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# Group Unified Coprodcuts and Related Quasitriangular Structures

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**Abstract.** For a group $\pi$, the objective of this paper is to construct a class of quasitriangular Hopf $\pi$-coalgebra. We first shall present the new tool called a group unified coprocut, followed by a classification result for $\pi$-unified coproducts in virtue of an algebra lazy 1-$\pi$-cycle which is the dual to that defined by Bichon and Kassel. Then, we discuss when a $\pi$-unified coproduct has a quasitriangular structure. Finally, some applications of our main results are considered.

## 1. Introduction

For a group $\pi$, Turaev[8] introduced the notion of a crossed $\pi$-category and showed that such a category gives rise to a three-dimensional homotopy quantum field theory with target space $K(\pi, 1)$. Virelizier[11] used crossed $\pi$-categories to construct Hennings-type invariants of principal $\pi$-bundles over complements of links in the 3-sphere. The crossed $\pi$-categories become quite delicate algebraic objects. Exploring a few general methods producing such categories is an interesting research subject. Recently, several new results are reported in constructions of crossed $\pi$-categories (cf. [10], [6], [14] and [15]).

Hopf $\pi$-coalgebra was introduced by [8] as the prototype algebraic structure. A systematic algebraic study of these new structures has been carried out in recent papers (cf.[5], [8, 9], [11], [12, 13] and [16]. etc). Quasitriangular semi-Hopf $\pi$-coalgebras (or Hopf $\pi$-coalgebras) are fundamental in the theory of Hopf $\pi$-coalgebras, which is a remarkable tool for constructing crossed $\pi$-categories and studying the quantum Yang-Baxter equation. Constructing a class of quasitriangular semi-Hopf $\pi$-coalgebras (or Hopf $\pi$-coalgebras) is the starting point of this paper.

In this paper, we shall introduce a new coproduct named $\pi$-unified coproduct as the generalization of unified coproduct introduced by the second author in [4] which is the dual of unified product introduced in [1] as an answer to the restricted extending structures problem for Hopf algebras. Then we discuss the its classification and quasitriangular stuctures.

The paper is organized as follows.

In Section 2, we recall some knowledge about Hopf $\pi$-coalgebras, quasitriangular semi-Hopf $\pi$-coalgebras (or Hopf $\pi$-coalgebras). We shall introduce the notion of a coextending $\pi$-datum and construct the $\pi$-unified coproduct in Section 3. In Section 4, we shall prove the classification theorem for $\pi$-unified coproduct.

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In Section 5, the notions of a generalized right (resp.left) dual compatible $\pi$-pair and a weak quasitriangular structure are constructed. It is shown that there exists an equivalence between the set of all quasitriangular stuctures on the $\pi$-unified coproduct $A \rtimes^{\pi} H$ and the set of all quadruples $(P, Q, U, V)$ satisfying some compatibility conditions. Some applications of our results are discussed in Section 6.

## 2. Preliminaries

Throughout the article, we let $\pi$ be a discrete group with unit $e$ and $k$ a field. All (co)algebras are supposed to be over $k$. All maps are $k$-linear. The tensor product $\otimes = \otimes_k$ is always assumed to be over $k$. If $U$ and $V$ are vector spaces, $T_{U,V} : U \otimes V \to V \otimes U$ will denote the flip map.

### 2.1. *π-coalgebras*

We recall from [8] that a *π-coalgebra* is a family of $k$-spaces $C = \{C_\alpha\}_{\alpha\in\pi}$ together with a family of linear maps $\Delta = \{\Delta_{\alpha,\beta} : C_{\alpha\beta} \to C_\alpha \otimes C_\beta\}_{\alpha,\beta\in\pi}$ (called a *comultiplication*) and a linear map $\varepsilon_e : C_e \to k$ (called a *counit*) such that $\Delta$ is coassociative in the sense that, for all $\alpha, \beta \in \pi$,

$$(\Delta_{\alpha,\beta} \otimes id) \circ \Delta_{\alpha\beta,\gamma} = (id \otimes \Delta_{\beta,\gamma}) \circ \Delta_{\alpha,\beta\gamma},$$

$$(id \otimes \varepsilon_e) \circ \Delta_{\alpha,e} = id = (\varepsilon_e \otimes id) \circ \Delta_{e,\alpha}.$$

Following the Sweedler's notation for $\pi$-coalgebras, for any $\alpha, \beta \in \pi$ and $c \in C_{\alpha\beta}$, one writes

$$\Delta_{\alpha,\beta}(c) = c_{(1,\alpha)} \otimes c_{(2,\beta)}.$$

Note that $(C_e, \Delta_{e,e}, \varepsilon_e)$ is a coalgebra in the usual sense of the word

### 2.2. *Hopf π-coalgebras*

Recall from [8] that a *semi-Hopf π-coalgebra* is a $\pi$-coalgebra $H = (\{H_\alpha\}, \Delta, \varepsilon_e)$ such that the following conditions hold:

- Each $H_\alpha$ is an algebra with multiplication $m_\alpha$ and unit $1_\alpha \in H_\alpha$,
- For all $\alpha, \beta \in \pi$, $\Delta_{\alpha,\beta}$ and $\varepsilon_e : H_e \to k$ are algebra maps.

A semi-Hopf $\pi$-coalgebra $H = (\{H_\alpha, m_\alpha, 1_\alpha\}, \Delta, \varepsilon_e)$ is called a *Hopf π-coalgebra*, if there exists a family of linear maps $S = \{S_\alpha : H_\alpha \to H_{\alpha^{-1}}\}_{\alpha\in\pi}$ (called an *antipode*) such that

$$m_\alpha \circ (id \otimes S_{\alpha^{-1}}) \circ \Delta_{\alpha,\alpha^{-1}} = \varepsilon_e 1_\alpha = m_\alpha \circ (S_{\alpha^{-1}} \otimes id) \circ \Delta_{\alpha^{-1},\alpha}.$$

A crossed Hopf $\pi$-coalgebra is a Hopf $\pi$-coalgebra with a family of algebra isomorphisms $\varphi = \{\varphi_\beta : H_\alpha \to H_{\beta\alpha\beta^{-1}}\}_{\alpha,\beta\in\pi}$ satisfying the following conditions: for any $\alpha, \beta, \gamma \in \pi$,

- $\varphi$ is multiplicative, i.e., $\varphi_\beta \circ \varphi_\gamma = \varphi_{\beta\gamma}$. it follows that $\varphi_e|_{H_\alpha} = id$,
- $\varphi$ is compatible with $\Delta$, i.e.,
$$(\varphi_\beta \otimes \varphi_\beta) \circ \Delta_{\alpha,\gamma} = \Delta_{\beta\alpha\beta^{-1},\beta\gamma\beta^{-1}} \circ \varphi_\beta,$$
- $\varepsilon_e \circ \varphi_\beta = \varepsilon_e$.

*2.3. Quasitriangular Hopf $\pi$-coalgebras*

Recall from [8] that a quasitriangular Hopf $\pi$-coalgebra is a crossed Hopf $\pi$-coalgebra $H = (\{H_\alpha\}, \Delta, \varepsilon, S, \varphi)$ endowed with a family of elements $R = \{R_{\alpha,\beta} = R^{1,\alpha} \otimes R^{2,\beta} \in H_\alpha \otimes H_\beta\}_{\alpha,\beta\in\pi}$ such that

(Q1) For any $\alpha, \beta \in \pi$ and $x \in H_{\alpha\beta}$,

$$R_{\alpha,\beta} \cdot \Delta_{\alpha,\beta}(x) = \tau_{\beta,\alpha} \circ (\varphi_{\alpha^{-1}} \otimes id) \circ \Delta_{\alpha\beta\alpha^{-1},\alpha}(x) \cdot R_{\alpha,\beta},$$

where $\tau_{\beta,\alpha}$ denotes the flip map $H_\beta \otimes H_\alpha \to H_\alpha \otimes H_\beta$,

(Q2) For all $\alpha, \beta, \gamma \in \pi$,

$$(id \otimes \Delta_{\beta,\gamma})(R_{\alpha,\beta\gamma}) = (R_{\alpha,\gamma})_{1\beta 3} \cdot (R_{\alpha,\beta})_{12\gamma},$$

(Q3) For all $\alpha, \beta, \gamma \in \pi$,

$$(\Delta_{\alpha,\beta} \otimes id)(R_{\alpha\beta,\delta}) = [(id \otimes \varphi_{\beta^{-1}})(R_{\alpha,\beta\gamma\beta^{-1}})]_{1\beta 3} \cdot (R_{\beta,\gamma})_{\alpha 23},$$

(Q4) For all $\alpha, \beta, \gamma \in \pi$,

$$(\varphi_\beta \otimes \varphi_\beta)(R_{\alpha,\gamma}) = R_{\beta\alpha\beta^{-1},\beta\gamma\beta^{-1}},$$

where, for vector spaces $P, Q$ and $r = \sum_j p_j \otimes q_j$, we set $r_{12\gamma} = r \otimes 1_\gamma \in P \otimes Q \otimes H_\gamma$, $r_{\alpha 23} = 1_\alpha \otimes r \in H_\alpha \otimes P \otimes Q$ and $r_{1\beta 3} = \sum_j p_j \otimes 1_\beta \otimes q_j \in P \otimes H_\beta \otimes Q$. Note that $R_{e,e}$ is a quasitriangular structure for the Hopf algebra $H_e$. Let $(H, R)$ be a quasitriangular Hopf $\pi$-coalgebra. Then the following identities hold:

(a) $(\varepsilon_e \otimes id)(R_{e,\alpha})) = 1_\alpha = (id \otimes \varepsilon_e)(R_{\alpha,e})$,
(b) $R^{-1}_{\alpha,\beta} = (S_{\alpha^{-1}} \circ \varphi_\alpha \otimes id)(R_{\alpha^{-1},\beta})$.

## 3. $\pi$-unified Coproducts

In this section, we shall introduce the $\pi$-unified coproduct as the natural generalization of unified coproduct introduced by the second author in [4]. First, a coextending $\pi$-datum is introduced as follows.

**Definition 3.1.** *Let $A$ be a bialgebra. A coextending $\pi$-datum of $A$ is a system $\Omega(A) = (H, \rho, \varrho, \omega)$ where:*

*(i) $H = (\{H_\alpha, \Delta_{\alpha,\beta}, \varepsilon_e\}, \Delta, \varepsilon_e)$ is a family of vector spaces over the field $k$ such that, for any $\alpha \in \pi$, each $H_\alpha$ is an associative algebra with the unit $1_\alpha$, $(\{H_\alpha, \Delta_{\alpha,\beta}, \varepsilon_e\})$ is a $\pi$-coalgebra which is not necessarily coassociative, such that*

$$\varepsilon_e(hg) = \varepsilon_e(h)\varepsilon_e(g) \tag{3.1}$$

*for all $h, g \in H_e$.*

*(ii) A family linear maps $\rho = \{\rho_\alpha : H_\alpha \to H_\alpha \otimes A\}_{\alpha\in\pi}$, $\varrho = \{\varrho_\alpha : A \to H_\alpha \otimes A\}_{\alpha\in\pi}$, $\omega = \{\omega_{\alpha,\beta} : A \to H_\alpha \otimes H_\beta\}_{\alpha,\beta\in\pi}$ are morphisms of algebras such that the following conditions hold:*

$$\varepsilon_A(h_{\langle 0,\alpha\rangle})h_{\langle -1,\alpha\rangle} = h, \ \varepsilon_e(a_{[-1,e]})a_{[0,e]} = a, \tag{3.2}$$

$$\varepsilon_e(g_{\langle -1,e\rangle})g_{\langle 0,e\rangle} = \varepsilon_e(h)1_A, \ \varepsilon_A(a_{[0,\alpha]})a_{[-1,\alpha]} = \varepsilon_A(a)1_H, \tag{3.3}$$

$$\varepsilon_e(\omega_{e,\alpha}(a)^{(1,e)})\omega_{e,\alpha}(a)^{(2,\alpha)} = \varepsilon_A(a)1_\alpha = \omega_{\alpha,e}(a)^{(1,\alpha)}\varepsilon_e(\omega_{\alpha,e}(a)^{(2,e)}) \tag{3.4}$$

*for all $a \in A$, $g \in H_e$ and $h \in H_\alpha$ with $\alpha \in \pi$. Here we adopt the notations: $\rho_\alpha(h) = h_{\langle -1,\alpha\rangle} \otimes h_{\langle 0,\alpha\rangle}$, $\varrho_\alpha(a) = a_{[-1,\alpha]} \otimes a_{[0,\alpha]}$ and $\omega_{\alpha,\beta}(a) = \omega_{\alpha,\beta}(a)^{(1,\alpha)} \otimes \omega_{\alpha,\beta}(a)^{(2,\beta)}$ for all $a \in A$ and $h \in H_\alpha$ with $\alpha, \beta \in \pi$.*

Let $A$ be a bialgebra and $\Omega(A) = (H, \rho, \varrho, \omega)$ a coextending $\pi$-datum of $A$. We denote by $A \rtimes^\pi H$ a family of vector spaces $\{A \otimes H_\alpha\}_{\alpha\in\pi}$ together with the comultiplication

$$\bar{\Delta}_{\alpha,\beta}(a \rtimes h) = a_{(1)} \rtimes a_{(2)[-1,\alpha]}\omega_{\alpha,\beta}(a_{(3)})^{(1,\alpha)}{}_{\langle -1,\alpha\rangle}h_{(1,\alpha)\langle -1,\alpha\rangle} \otimes a_{(2)[0,\alpha]}\omega_{\alpha,\beta}(a_{(3)})^{(1,\alpha)}{}_{\langle 0,\alpha\rangle}h_{(1,\alpha)\langle 0,\alpha\rangle} \rtimes \omega_{\alpha,\beta}(a_{(3)})^{(2,\beta)}h_{(2,\beta)}. \tag{3.5}$$

**Definition 3.2.** *Let $A$ be a bialgebra and $\Omega(A) = (H, \rho, \varrho, \omega)$ a coextending $\pi$-datum of $A$. The object $A \rtimes^{\pi} H$ introduced above is called the $\pi$-unified coproduct of $A$ and $\Omega(A)$, if $A \rtimes^{\pi} H$ is a semi-Hopf $\pi$-coalgebra with the comultiplication (3.5), the counit*

$$\bar{\varepsilon}_e(a \rtimes h) = \varepsilon_A(a)\varepsilon_e(h) \tag{3.6}$$

*for all $e \in H_e$ and $a \in A$ and the algebra structure given by the tensor product of algebras, i.e.,*

$$(a \rtimes h)(b \rtimes g) = ab \rtimes hg \tag{3.7}$$

*for all $a, b \in A$ and $h, g \in H_\alpha$ with $\alpha \in \pi$. In this case, the coextending $\pi$-datum $\Omega(A) = (H, \rho, \varrho, \omega)$ is called a semi-Hopf $\pi$-coalgebra coextending structure of $A$. A crossed semi-Hopf $\pi$-coalgebra coextending structure of $A$ is a semi-Hopf $\pi$-coalgebra coextending structure $\Omega(A) = \{H, \rho, \varrho, \omega\}$ satisfying that $H$ has a crossing $\varphi$.*

