{a}), \ldots), \tag{15}$$

where the exponential complete Bell polynomials are defined as

$$Y_n(x_1, x_2, \ldots) = \sum_{l_1+2l_2+\cdots=n} \frac{n!}{l_1! l_2! \cdots} \left(\frac{x_1}{1!}\right)^{l_1} \left(\frac{x_2}{2!}\right)^{l_2} \cdots,$$

and

$$U_{k,i}(\boldsymbol{a}) = (i-1)! \sum_{j=0}^{k} u_j^i, \quad i = 1, 2, \ldots l.$$

Comparing (2.8) *with* (2.9), *there holds*

$$\sum_{0\le i_1\le\cdots\le i_l\le k} \prod_{j=1}^{l} u_{i_j} = \sum_{l_1+2l_2+\cdots=l} \frac{1}{l_1! l_2! \cdots} \left(\frac{s_{k,1}(\boldsymbol{a})}{1}\right)^{l_1} \left(\frac{s_{k,2}(\boldsymbol{a})}{2}\right)^{l_2} \cdots, \tag{16}$$

where

$$s_{k,i}(\boldsymbol{a}) = \sum_{j=0}^{k} u_j^i, \quad i = 1, 2, \ldots l.$$

Therefore, Eq. (2.7) *can be rewritten as an alternative formula:*

$$\frac{1}{(x+a_n)^{l+1}} = \sum_{k=0}^{n} \frac{\prod_{i=0}^{k-1}(a_i - a_n)}{\prod_{i=0}^{k}(x+a_i)} \sum_{l_1+2l_2+\cdots=l} \prod_{i\ge 1} \frac{s_{k,i}(\boldsymbol{a})^{l_i}}{l_i! i^{l_i}}. \tag{17}$$

Remark 2.5. *Eq.* (2.7) *contains Chu-Yan's result, i.e., Eq.* (1.5)*. If we take $a_k = k$ for $k = 0, 1, \dots, n$, we can arrive at* (1.5) *by simple calculations. Actually, Eq.* (2.7) *also contains Prondinger's identity as a special case because* (1.4) *and* (1.5) *are equivalent with each other. In Eq.* (1.5)*, if we replace n by $n-1$ and x by $x+1$, then we can retrieve* (1.4)*.*

Let $a_i = q^{-i}$ for $i = 0, 1, \dots, n$ in Eq. (2.7). By direct calculating we obtain a q-analog of Chu-Yan's identity.

Corollary 2.6. *For $l \geq 1$, there holds*

$$\frac{q^{n(l+1)}}{(1+xq^n)^{l+1}} = \sum_{k=0}^{n} (-1)^k q^{\binom{k+1}{2}-kn} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(q;q)_k}{(-x;q)_{k+1}} \sum_{0 \leq i_1 \leq \cdots \leq i_l \leq k} \prod_{j=1}^{l} \frac{q^{i_j}}{1+xq^{i_j}}. \tag{18}$$

If we replace x by $-q^x$, we can obtain an alternative formula of (2.12) as follows.

Corollary 2.7. *For $l \geq 1$, there holds*

$$\frac{q^{n(l+1)}[x]_q}{[x+n]_q^{l+1}} = \sum_{k=0}^{n} (-1)^k q^{\binom{k+1}{2}-kn} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} x+k \\ k \end{bmatrix}_q^{-1} \sum_{0 \leq i_1 \leq \cdots \leq i_l \leq k} \prod_{j=1}^{l} \frac{q^{i_j}}{[x+i_j]_q}. \tag{19}$$

Remark 2.8. *In fact, Eq.* (2.13) *is equivalent to Eq.* (1.6)*. If we replace n by $n-1$ and x by $x+1$ and use the relationship* (2.10)*, we immediately arrive at* (1.6)*.*

If we replace x by $-qx$ and n by $n-1$ in (2.12), then we find an identity which is equivalent to Ismail-Stanton's identity (see Theorem 2.2 in [11]).

Corollary 2.9. *For $l \geq 1$, there holds*

$$\frac{q^{nl}}{(1-xq^n)^{l+1}} = \sum_{k=1}^{n} (-1)^{k-1} q^{\binom{k}{2}-k(n-1)} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \frac{(q;q)_{k-1}}{(xq;q)_k} \sum_{1 \leq i_1 \leq \cdots \leq i_l \leq k} \prod_{j=1}^{l} \frac{q^{i_j}}{1-xq^{i_j}}. \tag{20}$$

Remark 2.10. *In [11], Ismail and Stanton use the theory of basic hypergeometric functions and generalize many identities. One of those important identities is stated as follows.*

$$\begin{aligned} &\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^{k-1} q^{\binom{k}{2}+kl} \frac{1-q^k}{(1-xq^k)^{l+1}} \\ &\quad = \frac{(q;q)_n}{(xq;q)_n} \sum_{j_1+j_2+\cdots+j_n=l} \prod_{i=1}^{n} \frac{q^{ij_i}}{(1-xq^i)^{j_i}}. \end{aligned} \tag{21}$$

This identity reduces to the well-known Dilcher identity [9] when $x = 1$. (2.14) *and* (2.15) *are equivalent because there hold*

$$f_n = \sum_{k=1}^{n} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q g_k \Leftrightarrow g_n = \sum_{k=1}^{n} (-1)^k q^{\binom{k}{2}-k(n-1)} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q f_k$$

and

$$\sum_{1 \leq i_1 \leq \cdots \leq i_l \leq n} \prod_{j=1}^{l} \frac{q^{i_j}}{1-xq^{i_j}} = \sum_{j_1+j_2+\cdots+j_n=l} \prod_{i=1}^{n} \frac{q^{ij_i}}{(1-xq^i)^{j_i}}.$$

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