**Theorem 3.3.** Let $A$ be a bialgebra and $\Omega(A) = (H, \rho, \varrho, \omega)$ a coextending $\pi$-datum of $A$. The following statements are equivalent:

(1) $A \rtimes^{\pi} H$ is a $\pi$-unified coproduct,

(2) The following conditions hold:

(i) $\Delta_{\alpha,\beta} : H_{\alpha\beta} \to H_\alpha \otimes H_\beta$ and $\varepsilon_e : H_e \to k$ are algebra maps,

(ii) For all $\alpha, \beta, \gamma \in \pi$ and $a, b \in A$, we have

(a1) $\omega_{\alpha,\beta}(b)^{(1,\alpha)}h_{(1,\alpha)} \otimes \omega_{\alpha,\beta}(b)^{(2,\beta)}h_{(2,\beta)}$
$= h_{(1,\alpha)}\omega_{\alpha,\beta}(b)^{(1,\alpha)} \otimes h_{(2,\beta)}\omega_{\alpha,\beta}(b)^{(2,\beta)}$, all $h \in H_{\alpha\beta}$,

(a2) $b_{[-1,\alpha]}h_{\langle -1,\alpha\rangle} \otimes b_{[0,\alpha]}h_{\langle 0,\alpha\rangle} = h_{\langle -1,\alpha\rangle}b_{[-1,\alpha]} \otimes h_{\langle 0,\alpha\rangle}b_{[0,\alpha]}$, all $h \in H_\alpha$,

(a3) $h_{(1,\alpha\beta)(1,\alpha)} \otimes h_{(1,\alpha\beta)(2,\beta)} \otimes h_{(2,\gamma)}$
$= h_{(1,\alpha)\langle -1,\alpha\rangle} \otimes \omega_{\beta,\gamma}(h_{(1,\alpha)\langle 0,\alpha\rangle})^{(1,\beta)}h_{(2,\beta\gamma)(1,\beta)} \otimes \omega_{\beta,\gamma}(h_{(1,\alpha)\langle 0,\alpha\rangle})^{(2,\gamma)}h_{(2,\beta\gamma)(2,\gamma)}$, all $h \in H_{\alpha\beta\gamma}$,

(a4) $h_{\langle -1,\alpha\rangle} \otimes h_{\langle 0,\alpha\rangle(1)} \otimes h_{\langle 0,\alpha\rangle(2)} = h_{\langle -1,\alpha\rangle\langle -1,\alpha\rangle} \otimes h_{\langle -1,\alpha\rangle\langle 0,\alpha\rangle} \otimes h_{\langle 0,\alpha\rangle}$, all $h \in H_\alpha$,

(a5) $h_{\langle -1,\alpha\beta\rangle(1,\alpha)} \otimes h_{\langle -1,\alpha\beta\rangle(2,\beta)} \otimes h_{\langle 0,\alpha\beta\rangle}$
$= h_{(1,\alpha)\langle -1,\alpha\rangle} \otimes h_{(1,\alpha)\langle 0,\alpha\rangle[-1,\beta]}h_{(2,\beta)\langle -1,\beta\rangle} \otimes h_{(1,\alpha)\langle 0,\alpha\rangle[0,\beta]}h_{(2,\beta)\langle 0,\beta\rangle}$,
all $h \in H_{\alpha\beta}$,

(a6) $a_{[-1,\alpha]} \otimes a_{[0,\alpha](1)} \otimes a_{[0,\alpha](2)} = a_{(1)[-1,\alpha]}a_{(2)[-1,\alpha]\langle -1,\alpha\rangle} \otimes a_{(1)[0,\alpha]}a_{(2)[-1,\alpha]\langle 0,\alpha\rangle} \otimes a_{(2)[0,\alpha]}$,

(a7) $\omega_{\alpha,\beta}(a_{(1)})^{(1,\alpha)}a_{(2)[-1,\alpha\beta](1,\alpha)} \otimes \omega_{\alpha,\beta}(a_{(1)})^{(2,\beta)}a_{(2)[-1,\alpha\beta](2,\beta)} \otimes a_{(2)[0,\alpha\beta]}$
$= a_{(1)[-1,\alpha]}[\omega_{\alpha,\beta}(a_{(2)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle} \otimes a_{(1)[0,\alpha][-1,\beta]}[\omega_{\alpha,\beta}(a_{(2)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle[-1,\beta]}[\omega_{\alpha,\beta}(a_{(2)})^{(2,\beta)}]_{\langle -1,\beta\rangle}$
$\otimes a_{(1)[0,\alpha][0,\beta]}[\omega_{\alpha,\beta}(a_{(2)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle[0,\beta]}[\omega_{\alpha,\beta}(a_{(2)})^{(2,\beta)}]_{\langle 0,\beta\rangle}$,

(a8) $\omega_{\alpha,\beta}(a_{(1)})^{(1,\alpha)}[\omega_{\alpha\beta,\gamma}(a_{(2)})^{(1,\alpha\beta)}]_{(1,\alpha)} \otimes \omega_{\alpha,\beta}(a_{(1)})^{(2,\beta)}[\omega_{\alpha\beta,\gamma}(a_{(2)})^{(1,\alpha\beta)}]_{(2,\beta)} \otimes \omega_{\alpha\beta,\gamma}(a_{(2)})^{(2,\gamma)}$
$= a_{(1)[-1,\alpha]}[\omega_{\alpha,\beta\gamma}(a_{(2)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle} \otimes \omega_{\beta,\gamma}(a_{(1)[0,\alpha]}[\omega_{\alpha,\beta\gamma}(a_{(2)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle})^{(1,\beta)}[\omega_{\alpha,\beta\gamma}(a_{(2)})^{(2,\beta\gamma)}]_{(1,\beta)}$
$\otimes \omega_{\beta,\gamma}(a_{(1)[0,\alpha]}[\omega_{\alpha,\beta\gamma}(a_{(2)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle})^{(2,\gamma)}[\omega_{\alpha,\beta\gamma}(a_{(2)})^{(2,\beta\gamma)}]_{(2,\gamma)}$.

*Proof.* First, we prove that $\bar{\varepsilon}_e$ given by (3.6) is an algebra map if and only if $\varepsilon_e : H_e \to k$ is an algebra map. Observe that, for all $a, b \in A$ and $h, g \in H_e$, we have

$$\begin{aligned}
\bar{\varepsilon}_e((a \rtimes h)(b \rtimes g)) &= \bar{\varepsilon}_e(ab \rtimes hg) \\
&= \varepsilon_A(ab)\varepsilon_e(hg) \\
&\overset{(3.1)}{=} \varepsilon_A(a)\varepsilon_A(b)\varepsilon_e(h)\varepsilon_e(g) \\
&= \bar{\varepsilon}_e(a \rtimes h)\bar{\varepsilon}_e(b \rtimes g)
\end{aligned}$$

and $1 = \bar{\varepsilon}_e(1_A \rtimes 1_H) = \varepsilon_e(1_e)$. Thus $\varepsilon_e$ is an algebra map by considering $a = b = 1_A$ in the obove equation. Conversely, suppose that $\varepsilon_e$ is an algebra map, we can easily check that $\bar{\varepsilon}_e$ is an algebra map.

If each $\bar{\Delta}_{\alpha,\beta}$ is an algebra map, we have

$$1_A \rtimes 1_{e(1,e)\langle -1,e\rangle} \otimes 1_{e(1,e)\langle 0,e\rangle} \rtimes 1_{e(2,e)} = \bar{\Delta}_{e,e}(1_A \rtimes 1_e) = 1_A \rtimes 1_e \otimes 1_A \rtimes 1_e.$$

Applying $\varepsilon_A \otimes id \otimes \varepsilon_A \otimes id$ to both side of the above equation, we have

$$\Delta_{e,e}(1_e) = 1_e \otimes 1_e.$$

For all $h, g \in H_{\alpha\beta}$ with $\alpha, \beta \in \pi$, we observe that

$$\bar{\Delta}_{\alpha,\beta}((1_A \rtimes h)(1_A \rtimes g)) = \bar{\Delta}_{\alpha,\beta}(1_A \rtimes h)\bar{\Delta}_{\alpha,\beta}(1_A \rtimes g),$$

we have

$$\begin{aligned} & 1_A \rtimes (hg)_{(1,\alpha)\langle-1,\alpha\rangle} \otimes (hg)_{(1,\alpha)\langle 0,\alpha\rangle} \rtimes (hg)_{(2,\beta)} \\ = \;& 1_A \rtimes h_{(1,\alpha)\langle-1,\alpha\rangle} g_{(1,\alpha)\langle-1,\alpha\rangle} \otimes h_{(1,\alpha)\langle 0,\alpha\rangle} g_{(1,\alpha)\langle 0,\alpha\rangle} \rtimes h_{(2,\beta)} g_{(2,\beta)}. \end{aligned}$$

Applying $\varepsilon_A \otimes id \otimes \varepsilon_A \otimes id$ to both sides of the above equation, it follows that

$$(hg)_{(1,\alpha)} \otimes (hg)_{(2,\beta)} = h_{(1,\alpha)} g_{(1,\alpha)} \otimes h_{(2,\beta)} g_{(2,\beta)}.$$

Thus each $\Delta_{\alpha,\beta}$ is an algebra map. Observe that $\bar{\varepsilon}_e$ is the counit of $A \rtimes H$ by (i) and (3.4).

For all $b \in A$ and $h \in H_{\alpha\beta}$ with $\alpha, \beta \in \pi$, since

$$\bar{\Delta}_{\alpha,\beta}((1_A \rtimes h)(b \rtimes 1_{\alpha\beta})) = \bar{\Delta}_{\alpha,\beta}(1_A \rtimes h)\bar{\Delta}_{\alpha,\beta}(b \rtimes 1_{\alpha\beta}),$$

we have

$$\begin{aligned} & b_{(1)} \rtimes b_{(2)[-1,\alpha]} [\omega_{\alpha,\beta}(b_{(3)})^{(1,\alpha)}]_{\langle-1,\alpha\rangle} h_{(1,\alpha)\langle-1,\alpha\rangle} \\ & \quad \otimes b_{(2)[0,\alpha]} [\omega_{\alpha,\beta}(b_{(3)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle} h_{(1,\alpha)\langle 0,\alpha\rangle} \rtimes \omega_{\alpha,\beta}(b_{(3)})^{(2,\beta)} h_{(2,\beta)} \\ = \;& b_{(1)} \rtimes h_{(1,\alpha)\langle-1,\alpha\rangle} b_{(2)[-1,\alpha]} [\omega_{\alpha,\beta}(b_{(3)})^{(1,\alpha)}]_{\langle-1,\alpha\rangle} \\ & \otimes h_{(1,\alpha)\langle 0,\alpha\rangle} b_{(2)[0,\alpha]} [\omega_{\alpha,\beta}(b_{(3)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle} \rtimes h_{(2,\beta)} \omega_{\alpha,\beta}(b_{(3)})^{(2,\beta)}. \end{aligned}$$

The relations (a1) can be obtained by applying $\varepsilon_A \otimes id \otimes \varepsilon_A \otimes id$ to both sides of the above equation. By applying $\varepsilon_A \otimes id \otimes id \otimes \varepsilon_e$ to both sides of the above equation for $\beta = e$, it yields relation (a2).

Assume that each $\Delta_{\alpha,\beta}$ is an algebra map and relations (a1) and (a2) hold. For all $a, b \in A$ and $h, g \in H_{\alpha\beta}$ with $\alpha, \beta \in \pi$, we compute as follows:

$$\begin{aligned} & \bar{\Delta}_{\alpha,\beta}((a \rtimes h)(b \rtimes g)) \\ = \;& a_{(1)} b_{(1)} \rtimes a_{(2)[-1,\alpha]} b_{(2)[-1,\alpha]} [\omega_{\alpha,\beta}(a_{(3)} b_{(3)})^{(1,\alpha)}]_{\langle-1,\alpha\rangle} h_{(1,\alpha)\langle-1,\alpha\rangle} \\ & g_{(1,\alpha)\langle-1,\alpha\rangle} \otimes a_{(2)[0,\alpha]} b_{(2)[0,\alpha]} [\omega_{\alpha,\beta}(a_{(3)} b_{(3)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle} h_{(1,\alpha)\langle 0,\alpha\rangle} \\ & g_{(1,\alpha)\langle 0,\alpha\rangle} \rtimes \omega_{\alpha,\beta}(a_{(3)} b_{(3)})^{(2,\beta)} h_{(2,\beta)} g_{(2,\beta)} \\ = \;& a_{(1)} b_{(1)} \rtimes a_{(2)[-1,\alpha]} b_{(2)[-1,\alpha]} [\omega_{\alpha,\beta}(a_{(3)})^{(1,\alpha)}]_{\langle-1,\alpha\rangle} \\ & [\omega_{\alpha,\beta}(b_{(3)})^{(1,\alpha)}]_{\langle-1,\alpha\rangle} h_{(1,\alpha)\langle-1,\alpha\rangle} g_{(1,\alpha)\langle-1,\alpha\rangle} \\ & \otimes a_{(2)[0,\alpha]} b_{(2)[0,\alpha]} [\omega_{\alpha,\beta}(a_{(3)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle} [\omega_{\alpha,\beta}(b_{(3)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle} h_{(1,\alpha)\langle 0,\alpha\rangle} \\ & g_{(1,\alpha)\langle 0,\alpha\rangle} \rtimes \omega_{\alpha,\beta}(a_{(3)})^{(2,\beta)} \omega_{\alpha,\beta}(b_{(3)})^{(2,\beta)} h_{(2,\beta)} g_{(2,\beta)} \\ \overset{(a1),(a2)}{=} \;& a_{(1)} b_{(1)} \rtimes a_{(2)[-1,\alpha]} [\omega_{\alpha,\beta}(a_{(3)})^{(1,\alpha)}]_{\langle-1,\alpha\rangle} h_{(1,\alpha)\langle-1,\alpha\rangle} \\ & b_{(2)[-1,\alpha]} [\omega_{\alpha,\beta}(b_{(3)})^{(1,\alpha)}]_{\langle-1,\alpha\rangle} g_{(1,\alpha)\langle-1,\alpha\rangle} \\ & \otimes a_{(2)[0,\alpha]} [\omega_{\alpha,\beta}(a_{(3)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle} h_{(1,\alpha)\langle 0,\alpha\rangle} b_{(2)[0,\alpha]} [\omega_{\alpha,\beta}(b_{(3)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle} \\ & g_{(1,\alpha)\langle 0,\alpha\rangle} \rtimes \omega_{\alpha,\beta}(a_{(3)})^{(2,\beta)} h_{(2,\beta)} \omega_{\alpha,\beta}(b_{(3)})^{(2,\beta)} g_{(2,\beta)} \\ = \;& \bar{\Delta}_{\alpha,\beta}(a \rtimes h)\bar{\Delta}_{\alpha,\beta}(b \rtimes g). \end{aligned}$$

Thus, each $\Delta_{\alpha,\beta}$ is an algebra map.

Assume that $\Delta = \{\Delta_{\alpha,\beta}\}$ given by (3.5) is coassociative, that is,

$$(id \otimes \bar{\Delta}_{\beta,\gamma}) \circ \bar{\Delta}_{\alpha,\beta\gamma} = (\bar{\Delta}_{\alpha,\beta} \otimes id) \circ \bar{\Delta}_{\alpha\beta,\gamma}.$$

For all $h \in H_{\alpha\beta\gamma}$ with $\alpha, \beta, \gamma \in \pi$, from

$$(id \otimes \bar{\Delta}_{\beta,\gamma}) \circ \bar{\Delta}_{\alpha,\beta\gamma}(1_A \rtimes h) = (\bar{\Delta}_{\alpha,\beta} \otimes id) \circ \bar{\Delta}_{\alpha\beta,\gamma}(1_A \rtimes h),$$

it follows that

$$\begin{aligned}
& 1_A \otimes h_{(1,\alpha)\langle -1,\alpha\rangle} \otimes h_{(1,\alpha)\langle 0,\alpha\rangle(1)} \\
& \quad \otimes h_{(1,\alpha)\langle 0,\alpha\rangle(2)[-1,\beta]}[\omega_{\beta,\gamma}(h_{(1,\alpha)\langle 0,\alpha\rangle(3)})^{(1,\beta)}]_{\langle -1,\beta\rangle} \\
& \quad h_{(2,\beta\gamma)(1,\beta)\langle -1,\beta\rangle} \otimes h_{(1,\alpha)\langle 0,\alpha\rangle(2)[0,\beta]} \\
& \quad [\omega_{\beta,\gamma}(h_{(1,\alpha)\langle 0,\alpha\rangle(3)})^{(1,\beta)}]_{\langle 0,\beta\rangle} h_{(2,\beta\gamma)(1,\beta)\langle 0,\beta\rangle} \\
& \quad \otimes \omega_{\beta,\gamma}(h_{(1,\alpha)\langle 0,\alpha\rangle(3)})^{(2,\gamma)} h_{(2,\beta\gamma)(2,\gamma)} \\
= \;& 1_A \otimes h_{(1,\alpha\beta)\langle -1,\alpha\beta\rangle(1,\alpha)\langle -1,\alpha\rangle} \otimes h_{(1,\alpha\beta)\langle -1,\alpha\beta\rangle(1,\alpha)\langle 0,\alpha\rangle} \\
& \otimes h_{(1,\alpha\beta)\langle -1,\alpha\beta\rangle(2,\beta)} \otimes h_{(1,\alpha\beta)\langle 0,\alpha\beta\rangle} \otimes h_{(2,\gamma)}.
\end{aligned}$$

Then applying $\varepsilon_A \otimes id \otimes \varepsilon_A \otimes id \otimes \varepsilon_A \otimes id$ to both sides of the obove equation yields relation (a3). Applying $\varepsilon_A \otimes id \otimes id \otimes \varepsilon_e \otimes id \otimes \varepsilon_e$ to both sides of the same equation above for $\beta = \gamma = e$, we get relation (a4). Relation (a5) can be seen by applying $\varepsilon_A \otimes id \otimes \varepsilon_A \otimes id \otimes id \otimes \varepsilon_e$ to it again for $\gamma = e$.

For all $a \in A$ and $\alpha, \beta, \gamma \in \pi$, from the relation

$$(id \otimes \bar{\Delta}_{\beta,\gamma}) \circ \bar{\Delta}_{\alpha,\beta\gamma}(a \rtimes 1_{\alpha\beta\gamma}) = (\bar{\Delta}_{\alpha,\beta} \otimes id) \circ \bar{\Delta}_{\alpha\beta,\gamma}(a \rtimes 1_{\alpha\beta\gamma}),$$

we have

$$\begin{aligned}
& a_{(1)} \otimes a_{(2)[-1,\alpha]}[\omega_{\alpha,\beta\gamma}(a_{(3)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle} \otimes a_{(2)[0,\alpha](1)}[\omega_{\alpha,\beta\gamma}(a_{(3)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle(1)} \\
& \quad \otimes a_{(2)[0,\alpha](2)[-1,\beta]}[\omega_{\alpha,\beta\gamma}(a_{(3)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle(2)[-1,\beta]} \\
& \quad [\omega_{\beta,\gamma}(a_{(2)[0,\alpha](3)}[\omega_{\alpha,\beta\gamma}(a_{(3)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle(3)})^{(1,\beta)}]_{\langle -1,\beta\rangle} \\
& \quad [\omega_{\alpha,\beta\gamma}(a_{(3)})^{(2,\beta\gamma)}]_{(1,\beta)\langle -1,\beta\rangle} \\
& \quad \otimes a_{(2)[0,\alpha](2)[0,\beta]}[\omega_{\alpha,\beta\gamma}(a_{(3)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle(2)[0,\beta]} \\
& \quad [\omega_{\beta,\gamma}(a_{(2)[0,\alpha](3)}[\omega_{\alpha,\beta\gamma}(a_{(3)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle(3)})^{(1,\beta)}]_{\langle 0,\beta\rangle} \\
& \quad [\omega_{\alpha,\beta\gamma}(a_{(3)})^{(2,\beta\gamma)}]_{(1,\beta)\langle 0,\beta\rangle} \\
& \quad \otimes \omega_{\beta,\gamma}(a_{(2)[0,\alpha](3)}[\omega_{\alpha,\beta\gamma}(a_{(3)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle(3)})^{(2,\gamma)}[\omega_{\alpha,\beta\gamma}(a_{(3)})^{(2,\beta\gamma)}]_{(2,\gamma)} \\
= \;& a_{(1)} \otimes a_{(2)[-1,\alpha]}[\omega_{\alpha,\beta}(a_{(3)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle} a_{(4)[-1,\alpha\beta](1,\alpha)\langle -1,\alpha\rangle} \\
& [\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{\langle -1,\alpha\beta\rangle(1,\alpha)\langle -1,\alpha\rangle} \\
& \otimes a_{(2)[0,\alpha]}[\omega_{\alpha,\beta}(a_{(3)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle} a_{(4)[-1,\alpha\beta](1,\alpha)\langle 0,\alpha\rangle} \\
& [\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{\langle -1,\alpha\beta\rangle(1,\alpha)\langle 0,\alpha\rangle} \\
& \otimes \omega_{\alpha,\beta}(a_{(3)})^{(2,\beta)} a_{(4)[-1,\alpha\beta](2,\beta)}[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{\langle -1,\alpha\beta\rangle(2,\beta)} \\
& \otimes a_{(4)[0,\alpha\beta]}[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{\langle 0,\alpha\beta\rangle} \otimes \omega_{\alpha\beta,\gamma}(a_{(5)})^{(2,\gamma)}.
\end{aligned}$$

By considering $\beta = \gamma = e$ in the above equation and applying $\varepsilon_A \otimes id \otimes id \otimes \varepsilon_e \otimes id_A \otimes \varepsilon_e$ to both sides of the resulting equation, we can get relation (a6). Applying $\varepsilon_A \otimes id \otimes \varepsilon_A \otimes id \otimes id \otimes \varepsilon_e$ to the equation above for $\gamma = e$ yields relation (a7) and ($a$8) can be seen by applying $\varepsilon_A \otimes id_H \otimes \varepsilon_A \otimes id \otimes \varepsilon_A \otimes id$ to both sides of the same equation again.

Assume that relations (a1)-(a8) hold. Now we check that $\Delta$ is coassociative. Indeed, we compute as follows:

$$
\begin{aligned}
& (\bar{\Delta}_{\alpha,\beta}\otimes id)\bar{\Delta}_{\alpha\beta,\gamma}(a\rtimes h)\\
= \;& \bar{\Delta}_{\alpha,\beta}(a_{(1)}\rtimes a_{(2)[-1,\alpha\beta]}[\omega_{\alpha\beta,\gamma}(a_{(3)})^{(1,\alpha\beta)}]_{\langle-1,\alpha\beta\rangle}h_{(1,\alpha\beta)\langle-1,\alpha\beta\rangle})\\
& \otimes a_{(2)[0,\alpha\beta]}[\omega_{\alpha\beta,\gamma}(a_{(3)})^{(1,\alpha\beta)}]_{\langle 0,\alpha\beta\rangle}h_{(1,\alpha\beta)\langle 0,\alpha\beta\rangle}\rtimes\omega_{\alpha\beta,\gamma}(a_{(3)})^{(2,\gamma)}h_{(2,\gamma)}\\
= \;& a_{(1)}\rtimes a_{(2)[-1,\alpha]}[\omega_{\alpha,\beta}(a_{(3)})^{(1,\alpha)}]_{\langle-1,\alpha\rangle}\\
& a_{(4)[-1,\alpha\beta](1,\alpha)\langle-1,\alpha\rangle}[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{\langle-1,\alpha\beta\rangle(1,\alpha)\langle-1,\alpha\rangle}\\
& \underbrace{h_{(1,\alpha\beta)\langle-1,\alpha\beta\rangle(1,\alpha)\langle-1,\alpha\rangle}}\otimes a_{(2)[0,\alpha]}[\omega_{\alpha,\beta}(a_{(3)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle}\\
& a_{(4)[-1,\alpha\beta](1,\alpha)\langle 0,\alpha\rangle}[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{\langle-1,\alpha\beta\rangle(1,\alpha)\langle 0,\alpha\rangle}\underbrace{h_{(1,\alpha\beta)\langle-1,\alpha\beta\rangle(1,\alpha)\langle 0,\alpha\rangle}}\\
& \rtimes\omega_{\alpha,\beta}(a_{(3)})^{(2,\beta)}a_{(4)[-1,\alpha\beta](2,\beta)}[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{\langle-1,\alpha\beta\rangle(2,\beta)}\underbrace{h_{(1,\alpha\beta)\langle-1,\alpha\beta\rangle(2,\beta)}}\\
& \otimes a_{(4)[0,\alpha\beta]}[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{\langle 0,\alpha\beta\rangle}\underbrace{h_{(1,\alpha\beta)\langle 0,\alpha\beta\rangle}}\rtimes\omega_{\alpha\beta,\gamma}(a_{(5)})^{(2,\gamma)}h_{(2,\gamma)}\\
\overset{(a5)}{=} \;& a_{(1)}\rtimes a_{(2)[-1,\alpha]}[\omega_{\alpha,\beta}(a_{(3)})^{(1,\alpha)}]_{\langle-1,\alpha\rangle}\\
& a_{(4)[-1,\alpha\beta](1,\alpha)\langle-1,\alpha\rangle}[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{\langle-1,\alpha\beta\rangle(1,\alpha)\langle-1,\alpha\rangle}\\
& \underbrace{h_{(1,\alpha\beta)(1,\alpha)\langle-1,\alpha\rangle\langle-1,\alpha\rangle}}\otimes a_{(2)[0,\alpha]}[\omega_{\alpha,\beta}(a_{(3)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle}\\
& a_{(4)[-1,\alpha\beta](1,\alpha)\langle 0,\alpha\rangle}[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{\langle-1,\alpha\beta\rangle(1,\alpha)\langle 0,\alpha\rangle}\underbrace{h_{(1,\alpha\beta)(1,\alpha)\langle-1,\alpha\rangle\langle 0,\alpha\rangle}}\\
& \rtimes\omega_{\alpha,\beta}(a_{(3)})^{(2,\beta)}a_{(4)[-1,\alpha\beta](2,\beta)}[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{\langle-1,\alpha\beta\rangle(2,\beta)}\\
& \underbrace{h_{(1,\alpha\beta)(1,\alpha)\langle 0,\alpha\rangle[-1,\beta]}}h_{(1,\alpha\beta)(2,\beta)\langle-1,\beta\rangle}\otimes a_{(4)[0,\alpha\beta]}[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{\langle 0,\alpha\beta\rangle}\\
& \underbrace{h_{(1,\alpha\beta)(1,\alpha)\langle 0,\alpha\rangle[0,\beta]}}h_{(1,\alpha\beta)(2,\beta)\langle 0,\beta\rangle}\rtimes\omega_{\alpha\beta,\gamma}(a_{(5)})^{(2,\gamma)}h_{(2,\gamma)}\\
\overset{(a4)}{=} \;& a_{(1)}\rtimes a_{(2)[-1,\alpha]}[\omega_{\alpha,\beta}(a_{(3)})^{(1,\alpha)}]_{\langle-1,\alpha\rangle}\\
& a_{(4)[-1,\alpha\beta](1,\alpha)\langle-1,\alpha\rangle}[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{\langle-1,\alpha\beta\rangle(1,\alpha)\langle-1,\alpha\rangle}\\
& \underbrace{h_{(1,\alpha\beta)(1,\alpha)\langle-1,\alpha\rangle}}\otimes a_{(2)[0,\alpha]}[\omega_{\alpha,\beta}(a_{(3)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle}\\
& a_{(4)[-1,\alpha\beta](1,\alpha)\langle 0,\alpha\rangle}[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{\langle-1,\alpha\beta\rangle(1,\alpha)\langle 0,\alpha\rangle}\underbrace{h_{(1,\alpha\beta)(1,\alpha)\langle 0,\alpha\rangle(1)}}\\
& \rtimes\omega_{\alpha,\beta}(a_{(3)})^{(2,\beta)}a_{(4)[-1,\alpha\beta](2,\beta)}[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{\langle-1,\alpha\beta\rangle(2,\beta)}\\
& \underbrace{h_{(1,\alpha\beta)(1,\alpha)\langle 0,\alpha\rangle(2)[-1,\beta]}h_{(1,\alpha\beta)(2,\beta)\langle-1,\beta\rangle}}\otimes a_{(4)[0,\alpha\beta]}[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{\langle 0,\alpha\beta\rangle}\\
& \underbrace{h_{(1,\alpha\beta)(1,\alpha)\langle 0,\alpha\rangle(2)[0,\beta]}h_{(1,\alpha\beta)(2,\beta)\langle 0,\beta\rangle}}\rtimes\omega_{\alpha\beta,\gamma}(a_{(5)})^{(2,\gamma)}h_{(2,\gamma)}\\
\overset{(a3)}{=} \;& a_{(1)}\rtimes a_{(2)[-1,\alpha]}[\omega_{\alpha,\beta}(a_{(3)})^{(1,\alpha)}]_{\langle-1,\alpha\rangle}\\
& a_{(4)[-1,\alpha\beta](1,\alpha)\langle-1,\alpha\rangle}[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{\langle-1,\alpha\beta\rangle(1,\alpha)\langle-1,\alpha\rangle}\\
& \underbrace{h_{(1,\alpha)\langle-1,\alpha\rangle\langle-1,\alpha\rangle}}\otimes a_{(2)[0,\alpha]}[\omega_{\alpha,\beta}(a_{(3)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle}\\
& a_{(4)[-1,\alpha\beta](1,\alpha)\langle 0,\alpha\rangle}[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{\langle-1,\alpha\beta\rangle(1,\alpha)\langle 0,\alpha\rangle}\underbrace{h_{(1,\alpha)\langle-1,\alpha\rangle\langle 0,\alpha\rangle(1)}}
\end{aligned}
$$

$$
\begin{aligned}
&\rtimes\omega_{\alpha,\beta}(a_{(3)})^{(2,\beta)}a_{(4)[-1,\alpha\beta](2,\beta)}[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{\langle-1,\alpha\beta\rangle(2,\beta)}\\
&\underbrace{h_{(1,\alpha)\langle-1,\alpha\rangle\langle0,\alpha\rangle(2)[-1,\beta]}}[\omega_{\beta,\gamma}(\underbrace{h_{(1,\alpha)\langle0,\alpha\rangle}})^{(1,\beta)}]_{\langle-1,\beta\rangle}h_{(2,\beta\gamma)(1,\beta)\langle-1,\beta\rangle}\\
&\otimes a_{(4)[0,\alpha\beta]}[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{\langle0,\alpha\beta\rangle}\underbrace{h_{(1,\alpha)\langle-1,\alpha\rangle\langle0,\alpha\rangle(2)[0,\beta]}}\\
&[\omega_{\beta,\gamma}(\underbrace{h_{(1,\alpha)\langle0,\alpha\rangle}})^{(1,\beta)}]_{\langle0,\beta\rangle}h_{(2,\beta\gamma)(1,\beta)\langle0,\beta\rangle}\\
&\rtimes\omega_{\alpha\beta,\gamma}(a_{(5)})^{(2,\gamma)}\omega_{\beta,\gamma}(h_{(1,\alpha)\langle0,\alpha\rangle})^{(2,\gamma)}h_{(2,\beta\gamma)(2,\beta)}\\
\overset{(a3)}{=}\;&a_{(1)}\rtimes a_{(2)[-1,\alpha]}[\omega_{\alpha,\beta}(a_{(3)})^{(1,\alpha)}]_{\langle-1,\alpha\rangle}\\
&a_{(4)[-1,\alpha\beta](1,\alpha)\langle-1,\alpha\rangle}[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{\langle-1,\alpha\beta\rangle(1,\alpha)\langle-1,\alpha\rangle}\\
&h_{(1,\alpha)\langle-1,\alpha\rangle}\otimes a_{(2)[0,\alpha]}[\omega_{\alpha,\beta}(a_{(3)})^{(1,\alpha)}]_{\langle0,\alpha\rangle}\\
&a_{(4)[-1,\alpha\beta](1,\alpha)\langle0,\alpha\rangle}[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{\langle-1,\alpha\beta\rangle(1,\alpha)\langle0,\alpha\rangle}h_{(1,\alpha)\langle0,\alpha\rangle(1)}\\
&\rtimes\omega_{\alpha,\beta}(a_{(3)})^{(2,\beta)}a_{(4)[-1,\alpha\beta](2,\beta)}[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{\langle-1,\alpha\beta\rangle(2,\beta)}\\
&h_{(1,\alpha)\langle0,\alpha\rangle(2)[-1,\beta]}[\omega_{\beta,\gamma}(h_{(1,\alpha)\langle0,\alpha\rangle(3)})^{(1,\beta)}]_{\langle-1,\beta\rangle}h_{(2,\beta\gamma)(1,\beta)\langle-1,\beta\rangle}\\
&\otimes a_{(4)[0,\alpha\beta]}[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{\langle0,\alpha\beta\rangle}h_{(1,\alpha)\langle0,\alpha\rangle(2)[0,\beta]}\\
&[\omega_{\beta,\gamma}(h_{(1,\alpha)\langle0,\alpha\rangle(3)})^{(1,\beta)}]_{\langle0,\beta\rangle}h_{(2,\beta\gamma)(1,\beta)\langle0,\beta\rangle}\\
&\rtimes\omega_{\alpha\beta,\gamma}(a_{(5)})^{(2,\gamma)}\omega_{\beta,\gamma}(h_{(1,\alpha)\langle0,\alpha\rangle(3)})^{(2,\gamma)}h_{(2,\beta\gamma)(2,\beta)}.
\end{aligned}
$$

and

$$
\begin{aligned}
&(id\otimes\bar{\Delta}_{\beta,\gamma})\bar{\Delta}_{\alpha,\beta\gamma}(a\rtimes h)\\
=\;&a_{(1)}\rtimes a_{(2)[-1,\alpha]}[\omega_{\alpha,\beta\gamma}(a_{(3)})^{(1,\alpha)}]_{\langle-1,\alpha\rangle}h_{(1,\alpha)\langle-1,\alpha\rangle}\otimes\\
&\bar{\Delta}_{\beta,\gamma}(a_{(2)[0,\alpha]}[\omega_{\alpha,\beta\gamma}(a_{(3)})^{(1,\alpha)}]_{\langle0,\alpha\rangle}h_{(1,\alpha)\langle0,\alpha\rangle}\rtimes\omega_{\alpha,\beta\gamma}(a_{(3)})^{(2,\beta\gamma)}h_{(2,\beta\gamma)})\\
=\;&a_{(1)}\rtimes\underbrace{a_{(2)[-1,\alpha]}}[\omega_{\alpha,\beta\gamma}(a_{(3)})^{(1,\alpha)}]_{\langle-1,\alpha\rangle}h_{(1,\alpha)\langle-1,\alpha\rangle}\\
&\otimes\underbrace{a_{(2)[0,\alpha](1)}}[\omega_{\alpha,\beta\gamma}(a_{(3)})^{(1,\alpha)}]_{\langle0,\alpha\rangle(1)}h_{(1,\alpha)\langle0,\alpha\rangle(1)}\\
&\rtimes\underbrace{a_{(2)[0,\alpha](2)[-1,\beta]}}[\omega_{\alpha,\beta\gamma}(a_{(3)})^{(1,\alpha)}]_{\langle0,\alpha\rangle(2)[-1,\beta]}h_{(1,\alpha)\langle0,\alpha\rangle(2)[-1,\beta]}\\
&\underbrace{[\omega_{\beta,\gamma}(a_{(2)[0,\alpha](3)}}[\omega_{\alpha,\beta\gamma}(a_{(3)})^{(1,\alpha)}]_{\langle0,\alpha\rangle(3)}h_{(1,\alpha)\langle0,\alpha\rangle(3)})^{(1,\beta)}]_{\langle-1,\beta\rangle}\\
&[\omega_{\alpha,\beta\gamma}(a_{(3)})^{(2,\beta\gamma)}]_{(1,\beta)\langle-1,\beta\rangle}h_{(2,\beta\gamma)(1,\beta)\langle-1,\beta\rangle}\\
&\otimes\underbrace{a_{(2)[0,\alpha](2)[0,\beta]}}[\omega_{\alpha,\beta\gamma}(a_{(3)})^{(1,\alpha)}]_{\langle0,\alpha\rangle(2)[0,\beta]}h_{(1,\alpha)\langle0,\alpha\rangle(2)[0,\beta]}\\
&[\omega_{\beta,\gamma}(\underbrace{a_{(2)[0,\alpha](3)}}[\omega_{\alpha,\beta\gamma}(a_{(3)})^{(1,\alpha)}]_{\langle0,\alpha\rangle(3)}h_{(1,\alpha)\langle0,\alpha\rangle(3)})^{(1,\beta)}]_{\langle0,\beta\rangle}\\
&[\omega_{\alpha,\beta\gamma}(a_{(3)})^{(2,\beta\gamma)}]_{(1,\beta)\langle0,\beta\rangle}h_{(2,\beta\gamma)(1,\beta)\langle0,\beta\rangle}\\
&\rtimes\omega_{\beta,\gamma}(\underbrace{a_{(2)[0,\alpha](3)}}[\omega_{\alpha,\beta\gamma}(a_{(3)})^{(1,\alpha)}]_{\langle0,\alpha\rangle(3)}h_{(1,\alpha)\langle0,\alpha\rangle(3)})^{(2,\gamma)}\\
&[\omega_{\alpha,\beta\gamma}(a_{(3)})^{(2,\beta\gamma)}]_{(2,\gamma)}h_{(2,\beta\gamma)(2,\gamma)}\\
\overset{(a6)}{=}\;&a_{(1)}\rtimes a_{(2)[-1,\alpha]}a_{(3)[-1,\alpha]\langle-1,\alpha\rangle}\underbrace{[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]_{\langle-1,\alpha\rangle}}h_{(1,\alpha)\langle-1,\alpha\rangle}
\end{aligned}
$$

$$\otimes a_{(2)[0,\alpha](1)}a_{(3)[-1,\alpha]\langle 0,\alpha\rangle(1)}\underbrace{[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle(1)}}h_{(1,\alpha)\langle 0,\alpha\rangle(1)}$$
$$\rtimes a_{(2)[0,\alpha](2)[-1,\beta]}a_{(3)[-1,\alpha]\langle 0,\alpha\rangle(2)[-1,\beta]}$$
$$\underbrace{[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle(2)[-1,\beta]}}h_{(1,\alpha)\langle 0,\alpha\rangle(2)[-1,\beta]}$$
$$[\omega_{\beta,\gamma}(a_{(3)[0,\alpha]}\underbrace{[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle(3)}}h_{(1,\alpha)\langle 0,\alpha\rangle(3)})^{(1,\beta)}]_{\langle -1,\beta\rangle}$$
$$[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(2,\beta\gamma)}]_{(1,\beta)\langle -1,\beta\rangle}h_{(2,\beta\gamma)(1,\beta)\langle -1,\beta\rangle}$$
$$\otimes a_{(2)[0,\alpha](2)[0,\beta]}a_{(3)[-1,\alpha]\langle 0,\alpha\rangle(2)[0,\beta]}\underbrace{[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle(2)[0,\beta]}}h_{(1,\alpha)\langle 0,\alpha\rangle(2)[0,\beta]}$$
$$[\omega_{\beta,\gamma}(a_{(3)[0,\alpha]}\underbrace{[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle(3)}}h_{(1,\alpha)\langle 0,\alpha\rangle(3)})^{(1,\beta)}]_{\langle 0,\beta\rangle}$$
$$[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(2,\beta\gamma)}]_{(1,\beta)\langle 0,\beta\rangle}h_{(2,\beta\gamma)(1,\beta)\langle 0,\beta\rangle}$$
$$\rtimes\omega_{\beta,\gamma}(a_{(3)[0,\alpha]}\underbrace{[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]}_{\langle 0,\alpha\rangle(3)}h_{(1,\alpha)\langle 0,\alpha\rangle(3)})^{(2,\gamma)}$$
$$[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(2,\beta\gamma)}]_{(2,\gamma)}h_{(2,\beta\gamma)(2,\gamma)}$$

$$\overset{(a4)}{=} a_{(1)}\rtimes a_{(2)[-1,\alpha]}a_{(3)[-1,\alpha]\langle -1,\alpha\rangle}[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle\langle -1,\alpha\rangle}h_{(1,\alpha)\langle -1,\alpha\rangle}$$
$$\otimes a_{(2)[0,\alpha](1)}a_{(3)[-1,\alpha]\langle 0,\alpha\rangle(1)}[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle\langle 0,\alpha\rangle}h_{(1,\alpha)\langle 0,\alpha\rangle(1)}$$
$$\rtimes a_{(2)[0,\alpha](2)[-1,\beta]}a_{(3)[-1,\alpha]\langle 0,\alpha\rangle(2)[-1,\beta]}$$
$$[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle(1)[-1,\beta]}h_{(1,\alpha)\langle 0,\alpha\rangle(2)[-1,\beta]}$$
$$[\omega_{\beta,\gamma}(a_{(3)[0,\alpha]}[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle(2)}h_{(1,\alpha)\langle 0,\alpha\rangle(3)})^{(1,\beta)}]_{\langle -1,\beta\rangle}$$
$$[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(2,\beta\gamma)}]_{(1,\beta)\langle -1,\beta\rangle}h_{(2,\beta\gamma)(1,\beta)\langle -1,\beta\rangle}$$
$$\otimes a_{(2)[0,\alpha](2)[0,\beta]}a_{(3)[-1,\alpha]\langle 0,\alpha\rangle(2)[0,\beta]}[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle(1)[0,\beta]}h_{(1,\alpha)\langle 0,\alpha\rangle(2)[0,\beta]}$$
$$[\omega_{\beta,\gamma}(a_{(3)[0,\alpha]}[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle(2)}h_{(1,\alpha)\langle 0,\alpha\rangle(3)})^{(1,\beta)}]_{\langle 0,\beta\rangle}$$
$$[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(2,\beta\gamma)}]_{(1,\beta)\langle 0,\beta\rangle}h_{(2,\beta\gamma)(1,\beta)\langle 0,\beta\rangle}$$
$$\rtimes\omega_{\beta,\gamma}(a_{(3)[0,\alpha]}[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle(2)}h_{(1,\alpha)\langle 0,\alpha\rangle(3)})^{(2,\gamma)}$$
$$[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(2,\beta\gamma)}]_{(2,\gamma)}h_{(2,\beta\gamma)(2,\gamma)}$$

$$\overset{(a4)}{=} a_{(1)}\rtimes a_{(2)[-1,\alpha]}a_{(3)[-1,\alpha]\langle -1,\alpha\rangle}[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle\langle -1,\alpha\rangle\langle -1,\alpha\rangle}h_{(1,\alpha)\langle -1,\alpha\rangle}$$
$$\otimes a_{(2)[0,\alpha](1)}a_{(3)[-1,\alpha]\langle 0,\alpha\rangle(1)}[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle\langle -1,\alpha\rangle\langle 0,\alpha\rangle}h_{(1,\alpha)\langle 0,\alpha\rangle(1)}$$
$$\rtimes a_{(2)[0,\alpha](2)[-1,\beta]}a_{(3)[-1,\alpha]\langle 0,\alpha\rangle(2)[-1,\beta]}$$
$$[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle\langle 0,\alpha\rangle[-1,\beta]}h_{(1,\alpha)\langle 0,\alpha\rangle(2)[-1,\beta]}$$
$$\underbrace{[\omega_{\beta,\gamma}(a_{(3)[0,\alpha]}[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle}h_{(1,\alpha)\langle 0,\alpha\rangle(3)})^{(1,\beta)}]_{\langle -1,\beta\rangle}}$$
$$[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(2,\beta\gamma)}]_{(1,\beta)\langle -1,\beta\rangle}h_{(2,\beta\gamma)(1,\beta)\langle -1,\beta\rangle}$$
$$\otimes a_{(2)[0,\alpha](2)[0,\beta]}a_{(3)[-1,\alpha]\langle 0,\alpha\rangle(2)[0,\beta]}[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle\langle 0,\alpha\rangle[0,\beta]}h_{(1,\alpha)\langle 0,\alpha\rangle(2)[0,\beta]}$$
$$\underbrace{[\omega_{\beta,\gamma}(a_{(3)[0,\alpha]}[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle}h_{(1,\alpha)\langle 0,\alpha\rangle(3)})^{(1,\beta)}]_{\langle 0,\beta\rangle}}$$
$$[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(2,\beta\gamma)}]_{(1,\beta)\langle 0,\beta\rangle}h_{(2,\beta\gamma)(1,\beta)\langle 0,\beta\rangle}$$
$$\rtimes\underbrace{\omega_{\beta,\gamma}(a_{(3)[0,\alpha]}[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle}h_{(1,\alpha)\langle 0,\alpha\rangle(3)})^{(2,\gamma)}}$$
$$[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(2,\beta\gamma)}]_{(2,\gamma)}h_{(2,\beta\gamma)(2,\gamma)}$$

$$= a_{(1)}\rtimes a_{(2)[-1,\alpha]}a_{(3)[-1,\alpha]\langle -1,\alpha\rangle}[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle\langle -1,\alpha\rangle\langle -1,\alpha\rangle}h_{(1,\alpha)\langle -1,\alpha\rangle}$$
$$\otimes a_{(2)[0,\alpha](1)}a_{(3)[-1,\alpha]\langle 0,\alpha\rangle(1)}[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle\langle -1,\alpha\rangle\langle 0,\alpha\rangle}h_{(1,\alpha)\langle 0,\alpha\rangle(1)}$$
$$\rtimes a_{(2)[0,\alpha](2)[-1,\beta]}a_{(3)[-1,\alpha]\langle 0,\alpha\rangle(2)[-1,\beta]}[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle\langle 0,\alpha\rangle[-1,\beta]}h_{(1,\alpha)\langle 0,\alpha\rangle(2)[-1,\beta]}$$

$$[\omega_{\beta,\gamma}(a_{(3)[0,\alpha]}[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle})^{(1,\beta)}]_{\langle -1,\beta\rangle}\underbrace{[\omega_{\beta,\gamma}(h_{(1,\alpha)\langle 0,\alpha\rangle(3)})^{(1,\beta)}]_{\langle -1,\beta\rangle}}$$
$$[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(2,\beta\gamma)}]_{(1,\beta)\langle -1,\beta\rangle}h_{(2,\beta\gamma)(1,\beta)\langle -1,\beta\rangle}$$
$$\otimes a_{(2)[0,\alpha](2)[0,\beta]}a_{(3)[-1,\alpha]\langle 0,\alpha\rangle(2)[0,\beta]}[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle\langle 0,\alpha\rangle[0,\beta]}h_{(1,\alpha)\langle 0,\alpha\rangle(2)[0,\beta]}$$
$$[\omega_{\beta,\gamma}(a_{(3)[0,\alpha]}[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle})^{(1,\beta)}]_{\langle 0,\beta\rangle}\underbrace{[\omega_{\beta,\gamma}(h_{(1,\alpha)\langle 0,\alpha\rangle(3)})^{(1,\beta)}]_{\langle 0,\beta\rangle}}$$
$$\underbrace{[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(2,\beta\gamma)}]_{(1,\beta)\langle 0,\beta\rangle}}h_{(2,\beta\gamma)(1,\beta)\langle 0,\beta\rangle}$$
$$\rtimes\omega_{\beta,\gamma}(a_{(3)[0,\alpha]}[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle})^{(2,\gamma)}\ \underbrace{\omega_{\beta,\gamma}(h_{(1,\alpha)\langle 0,\alpha\rangle(3)})^{(2,\gamma)}}$$
$$\underbrace{[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(2,\beta\gamma)}]_{(2,\gamma)}}h_{(2,\beta\gamma)(2,\gamma)}$$

$$\overset{(a1)}{=}\ a_{(1)}\rtimes a_{(2)[-1,\alpha]}a_{(3)[-1,\alpha]\langle -1,\alpha\rangle}[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle\langle -1,\alpha\rangle\langle -1,\alpha\rangle}h_{(1,\alpha)\langle -1,\alpha\rangle}$$
$$\otimes a_{(2)[0,\alpha](1)}a_{(3)[-1,\alpha]\langle 0,\alpha\rangle(1)}[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle\langle -1,\alpha\rangle\langle 0,\alpha\rangle}h_{(1,\alpha)\langle 0,\alpha\rangle(1)}$$
$$\rtimes a_{(2)[0,\alpha](2)[-1,\beta]}a_{(3)[-1,\alpha]\langle 0,\alpha\rangle(2)[-1,\beta]}$$
$$[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle\langle 0,\alpha\rangle[-1,\beta]}h_{(1,\alpha)\langle 0,\alpha\rangle(2)[-1,\beta]}$$
$$[\omega_{\beta,\gamma}(a_{(3)[0,\alpha]}[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle})^{(1,\beta)}]_{\langle -1,\beta\rangle}[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(2,\beta\gamma)}]_{(1,\beta)\langle -1,\beta\rangle}$$
$$[\omega_{\beta,\gamma}(h_{(1,\alpha)\langle 0,\alpha\rangle(3)})^{(1,\beta)}]_{\langle -1,\beta\rangle}h_{(2,\beta\gamma)(1,\beta)\langle -1,\beta\rangle}$$
$$\otimes a_{(2)[0,\alpha](2)[0,\beta]}a_{(3)[-1,\alpha]\langle 0,\alpha\rangle(2)[0,\beta]}[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle\langle 0,\alpha\rangle[0,\beta]}h_{(1,\alpha)\langle 0,\alpha\rangle(2)[0,\beta]}$$
$$[\omega_{\beta,\gamma}(a_{(3)[0,\alpha]}[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle})^{(1,\beta)}]_{\langle 0,\beta\rangle}[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(2,\beta\gamma)}]_{(1,\beta)\langle 0,\beta\rangle}$$
$$[\omega_{\beta,\gamma}(h_{(1,\alpha)\langle 0,\alpha\rangle(3)})^{(1,\beta)}]_{\langle 0,\beta\rangle}h_{(2,\beta\gamma)(1,\beta)\langle 0,\beta\rangle}$$
$$\rtimes\omega_{\beta,\gamma}(a_{(3)[0,\alpha]}[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle})^{(2,\gamma)}[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(2,\beta\gamma)}]_{(2,\gamma)}$$
$$\omega_{\beta,\gamma}(h_{(1,\alpha)\langle 0,\alpha\rangle(3)})^{(2,\gamma)}h_{(2,\beta\gamma)(2,\gamma)}$$

$$\overset{(a4)}{=}\ a_{(1)}\rtimes a_{(2)[-1,\alpha]}a_{(3)[-1,\alpha]\langle -1,\alpha\rangle\langle -1,\alpha\rangle}[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle\langle -1,\alpha\rangle\langle -1,\alpha\rangle}h_{(1,\alpha)\langle -1,\alpha\rangle}$$
$$\otimes a_{(2)[0,\alpha](1)}a_{(3)[-1,\alpha]\langle -1,\alpha\rangle\langle 0,\alpha\rangle}[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle\langle -1,\alpha\rangle\langle 0,\alpha\rangle}h_{(1,\alpha)\langle 0,\alpha\rangle(1)}$$
$$\rtimes a_{(2)[0,\alpha](2)[-1,\beta]}a_{(3)[-1,\alpha]\langle 0,\alpha\rangle[-1,\beta]}$$
$$[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle\langle 0,\alpha\rangle[-1,\beta]}h_{(1,\alpha)\langle 0,\alpha\rangle(2)[-1,\beta]}$$
$$[\omega_{\beta,\gamma}(a_{(3)[0,\alpha]}[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle})^{(1,\beta)}]_{\langle -1,\beta\rangle}[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(2,\beta\gamma)}]_{(1,\beta)\langle -1,\beta\rangle}$$
$$[\omega_{\beta,\gamma}(h_{(1,\alpha)\langle 0,\alpha\rangle(3)})^{(1,\beta)}]_{\langle -1,\beta\rangle}h_{(2,\beta\gamma)(1,\beta)\langle -1,\beta\rangle}$$
$$\otimes a_{(2)[0,\alpha](2)[0,\beta]}a_{(3)[-1,\alpha]\langle 0,\alpha\rangle[0,\beta]}[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle\langle 0,\alpha\rangle[0,\beta]}h_{(1,\alpha)\langle 0,\alpha\rangle(2)[0,\beta]}$$
$$[\omega_{\beta,\gamma}(a_{(3)[0,\alpha]}[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle})^{(1,\beta)}]_{\langle 0,\beta\rangle}[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(2,\beta\gamma)}]_{(1,\beta)\langle 0,\beta\rangle}$$
$$[\omega_{\beta,\gamma}(h_{(1,\alpha)\langle 0,\alpha\rangle(3)})^{(1,\beta)}]_{\langle 0,\beta\rangle}h_{(2,\beta\gamma)(1,\beta)\langle 0,\beta\rangle}$$
$$\rtimes\omega_{\beta,\gamma}(a_{(3)[0,\alpha]}[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle})^{(2,\gamma)}[\omega_{\alpha,\beta\gamma}(a_{(4)})^{(2,\beta\gamma)}]_{(2,\gamma)}$$
$$\omega_{\beta,\gamma}(h_{(1,\alpha)\langle 0,\alpha\rangle(3)})^{(2,\gamma)}h_{(2,\beta\gamma)(2,\gamma)}$$

$$\overset{(a8)}{=}\ a_{(1)}\rtimes a_{(2)[-1,\alpha]}[\omega_{\alpha,\beta}(a_{(3)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle\langle -1,\alpha\rangle}$$
$$[\omega_{\alpha\beta,\gamma}(a_{(4)})^{(1,\alpha\beta)}]_{(1,\alpha)\langle -1,\alpha\rangle\langle -1,\alpha\rangle}h_{(1,\alpha)\langle -1,\alpha\rangle}$$
$$\otimes a_{(2)[0,\alpha](1)}[\omega_{\alpha,\beta}(a_{(3)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle\langle 0,\alpha\rangle}[\omega_{\alpha\beta,\gamma}(a_{(4)})^{(1,\alpha\beta)}]_{(1,\alpha)\langle -1,\alpha\rangle\langle 0,\alpha\rangle}h_{(1,\alpha)\langle 0,\alpha\rangle(1)}$$
$$\rtimes a_{(2)[0,\alpha](2)[-1,\beta]}[\omega_{\alpha,\beta}(a_{(3)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle[-1,\beta]}$$
$$[\omega_{\alpha\beta,\gamma}(a_{(4)})^{(1,\alpha\beta)}]_{(1,\alpha)\langle 0,\alpha\rangle[-1,\beta]}h_{(1,\alpha)\langle 0,\alpha\rangle(2)[-1,\beta]}$$
$$[\omega_{\alpha,\beta}(a_{(3)})^{(2,\beta)}]_{\langle -1,\beta\rangle}[\omega_{\alpha\beta,\gamma}(a_{(4)})^{(1,\alpha\beta)}]_{(2,\beta)\langle -1,\beta\rangle}$$
$$[\omega_{\beta,\gamma}(h_{(1,\alpha)\langle 0,\alpha\rangle(3)})^{(1,\beta)}]_{\langle -1,\beta\rangle}h_{(2,\beta\gamma)(1,\beta)\langle -1,\beta\rangle}$$

$$\otimes a_{(2)[0,\alpha](2)[0,\beta]}[\omega_{\alpha,\beta}(a_{(3)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle[0,\beta]}[\omega_{\alpha\beta,\gamma}(a_{(4)})^{(1,\alpha\beta)}]_{(1,\alpha)\langle 0,\alpha\rangle[0,\beta]}h_{(1,\alpha)\langle 0,\alpha\rangle(2)[0,\beta]}$$
$$[\omega_{\alpha,\beta}(a_{(3)})^{(2,\beta)}]_{\langle 0,\beta\rangle}[\omega_{\alpha\beta,\gamma}(a_{(4)})^{(1,\alpha\beta)}]_{(2,\beta)\langle 0,\beta\rangle}$$
$$[\omega_{\beta,\gamma}(h_{(1,\alpha)\langle 0,\alpha\rangle(3)})^{(1,\beta)}]_{\langle 0,\beta\rangle}h_{(2,\beta\gamma)(1,\beta)\langle 0,\beta\rangle}$$
$$\rtimes\omega_{\alpha\beta,\gamma}(a_{(4)})^{(2,\gamma)}\omega_{\beta,\gamma}(h_{(1,\alpha)\langle 0,\alpha\rangle(3)})^{(2,\gamma)}h_{(2,\beta\gamma)(2,\gamma)}$$

$$\overset{(a6)}{=} a_{(1)}\rtimes a_{(2)[-1,\alpha]}a_{(3)[-1,\alpha]\langle -1,\alpha\rangle}[\omega_{\alpha,\beta}(a_{(4)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle\langle -1,\alpha\rangle}$$
$$[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{(1,\alpha)\langle -1,\alpha\rangle\langle -1,\alpha\rangle}h_{(1,\alpha)\langle -1,\alpha\rangle}$$
$$\otimes a_{(2)[0,\alpha]}a_{(3)[-1,\alpha]\langle 0,\alpha\rangle}[\omega_{\alpha,\beta}(a_{(4)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle\langle 0,\alpha\rangle}$$
$$[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{(1,\alpha)\langle -1,\alpha\rangle\langle 0,\alpha\rangle}h_{(1,\alpha)\langle 0,\alpha\rangle(1)}$$
$$\rtimes a_{(3)[0,\alpha][-1,\beta]}[\omega_{\alpha,\beta}(a_{(4)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle[-1,\beta]}$$
$$[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{(1,\alpha)\langle 0,\alpha\rangle[-1,\beta]}h_{(1,\alpha)\langle 0,\alpha\rangle(2)[-1,\beta]}$$
$$[\omega_{\alpha,\beta}(a_{(4)})^{(2,\beta)}]_{\langle -1,\beta\rangle}[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{(2,\beta)\langle -1,\beta\rangle}$$
$$[\omega_{\beta,\gamma}(h_{(1,\alpha)\langle 0,\alpha\rangle(3)})^{(1,\beta)}]_{\langle -1,\beta\rangle}h_{(2,\beta\gamma)(1,\beta)\langle -1,\beta\rangle}$$
$$\otimes a_{(3)[0,\alpha][0,\beta]}[\omega_{\alpha,\beta}(a_{(4)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle[0,\beta]}[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{(1,\alpha)\langle 0,\alpha\rangle[0,\beta]}h_{(1,\alpha)\langle 0,\alpha\rangle(2)[0,\beta]}$$
$$[\omega_{\alpha,\beta}(a_{(4)})^{(2,\beta)}]_{\langle 0,\beta\rangle}[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{(2,\beta)\langle 0,\beta\rangle}$$
$$[\omega_{\beta,\gamma}(h_{(1,\alpha)\langle 0,\alpha\rangle(3)})^{(1,\beta)}]_{\langle 0,\beta\rangle}h_{(2,\beta\gamma)(1,\beta)\langle 0,\beta\rangle}$$
$$\rtimes\omega_{\alpha\beta,\gamma}(a_{(5)})^{(2,\gamma)}\omega_{\beta,\gamma}(h_{(1,\alpha)\langle 0,\alpha\rangle(3)})^{(2,\gamma)}h_{(2,\beta\gamma)(2,\gamma)}$$

$$\overset{(a2)}{=} a_{(1)}\rtimes a_{(2)[-1,\alpha]}a_{(3)[-1,\alpha]\langle -1,\alpha\rangle}[\omega_{\alpha,\beta}(a_{(4)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle\langle -1,\alpha\rangle}$$
$$[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{(1,\alpha)\langle -1,\alpha\rangle\langle -1,\alpha\rangle}h_{(1,\alpha)\langle -1,\alpha\rangle}$$
$$\otimes a_{(2)[0,\alpha]}a_{(3)[-1,\alpha]\langle 0,\alpha\rangle}[\omega_{\alpha,\beta}(a_{(4)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle\langle 0,\alpha\rangle}$$
$$[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{(1,\alpha)\langle -1,\alpha\rangle\langle 0,\alpha\rangle}h_{(1,\alpha)\langle 0,\alpha\rangle(1)}$$
$$\rtimes a_{(3)[0,\alpha][-1,\beta]}[\omega_{\alpha,\beta}(a_{(4)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle[-1,\beta]}[\omega_{\alpha,\beta}(a_{(4)})^{(2,\beta)}]_{\langle -1,\beta\rangle}$$
$$[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{(1,\alpha)\langle 0,\alpha\rangle[-1,\beta]}$$
$$h_{(1,\alpha)\langle 0,\alpha\rangle(2)[-1,\beta]}[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{(2,\beta)\langle -1,\beta\rangle}$$
$$[\omega_{\beta,\gamma}(h_{(1,\alpha)\langle 0,\alpha\rangle(3)})^{(1,\beta)}]_{\langle -1,\beta\rangle}h_{(2,\beta\gamma)(1,\beta)\langle -1,\beta\rangle}$$
$$\otimes a_{(3)[0,\alpha][0,\beta]}[\omega_{\alpha,\beta}(a_{(4)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle[0,\beta]}[\omega_{\alpha,\beta}(a_{(4)})^{(2,\beta)}]_{\langle 0,\beta\rangle}$$
$$[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{(1,\alpha)\langle 0,\alpha\rangle[0,\beta]}$$
$$h_{(1,\alpha)\langle 0,\alpha\rangle(2)[0,\beta]}[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{(2,\beta)\langle 0,\beta\rangle}$$
$$[\omega_{\beta,\gamma}(h_{(1,\alpha)\langle 0,\alpha\rangle(3)})^{(1,\beta)}]_{\langle 0,\beta\rangle}h_{(2,\beta\gamma)(1,\beta)\langle 0,\beta\rangle}$$
$$\rtimes\omega_{\alpha\beta,\gamma}(a_{(5)})^{(2,\gamma)}\omega_{\beta,\gamma}(h_{(1,\alpha)\langle 0,\alpha\rangle(3)})^{(2,\gamma)}h_{(2,\beta\gamma)(2,\gamma)}$$

$$\overset{(a7)}{=} a_{(1)}\rtimes a_{(2)[-1,\alpha]}[\omega_{\alpha,\beta}(a_{(3)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle}a_{(4)[-1,\alpha\beta](1,\alpha)\langle -1,\alpha\rangle}$$
$$[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{(1,\alpha)\langle -1,\alpha\rangle\langle -1,\alpha\rangle}h_{(1,\alpha)\langle -1,\alpha\rangle}\otimes a_{(2)[0,\alpha]}[\omega_{\alpha,\beta}(a_{(3)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle}a_{(4)[-1,\alpha\beta](1,\alpha)\langle 0,\alpha\rangle}$$
$$[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{(1,\alpha)\langle -1,\alpha\rangle\langle 0,\alpha\rangle}h_{(1,\alpha)\langle 0,\alpha\rangle(1)}$$
$$\rtimes\omega_{\alpha,\beta}(a_{(3)})^{(2,\beta)}a_{(4)[-1,\alpha\beta](2,\beta)}[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{(1,\alpha)\langle 0,\alpha\rangle[-1,\beta]}$$
$$h_{(1,\alpha)\langle 0,\alpha\rangle(2)[-1,\beta]}[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{(2,\beta)\langle -1,\beta\rangle}$$
$$[\omega_{\beta,\gamma}(h_{(1,\alpha)\langle 0,\alpha\rangle(3)})^{(1,\beta)}]_{\langle -1,\beta\rangle}h_{(2,\beta\gamma)(1,\beta)\langle -1,\beta\rangle}$$
$$\otimes a_{(4)[0,\alpha\beta]}[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{(1,\alpha)\langle 0,\alpha\rangle[0,\beta]}$$
$$h_{(1,\alpha)\langle 0,\alpha\rangle(2)[0,\beta]}[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{(2,\beta)\langle 0,\beta\rangle}$$
$$[\omega_{\beta,\gamma}(h_{(1,\alpha)\langle 0,\alpha\rangle(3)})^{(1,\beta)}]_{\langle 0,\beta\rangle}h_{(2,\beta\gamma)(1,\beta)\langle 0,\beta\rangle}$$
$$\rtimes\omega_{\alpha\beta,\gamma}(a_{(5)})^{(2,\gamma)}\omega_{\beta,\gamma}(h_{(1,\alpha)\langle 0,\alpha\rangle(3)})^{(2,\gamma)}h_{(2,\beta\gamma)(2,\gamma)}$$

$$
\begin{aligned}
\overset{(a2)}{=}\ & a_{(1)} \rtimes a_{(2)[-1,\alpha]}[\omega_{\alpha,\beta}(a_{(3)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle}a_{(4)[-1,\alpha\beta](1,\alpha)\langle -1,\alpha\rangle}\\
& [\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{(1,\alpha)\langle -1,\alpha\rangle\langle -1,\alpha\rangle}h_{(1,\alpha)\langle -1,\alpha\rangle} \otimes a_{(2)[0,\alpha]}[\omega_{\alpha,\beta}(a_{(3)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle}a_{(4)[-1,\alpha\beta](1,\alpha)\langle 0,\alpha\rangle}\\
& [\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{(1,\alpha)\langle -1,\alpha\rangle\langle 0,\alpha\rangle}h_{(1,\alpha)\langle 0,\alpha\rangle(1)} \rtimes \omega_{\alpha,\beta}(a_{(3)})^{(2,\beta)}a_{(4)[-1,\alpha\beta](2,\beta)}[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{(1,\alpha)\langle 0,\alpha\rangle[-1,\beta]}\\
& [\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{(2,\beta)\langle -1,\beta\rangle}h_{(1,\alpha)\langle 0,\alpha\rangle(2)[-1,\beta]}[\omega_{\beta,\gamma}(h_{(1,\alpha)\langle 0,\alpha\rangle(3)})^{(1,\beta)}]_{\langle -1,\beta\rangle}h_{(2,\beta\gamma)(1,\beta)\langle -1,\beta\rangle}\\
& \otimes a_{(4)[0,\alpha\beta]}[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{(1,\alpha)\langle 0,\alpha\rangle[0,\beta]}\\
& [\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{(2,\beta)\langle 0,\beta\rangle}h_{(1,\alpha)\langle 0,\alpha\rangle(2)[0,\beta]}[\omega_{\beta,\gamma}(h_{(1,\alpha)\langle 0,\alpha\rangle(3)})^{(1,\beta)}]_{\langle 0,\beta\rangle}h_{(2,\beta\gamma)(1,\beta)\langle 0,\beta\rangle}\\
& \rtimes \omega_{\alpha\beta,\gamma}(a_{(5)})^{(2,\gamma)}\omega_{\beta,\gamma}(h_{(1,\alpha)\langle 0,\alpha\rangle(3)})^{(2,\gamma)}h_{(2,\beta\gamma)(2,\gamma)}\\
\overset{(a5)}{=}\ & a_{(1)} \rtimes a_{(2)[-1,\alpha]}[\omega_{\alpha,\beta}(a_{(3)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle}\\
& a_{(4)[-1,\alpha\beta](1,\alpha)\langle -1,\alpha\rangle}[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{\langle -1,\alpha\beta\rangle(1,\alpha)\langle -1,\alpha\rangle}\\
& h_{(1,\alpha)\langle -1,\alpha\rangle} \otimes a_{(2)[0,\alpha]}[\omega_{\alpha,\beta}(a_{(3)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle}\\
& a_{(4)[-1,\alpha\beta](1,\alpha)\langle 0,\alpha\rangle}[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{\langle -1,\alpha\beta\rangle(1,\alpha)\langle 0,\alpha\rangle}h_{(1,\alpha)\langle 0,\alpha\rangle(1)}\\
& \rtimes \omega_{\alpha,\beta}(a_{(3)})^{(2,\beta)}a_{(4)[-1,\alpha\beta](2,\beta)}[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{\langle -1,\alpha\beta\rangle(2,\beta)}\\
& h_{(1,\alpha)\langle 0,\alpha\rangle(2)[-1,\beta]}[\omega_{\beta,\gamma}(h_{(1,\alpha)\langle 0,\alpha\rangle(3)})^{(1,\beta)}]_{\langle -1,\beta\rangle}h_{(2,\beta\gamma)(1,\beta)\langle -1,\beta\rangle}\\
& \otimes a_{(4)[0,\alpha\beta]}[\omega_{\alpha\beta,\gamma}(a_{(5)})^{(1,\alpha\beta)}]_{\langle 0,\alpha\beta\rangle}h_{(1,\alpha)\langle 0,\alpha\rangle(2)[0,\beta]}\\
& [\omega_{\beta,\gamma}(h_{(1,\alpha)\langle 0,\alpha\rangle(3)})^{(1,\beta)}]_{\langle 0,\beta\rangle}h_{(2,\beta\gamma)(1,\beta)\langle 0,\beta\rangle}\\
& \rtimes \omega_{\alpha\beta,\gamma}(a_{(5)})^{(2,\gamma)}\omega_{\beta,\gamma}(h_{(1,\alpha)\langle 0,\alpha\rangle(3)})^{(2,\gamma)}h_{(2,\beta\gamma)(2,\beta)}.
\end{aligned}
$$

This ends the proof. ■

## 4. The Classification of $\pi$-unified Coproducts

In this section, we shall prove the classification theorem for $\pi$-unified coproducts which is the dual result of [1] for the trivial group.

Let $\Omega(A) = (H, \rho, \varrho, \omega)$ be a semi-Hopf $\pi$-coalgebra coextending structures of $A$. The $\pi$-unified coproduct $A \rtimes^{\pi} H$ is a right $H$-module via the action: for all $a \in A$, $h, g \in H_\alpha$ with $\alpha \in \pi$,

$$(a \rtimes h) \triangleleft g = a \rtimes hg,$$

and a left $A$-comodule via the coaction

$$a \rtimes h \mapsto a_{(1)} \otimes a_{(2)} \rtimes h.$$

From now on, the Hopf algebra structure on $A$ and the algebra structure on $H$ will be set. First, we need the following result.

**Lemma 4.1.** *Let $A$ be a Hopf algebra, $\Omega(A) = (H, \rho, \varrho, \omega)$ and $\Omega(A) = (H, \rho', \varrho', \omega')$ two semi-Hopf $\pi$-coalgebra coextending structures of $A$. The a family of linear maps*

$$\varphi = \{\varphi_\alpha : A \rtimes H_\alpha \to A \rtimes' H_\alpha\}_{\alpha\in\pi}$$

*is a left $A$-comodule, a right $H$-module and an algebra morphism if and only if there exists a family of algebra morphisms $\vartheta = \{\vartheta_\alpha : A \to H_\alpha\}_{\alpha\in\pi}$ such that $\vartheta_\alpha(A) \subseteq C(H_\alpha)$ (center of $H_\alpha$) and $\varphi_\alpha$ is given by*

$$\varphi_\alpha(a \rtimes h) = a_{(1)} \rtimes' \vartheta_\alpha(a_{(2)})h, \tag{4.1}$$

*for all $a \in A$ and $h \in H_\alpha$ with $\alpha \in \pi$. Furthermore, any such morphism $\varphi$ is an isomorphism with the inverse $\psi = \{\psi_\alpha\}_{\alpha\in\pi}$ given by*

$$\psi_\alpha : A \rtimes' H_\alpha \to A \rtimes H_\alpha,\ \psi(a \rtimes' h) = a_{(1)} \rtimes \vartheta_\alpha(S_A(a_{(2)}))h$$

*for all $a \in A$ and $h \in H_\alpha$.*

*Proof.* " $\Longrightarrow$ " Since each $\varphi_\alpha$ is a right $H_\alpha$-module, we have

$$\varphi_\alpha(a \rtimes h) = \varphi_\alpha(a \rtimes 1_\alpha) \triangleleft g = a^A \rtimes' a^{H_\alpha} g,$$

where we adopt the notation

$$\varphi_\alpha(a \rtimes 1_\alpha) = a^A \rtimes' a^{H_\alpha}$$

for all $a \in A$ and $h \in H_\alpha$. From $\varphi_\alpha$ being a left $A$-comodule map, we have

$$(a^A)_{(1)} \otimes (a^A)_{(2)} \otimes a^{H_\alpha} h = a_{(1)} \otimes (a_{(2)})^A \otimes (a_{(2)})^{H_\alpha} h$$

for all $a \in A$ and $h \in H_\alpha$. Applying $id \otimes \varepsilon_A \otimes id$ to the above equality, we obtain

$$\varphi_\alpha(a \rtimes h) = a_{(1)} \rtimes' \varepsilon_A((a_{(2)})^A)(a_{(2)})^{H_\alpha} h.$$

Now, we define

$$\vartheta_\alpha : A \to H_\alpha,\ \vartheta_\alpha(a) = \varepsilon_A(a^A) a^{H_\alpha}$$

for all $a \in A$, it follows that (4.1) holds. Notice that $\vartheta_\alpha(1_A) = 1_{H_\alpha}$ followed from $\varphi_\alpha(1_A \rtimes 1_\alpha) = 1_A \rtimes' 1_\alpha$. Considering that $\varphi_\alpha$ is an algebra morphism yields

$$a_{(1)} b_{(1)} \otimes \vartheta_\alpha(a_{(2)} b_{(2)}) hg = a_{(1)} b_{(1)} \otimes \vartheta_\alpha(a_{(2)}) h \vartheta_\alpha(b_{(2)}) g, \tag{4.2}$$

for all $a, b \in A$ and $h, g \in H_\alpha$. Applying $\varepsilon_A \otimes id$ to (4.2) for $a = 1_A$, $g = 1_\alpha$, one implies $\vartheta_\alpha(A)$ belongs to the center of $H_\alpha$. Considering $h = g = 1_\alpha$ in (4.2) and applying $\varepsilon_A \otimes id$ yields $\vartheta_\alpha(ab) = \vartheta_\alpha(a)\vartheta_\alpha(b)$. Thus, $\vartheta_\alpha$ is an algebra morphism.

" $\Longleftarrow$ " With $\varphi$ given by (4.1). Assume that $\varphi_\alpha(A) \in C(H_\alpha)$ and $\vartheta_\alpha$ is an algebra morphism. Then we can check that each $\varphi_\alpha$ is an algebra morphism in a direct way. Notice that that $\varphi$ and $\psi$ are the inverse of each other.

This ends the proof. □

**Definition 4.2.** *Let $A$ be a Hopf algebra, $\Omega(A) = (H, \rho, \varrho, \omega)$ a semi-Hopf coalgebra coextending structures of $A$. A family of algebra morphisms $u = \{\vartheta_\alpha : A \to H_\alpha\}_{\alpha \in \pi}$ is called an* algebra lazy 1-$\pi$-cycle, *if $\varepsilon_e \circ \vartheta_e = \varepsilon_A$ and $\vartheta_\alpha(A) \subseteq C(H_\alpha)$ for each $\alpha \in \pi$. We denote by $H_l^{1,\pi}(H, A)$ the group of all algebra lazy 1-$\pi$-cycles of $H$ with coefficients in $A$.*

$H_l^{1,\pi}(H, A)$ is a group with respect to the convolution product. We have to prove that if $u$ and $v \in H_l^{1,\pi}(H, A)$, then $u * v \in H_l^{1,\pi}(H, A)$. Indeed, for all $a, b \in A$ and $\alpha \in \pi$, we have

$$\begin{aligned}
\vartheta_\alpha * v_\alpha(ab) &= \vartheta_\alpha(a_{(1)} b_{(1)}) v_\alpha(a_{(2)} b_{(2)}) \\
&= \vartheta_\alpha(a_{(1)}) \vartheta_\alpha(b_{(1)}) v_\alpha(a_{(2)}) v_\alpha(b_{(2)}) \\
&= \vartheta_\alpha(a_{(1)}) v_\alpha(a_{(2)}) \vartheta_\alpha(b_{(1)}) v_\alpha(b_{(2)}) \\
&= (\vartheta_\alpha * v_\alpha(a))(\vartheta_\alpha * v_\alpha(b)).
\end{aligned}$$

Notice that $\vartheta_\alpha * v_\alpha(1_A) = 1_\alpha$. Thus, $\vartheta_\alpha * v_\alpha$ is an algebra morphism. It is straightforward to prove that $\vartheta_\alpha * v_\alpha(A) \subseteq C(A)$. For all $a \in A$, we have

$$\begin{aligned}
\varepsilon_e \circ (\vartheta_e * v_e)(a) &= \varepsilon_e(\vartheta_e(a_{(1)})) \varepsilon_e(v_e(a_{(2)})) \\
&= \varepsilon_A(a_{(1)}) \varepsilon_A(a_{(2)}) = \varepsilon_A(a).
\end{aligned}$$

This shows that $\varepsilon_e \circ (\vartheta_e * v_e) = \varepsilon_A$.

Now, we shall give the main theorem in this section.

**Theorem 4.3.** *Let $A$ be a Hopf algebra, $\Omega(A) = (H, \rho, \varrho, \omega)$ and $\Omega(A) = (H, \rho', \varrho', \omega')$ two semi-Hopf $\pi$-coalgebra coextending structures of $A$. Then there exists a family of linear map*

$$\varphi = \{\varphi_\alpha : A \rtimes H_\alpha \to A \rtimes' H_\alpha\}_{\alpha\in\pi}$$

*a left $A$-comodule, a right $H$-module and a semi-Hopf $\pi$-coalgebra map if and only if $\rho = \rho'$ and there exists an algebra lazy 1-$\pi$-cycle $u \in H_l^{1,\pi}(H, A)$ such that:*

$$h_{(1,\alpha)'} \otimes h_{(2,\beta)'} = h_{(1,\alpha)\langle -1,\alpha\rangle} \otimes \vartheta_\beta(h_{(1,\alpha)\langle 0,\alpha\rangle})h_{(2,\beta)}, \tag{4.3}$$

$$a_{[-1,\alpha]'} \otimes a_{[0,\alpha]'} = \vartheta_\alpha(a_{(1)})a_{(2)[-1,\alpha]}\vartheta_\alpha(S_A(a_{(3)}))_{\langle -1,\alpha\rangle} \otimes a_{(2)[0,\alpha]}\vartheta_\alpha(S_A(a_{(3)}))_{\langle 0,\alpha\rangle}, \tag{4.4}$$

$$\begin{aligned}\omega'_{\alpha,\beta}(a)^{(1,\alpha)} \otimes \omega'_{\alpha,\beta}(a)^{(2,\beta)} = {}& \vartheta_\alpha(a_{(1)})a_{(2)[-1,\alpha]}[\omega_{\alpha,\beta}(a_{(3)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle}\vartheta_{\alpha\beta}(S_A(a_{(4)}))_{(1,\alpha)'} \\ &\otimes\vartheta_\beta(a_{(2)[0,\alpha]}[\omega_{\alpha,\beta}(a_{(3)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle})\omega_{\alpha,\beta}(a_{(3)})^{(2,\beta)}\vartheta_{\alpha\beta}(S_A(a_{(4)}))_{(2,\beta)'},\end{aligned} \tag{4.5}$$

*for all $a \in A$ and $h \in H_{\alpha\beta}$. In this case, $\varphi$ is given by (4.1) and is an isomorphism.*

*Proof.* " $\Longrightarrow$ " From Lemma 4.1, there exists an algebra lazy 1-$\pi$-cycle $u \in H_l^{1,\pi}(H, A)$ with $\vartheta_\alpha^{-1} = \vartheta_\alpha \circ S_A$. Since $\varphi$ is a coalgebra morphism,

$$\begin{aligned}& a_{(1)} \otimes \vartheta_\alpha(a_{(2)})a_{(3)[-1,\alpha]}[\omega_{\alpha,\beta}(a_{(4)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle}h_{(1,\alpha)\langle -1,\alpha\rangle} \\ &\otimes a_{(3)[0,\alpha](1)}[\omega_{\alpha,\beta}(a_{(4)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle(1)}h_{(1,\alpha)\langle 0,\alpha\rangle(1)} \\ &\otimes\vartheta_\beta(a_{(3)[0,\alpha](2)}[\omega_{\alpha,\beta}(a_{(4)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle(2)}h_{(1,\alpha)\langle 0,\alpha\rangle(2)})\omega_{\alpha,\beta}(a_{(4)})^{(2,\beta)}h_{(2,\beta)} \\ = {}& a_{(1)} \otimes a_{(2)[-1,\alpha]'}[\omega'_{\alpha,\beta}(a_{(3)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle'}\vartheta_{\alpha\beta}(a_{(4)})_{(1,\alpha)'\langle -1,\alpha\rangle'}h_{(1,\alpha)'\langle -1,\alpha\rangle'} \\ &\otimes a_{(2)[0,\alpha]'}[\omega'_{\alpha,\beta}(a_{(3)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle'}\vartheta_{\alpha\beta}(a_{(4)})_{(1,\alpha)'\langle 0,\alpha\rangle'}h_{(1,\alpha)'\langle 0,\alpha\rangle'} \\ &\otimes\omega'_{\alpha,\beta}(a_{(3)})^{(2,\beta)}\vartheta_{\alpha\beta}(a_{(4)})_{(2,\beta)'}h_{(2,\beta)'}.\end{aligned}$$

Applying $\varepsilon_A \otimes id \otimes \varepsilon_A \otimes id$ to both sides of the equation above for $a = 1_A$ , we obtain relation (4.3). Also, by considering $a = 1_A$, $\beta = e$ and applying $\varepsilon_A \otimes id \otimes id \otimes \varepsilon_e$ to both sides of the resulting equation, we have $\rho = \rho'$. If we apply $\varepsilon_A \otimes id \otimes id \otimes \varepsilon_e$ to the same equation obove for $h = 1_e$, it yields relation (4.4). Relation (4.5) can be seen by applying $\varepsilon_A \otimes id \otimes \varepsilon_A \otimes id$ to the equality above for $h = 1_{\alpha\beta}$ respectively.

" $\Longleftarrow$ " Suppose that there exists an algebra lazy 1-$\pi$-cycle $\vartheta$ such that relations (4.3)-(4.5) are fulfilled. Then we shall check that $\varphi$ is a coalgebra morphism. As a matter of fact, for all $a \in A$ and $h \in H_{\alpha\beta}$, we have

$$\begin{aligned}& a_{(1)} \otimes a_{(2)[-1,\alpha]'}[\omega'_{\alpha,\beta}(a_{(3)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle'}\vartheta_{\alpha\beta}(a_{(4)})_{(1,\alpha)'\langle -1,\alpha\rangle'}h_{(1,\alpha)'\langle -1,\alpha\rangle'} \\ &\otimes a_{(2)[0,\alpha]'}[\omega'_{\alpha,\beta}(a_{(3)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle'}\vartheta_{\alpha\beta}(a_{(4)})_{(1,\alpha)'\langle 0,\alpha\rangle'}h_{(1,\alpha)'\langle 0,\alpha\rangle'} \\ &\otimes\omega'_{\alpha,\beta}(a_{(3)})^{(2,\beta)}\vartheta_{\alpha\beta}(a_{(4)})_{(2,\beta)'}h_{(2,\beta)'} \\ = {}& a_{(1)} \otimes \vartheta_\alpha(a_{(2)})a_{(3)[-1,\alpha]}\underbrace{\vartheta_\alpha(S_A(a_{(4)}))_{\langle -1,\alpha\rangle}\vartheta_\alpha(a_{(5)})_{\langle -1,\alpha\rangle}} \\ & a_{(6)[-1,\alpha]\langle -1,\alpha\rangle}[\omega_{\alpha,\beta}(a_{(7)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle\langle -1,\alpha\rangle} \\ & \underbrace{\vartheta_{\alpha\beta}(S_A(a_{(8)}))_{(1,\alpha)'\langle -1,\alpha\rangle}\vartheta_{\alpha\beta}(a_{(9)})_{(1,\alpha)'\langle -1,\alpha\rangle}}h_{(1,\alpha)'\langle -1,\alpha\rangle} \\ & \otimes a_{(3)[0,\alpha]}\underbrace{\vartheta_\alpha(S_A(a_{(4)}))_{\langle 0,\alpha\rangle}\vartheta_\alpha(a_{(5)})_{\langle 0,\alpha\rangle}}a_{(6)[-1,\alpha]\langle 0,\alpha\rangle} \\ & [\omega_{\alpha,\beta}(a_{(7)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle\langle 0,\alpha\rangle}\underbrace{\vartheta_{\alpha\beta}(S_A(a_{(8)}))_{(1,\alpha)'\langle 0,\alpha\rangle}\vartheta_{\alpha\beta}(a_{(9)})_{(1,\alpha)'\langle 0,\alpha\rangle}}\end{aligned}$$

$$h_{(1,\alpha)'\langle 0,\alpha\rangle}\otimes\vartheta_\beta(a_{(6)[0,\alpha]}[\omega_{\alpha,\beta}(a_{(7)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle})\omega_{\alpha,\beta}(a_{(7)})^{(2,\beta)}$$
$$\underbrace{\vartheta_{\alpha\beta}(S_A(a_{(8)}))_{(2,\beta)'}\vartheta_{\alpha\beta}(a_{(9)})_{(2,\beta)'}}h_{(2,\beta)'}$$

$$= a_{(1)}\otimes\vartheta_\alpha(a_{(2)})\underbrace{a_{(3)[-1,\alpha]}a_{(4)[-1,\alpha]\langle -1,\alpha\rangle}}[\omega_{\alpha,\beta}(a_{(5)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle\langle -1,\alpha\rangle}$$
$$h_{(1,\alpha)'\langle -1,\alpha\rangle}\otimes\underbrace{a_{(3)[0,\alpha]}a_{(4)[-1,\alpha]\langle 0,\alpha\rangle}}[\omega_{\alpha,\beta}(a_{(5)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle\langle 0,\alpha\rangle}$$
$$h_{(1,\alpha)'\langle 0,\alpha\rangle}\otimes\vartheta_\beta(\underbrace{a_{(4)[0,\alpha]}}[\omega_{\alpha,\beta}(a_{(5)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle})\omega_{\alpha,\beta}(a_{(5)})^{(2,\beta)}h_{(2,\beta)'}$$

$$\overset{(a6)}{=} a_{(1)}\otimes\vartheta_\alpha(a_{(2)})a_{(3)[-1,\alpha]}\underbrace{[\omega_{\alpha,\beta}(a_{(4)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle\langle -1,\alpha\rangle}}h_{(1,\alpha)'\langle -1,\alpha\rangle}$$
$$\otimes a_{(3)[0,\alpha](1)}\underbrace{[\omega_{\alpha,\beta}(a_{(4)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle\langle 0,\alpha\rangle}}h_{(1,\alpha)'\langle 0,\alpha\rangle}$$
$$\otimes\vartheta_\beta(a_{(3)[0,\alpha](2)}\underbrace{[\omega_{\alpha,\beta}(a_{(4)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle}})\omega_{\alpha,\beta}(a_{(4)})^{(2,\beta)}h_{(2,\beta)'}$$

$$\overset{(a4)}{=} a_{(1)}\otimes\vartheta_\alpha(a_{(2)})a_{(3)[-1,\alpha]}[\omega_{\alpha,\beta}(a_{(4)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle}\underbrace{h_{(1,\alpha)'\langle -1,\alpha\rangle}}$$
$$\otimes a_{(3)[0,\alpha](1)}[\omega_{\alpha,\beta}(a_{(4)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle(1)}\underbrace{h_{(1,\alpha)'\langle 0,\alpha\rangle}}$$
$$\otimes\vartheta_\beta(a_{(3)[0,\alpha](2)}[\omega_{\alpha,\beta}(a_{(4)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle(2)})\omega_{\alpha,\beta}(a_{(4)})^{(2,\beta)}\underbrace{h_{(2,\beta)'}}$$

$$\overset{(4.3)}{=} a_{(1)}\otimes\vartheta_\alpha(a_{(2)})a_{(3)[-1,\alpha]}[\omega_{\alpha,\beta}(a_{(4)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle}\underbrace{h_{(1,\alpha)\langle -1,\alpha\rangle\langle -1,\alpha\rangle}}$$
$$\otimes a_{(3)[0,\alpha](1)}[\omega_{\alpha,\beta}(a_{(4)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle(1)}\underbrace{h_{(1,\alpha)\langle -1,\alpha\rangle\langle 0,\alpha\rangle}}$$
$$\otimes\vartheta_\beta(a_{(3)[0,\alpha](2)}[\omega_{\alpha,\beta}(a_{(4)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle(2)})\omega_{\alpha,\beta}(a_{(4)})^{(2,\beta)}\vartheta_\beta(\underbrace{h_{(1,\alpha)\langle 0,\alpha\rangle}})h_{(2,\beta)}$$

$$\overset{(a4)}{=} a_{(1)}\otimes\vartheta_\alpha(a_{(2)})a_{(3)[-1,\alpha]}[\omega_{\alpha,\beta}(a_{(4)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle}h_{(1,\alpha)\langle -1,\alpha\rangle}$$
$$\otimes a_{(3)[0,\alpha](1)}[\omega_{\alpha,\beta}(a_{(4)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle(1)}h_{(1,\alpha)\langle 0,\alpha\rangle(1)}$$
$$\otimes\vartheta_\beta(a_{(3)[0,\alpha](2)}[\omega_{\alpha,\beta}(a_{(4)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle(2)})\omega_{\alpha,\beta}(a_{(4)})^{(2,\beta)}\vartheta_\beta(h_{(1,\alpha)\langle 0,\alpha\rangle(2)})h_{(2,\beta)}$$
$$= a_{(1)}\otimes\vartheta_\alpha(a_{(2)})a_{(3)[-1,\alpha]}[\omega_{\alpha,\beta}(a_{(4)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle}h_{(1,\alpha)\langle -1,\alpha\rangle}$$
$$\otimes a_{(3)[0,\alpha](1)}[\omega_{\alpha,\beta}(a_{(4)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle(1)}h_{(1,\alpha)\langle 0,\alpha\rangle(1)}$$
$$\otimes\vartheta_\beta(a_{(3)[0,\alpha](2)}[\omega_{\alpha,\beta}(a_{(4)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle(2)})\vartheta_\beta(h_{(1,\alpha)\langle 0,\alpha\rangle(2)})\omega_{\alpha,\beta}(a_{(4)})^{(2,\beta)}h_{(2,\beta)}$$

This is what we exactly show. The proof is now finished. □

## 5. Quasitriangular Structures of the $\pi$-Unified Coproducts

In this section, we shall study the quasitriangular structures on the $\pi$-unified coproduct $A \rtimes^\pi H$.

In what follows, we always let $A$ be a Hopf algebra and $\Omega(A) = (H, \rho, \varrho, \omega)$ a crossed semi-Hopf $\pi$-coalgebra coextending structure of $A$ with the crossing $\varphi$. First, we introduce some new definitions which are both the group version of the concepts of [4] and also the dual concepts of Definition 2.1, 2.2 and 2.4 in [2].

**Definition 5.1.** *Let $(A, P)$ be a quasitriangular Hopf algebra. We say that $(A, H, U)$ is a generalized right dual compatible $\pi$-pair, if there exists a family of elements $U = \{U_\alpha = U^{1,\alpha} \otimes U^{2,\alpha} \in A \otimes H_\alpha\}_{\alpha\in\pi}$ satisfying the following conditions: for all $\alpha, \beta \in \pi$,*

*(R1)* $\Delta_A(U^{1,\alpha}) \otimes U^{2,\alpha} = U^{1,\alpha} \otimes u^{1,\alpha} \otimes U^{2,\alpha}u^{2,\alpha}$,

*(R2)* $U^{1,\alpha\beta}P^1 \otimes \omega_{\alpha,\beta}(P^2)^{(1,\alpha)}[U^{2,\alpha\beta}]_{(1,\alpha)} \otimes \omega_{\alpha,\beta}(P^2)^{(2,\beta)}[U^{2,\alpha\beta}]_{(2,\beta)}$
$= U^{1,\beta}u^{1,\alpha} \otimes u^{2,\alpha} \otimes U^{2,\beta}$,

*(R3)* $\varepsilon_A(U^{1,\alpha})U^{2,\alpha} = 1_\alpha$, $\varepsilon_e(U^{2,e})U^{1,e} = 1_A$,

*(R4)* $U^{1,\alpha} \otimes \varphi_\beta(U^{2,\alpha}) = U^{1,\beta\alpha\beta^{-1}} \otimes U^{2,\beta\alpha\beta^{-1}}$.

**Definition 5.2.** *Let $(A, P)$ be a quasitriangular Hopf algebra. We say that $(H, A, V)$ is a generalized left dual compatible $\pi$-pair, if there exists a family of elements $V = \{V_\alpha = V^{1,\alpha} \otimes V^{2,\alpha} \in H_\alpha \otimes A\}_{\alpha\in\pi}$ satisfying the following conditions: for any $\alpha, \beta \in \pi$,*

*(L1)* $\omega_{\alpha,\beta}(P^1)^{(1,\alpha)}[V^{1,\alpha\beta}]_{(1,\alpha)} \otimes \omega_{\alpha,\beta}(P^1)^{(2,\beta)}[V^{1,\alpha\beta}]_{(2,\beta)} \otimes P^2V^{2,\alpha\beta}$
$= V^{1,\alpha} \otimes v^{1,\beta} \otimes V^{2,\alpha}v^{2,\beta}$,

*(L2)* $V^{1,\alpha} \otimes \Delta_A(V^{2,\alpha}) = V^{1,\alpha}v^{1,\alpha} \otimes v^{2,\alpha} \otimes V^{2,\alpha}$,

*(L3)* $\varepsilon_e(V^{1,e})V^{2,e} = 1_A$, $\varepsilon_A(V^{2,\alpha})V^{1,\alpha} = 1_\alpha$,

*(L4)* $\varphi_\beta(V^{1,\alpha}) \otimes V^{2,\alpha} = V^{1,\beta\alpha\beta^{-1}} \otimes V^{2,\beta\alpha\beta^{-1}}$.

**Definition 5.3.** *Let $(A, P)$ be a quasitriangular Hopf algebra, $(A, H, U)$ a generalized right dual compatible $\pi$-pair and $(H, A, V)$ a generalized left dual compatible $\pi$-pair. We say that $(H, Q)$ is a weak quasitriangular structure, if there exists a family of elements $Q = \{Q_{\alpha,\beta} = Q^{1,\alpha} \otimes Q^{2,\beta} \in H_\alpha \otimes H_\beta\}_{\alpha,\beta\in\pi}$ satisfying the following conditions: for all $\alpha, \beta, \gamma \in \pi$, $h \in H_{\alpha\beta}$,*

*(W1)* $Q^{1,\alpha}h_{(1,\alpha)} \otimes Q^{2,\beta}h_{(2,\beta)} = h_{(2,\alpha)}Q^{1,\alpha} \otimes \varphi_{\alpha^{-1}}(h_{(1,\alpha\beta\alpha^{-1})})Q^{2,\beta}$,

*(W2)* $\omega_{\alpha,\beta}(U^{1,\gamma})^{(1,\alpha)}[Q^{1,\alpha\beta}]_{(1,\alpha)} \otimes \omega_{\alpha,\beta}(U^{1,\gamma})^{(2,\beta)}[Q^{1,\alpha\beta}]_{(2,\beta)} \otimes U^{2,\gamma}Q^{2,\gamma}$
$= Q^{1,\alpha} \otimes q^{1,\beta} \otimes \varphi_{\beta^{-1}}(Q^{2,\beta\gamma\beta^{-1}})q^{2,\gamma}$,

*(W3)* $Q^{1,\alpha}q^{1,\alpha} \otimes q^{2,\beta} \otimes Q^{2,\gamma}$
$= Q^{1,\alpha}V^{1,\alpha} \otimes \omega_{\beta,\gamma}(V^{2,\alpha})^{(1,\beta)}[Q^{2,\beta\gamma}]_{(1,\beta)} \otimes \omega_{\beta,\gamma}(V^{2,\alpha})^{(2,\gamma)}[Q^{2,\beta\gamma}]_{(2,\gamma)}$,

*(W4)* $\varepsilon_e(Q^{1,e})Q^{2,\alpha} = Q^{1,\alpha}\varepsilon_e(Q^{2,e}) = 1_\alpha$,

*(W5)* $\varphi_\beta(Q^{1,\alpha}) \otimes \varphi_\beta(Q^{2,\gamma}) = Q^{1,\beta\alpha\beta^{-1}} \otimes Q^{1,\beta\gamma\beta^{-1}}$.

**Remark 5.4.** *If $\omega$ is trivial, i.e., $\omega_{\alpha,\beta}(a) = \varepsilon_A(a)1_\alpha \otimes 1_\beta$. Then weak quasitriangular structure is just the ordinary quasitriangular structure in Hopf $\pi$-coalgebra setting. Also, Definitions (5.1)-(5.2) take the following forms:*

*(a) We say that $(A, H, U)$ is a right dual compatible $\pi$-pair, if there exists a family of elements $U = \{U_\alpha = U^{1,\alpha} \otimes U^{2,\alpha} \in A \otimes H_\alpha\}_{\alpha\in\pi}$ satisfying the following conditions: for all $\alpha, \beta \in \pi$,*

*($\mathcal{R}1$)* $\Delta_A(U^{1,\alpha}) \otimes U^{2,\alpha} = U^{1,\alpha} \otimes u^{1,\alpha} \otimes U^{2,\alpha}u^{2,\alpha}$,

*($\mathcal{R}2$)* $U^{1,\alpha\beta} \otimes [U^{2,\alpha\beta}]_{(1,\alpha)} \otimes [U^{2,\alpha\beta}]_{(2,\beta)} = U^{1,\beta}u^{1,\alpha} \otimes u^{2,\alpha} \otimes U^{2,\beta}$,

*($\mathcal{R}3$)* $\varepsilon_A(U^{1,\alpha})U^{2,\alpha} = 1_\alpha$, $\varepsilon_e(U^{2,e})U^{1,e} = 1_A$,

*($\mathcal{R}4$)* $U^{1,\alpha} \otimes \varphi_\beta(U^{2,\alpha}) = U^{1,\beta\alpha\beta^{-1}} \otimes U^{2,\beta\alpha\beta^{-1}}$.

*(b) We say that $(H, A, V)$ is a left dual compatible $\pi$-pair, if there exists a family of elements $V = \{V_\alpha = V^{1,\alpha} \otimes V^{2,\alpha} \in H_\alpha \otimes A\}_{\alpha\in\pi}$ satisfying the following conditions: for any $\alpha, \beta \in \pi$,*

*($\mathcal{L}1$)* $[V^{1,\alpha\beta}]_{(1,\alpha)} \otimes [V^{1,\alpha\beta}]_{(2,\beta)} \otimes V^{2,\alpha\beta} = V^{1,\alpha} \otimes v^{1,\beta} \otimes V^{2,\alpha}v^{2,\beta}$,

*($\mathcal{L}2$)* $V^{1,\alpha} \otimes \Delta_A(V^{2,\alpha}) = V^{1,\alpha}v^{1,\alpha} \otimes v^{2,\alpha} \otimes V^{2,\alpha}$,

*($\mathcal{L}3$)* $\varepsilon_e(V^{1,e})V^{2,e} = 1_A$, $\varepsilon_A(V^{2,\alpha})V^{1,\alpha} = 1_\alpha$,

*($\mathcal{L}4$)* $\varphi_\beta(V^{1,\alpha}) \otimes V^{2,\alpha} = V^{1,\beta\alpha\beta^{-1}} \otimes V^{2,\beta\alpha\beta^{-1}}$.

In what follows, we shall investigate when $A \rtimes^{\pi} H$ forms a quasitriangular Hopf $\pi$-coalgebra.
For any $\alpha, \beta \in \pi$, Let

$$R_{\alpha,\beta} = R^{1,\alpha} \otimes R^{2,\alpha} \otimes R^{3,\beta} \otimes R^{4,\beta} \in A \rtimes^{\pi} H_{\alpha} \otimes A \rtimes^{\pi} H_{\beta},$$

we will use the following notations:

$$\begin{aligned}
P &= P^1 \otimes P^2 = R^{1,e}\varepsilon_e(R^{2,e}) \otimes R^{3,e}\varepsilon_e(R^{4,e}) \in A \otimes A,\\
Q_{\alpha,\beta} &= Q^{1,\alpha} \otimes Q^{2,\beta} = R^{2,\alpha}\varepsilon_A(R^{1,\alpha}) \otimes R^{4,\beta}\varepsilon_A(R^{3,\beta}) \in H_{\alpha} \otimes H_{\beta},\\
U_{\alpha} &= U^{1,\alpha} \otimes U^{2,\alpha} = R^{1,e}\varepsilon_e(R^{2,e}) \otimes R^{4,\alpha}\varepsilon_A(R^{3,\alpha}) \in A \otimes H_{\alpha},\\
V_{\alpha} &= V^{1,\alpha} \otimes U^{2,\alpha} = R^{2,\alpha}\varepsilon_A(R^{1,\alpha}) \otimes R^{3,e}\varepsilon_e(R^{4,e}) \in H_{\alpha} \otimes A.
\end{aligned}$$

Let $(A \rtimes^{\pi} H, R)$ be a quasitriangular Hopf $\pi$-coalgebra. Then we have the following results.

**Proposition 5.5.** *With the notations as above. We have*

*(A1)* $\varepsilon_A(P^1)P^2 = 1_A = P^1\varepsilon_A(P^2)$,
*(A2)* $\varepsilon_e(Q^{1,e})Q^{2,\alpha} = Q^{1,\alpha}\varepsilon_e(Q^{2,e}) = 1_{\alpha}$, $\varphi_{\beta}(Q^{1,\alpha}) \otimes \varphi_{\beta}(Q^{2,\gamma}) = Q^{1,\beta\alpha\beta^{-1}} \otimes Q^{1,\beta\gamma\beta^{-1}}$,
*(A3)* $\varepsilon_A(U^{1,\alpha})U^{2,\alpha} = 1_{\alpha}$, $\varepsilon_e(U^{2,e})U^{1,e} = 1_A$, $U^{1,\alpha} \otimes \varphi_{\beta}(U^{2,\alpha}) = U^{1,\beta\alpha\beta^{-1}} \otimes U^{2,\beta\alpha\beta^{-1}}$,
*(A4)* $\varepsilon_e(V^{1,e})V^{2,e} = 1_A$, $\varepsilon_A(V^{2,\alpha})V^{1,\alpha} = 1_{\alpha}$, $\varphi_{\beta}(V^{1,\alpha}) \otimes V^{2,\alpha} = V^{1,\beta\alpha\beta^{-1}} \otimes V^{2,\beta\alpha\beta^{-1}}$.

**Proposition 5.6.** *With the notations as above. Then:*

*(B1) The quasitriangular $R_{\alpha,\beta}$ is given by*

$$R_{\alpha,\beta} = U^{1,\beta}P^1 \otimes Q^{1,\alpha}V^{1,\alpha} \otimes P^2V^{2,\alpha} \otimes U^{2,\beta}Q^{2,\beta}$$

*(B2) $(A, P)$ is a quasitriangular Hopf algebra,*
*(B3) $(A, H, U)$ is a generalized right dual compatible $\pi$-pair,*
*(B4) $(H, A, V)$ is a generalized left dual compatible $\pi$-pair,*
*(B5) $(H, Q)$ is a weak quasitriangular structure.*

*Proof.* By the definition of a quasitriangular Hopf $\pi$-coalgebra, we have the above results (B2)-(B5).

For (B1), let $R = \{R_{\alpha,\beta}\}_{\alpha,\beta\in\pi}$ be a quasitriangular structure on $A \rtimes^{\pi} H$. For all $\alpha, \beta, \gamma, \delta \in \pi$, by equations (Q2) and (Q3), we compute as follows

$$\begin{aligned}
&(\bar{\Delta}_{\alpha,\beta} \otimes \bar{\Delta}_{\gamma,\delta})(R^{1,\alpha\beta} \otimes R^{2,\alpha\beta} \otimes R^{3,\gamma\delta} \otimes R^{4,\gamma\delta})\\
= \quad &(R^{1,\alpha\beta})_{(1)} \otimes (R^{1,\alpha\beta})_{(2)[-1,\alpha]}[\omega_{\alpha,\beta}((R^{1,\alpha\beta})_{(3)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle}(R^{2,\alpha\beta})_{(1,\alpha)\langle -1,\alpha\rangle}\\
&\otimes(R^{1,\alpha\beta})_{(2)[0,\alpha]}[\omega_{\alpha,\beta}((R^{1,\alpha\beta})_{(3)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle}(R^{2,\alpha\beta})_{(1,\alpha)\langle 0,\alpha\rangle}\\
&\otimes\omega_{\alpha,\beta}((R^{1,\alpha\beta})_{(3)})^{(2,\beta)}(R^{2,\alpha\beta})_{(2,\beta)}\\
&\otimes(R^{3,\gamma\delta})_{(1)} \otimes (R^{3,\gamma\delta})_{(2)[-1,\gamma]}[\omega_{\gamma,\delta}((R^{3,\gamma\delta})_{(3)})^{(1,\gamma)}]_{\langle -1,\gamma\rangle}(R^{4,\gamma\delta})_{(1,\gamma)\langle -1,\gamma\rangle}\\
&\otimes(R^{3,\gamma\delta})_{(2)[0,\gamma]}[\omega_{\gamma,\delta}((R^{3,\gamma\delta})_{(3)})^{(1,\gamma)}]_{\langle 0,\gamma\rangle}(R^{4,\gamma\delta})_{(1,\gamma)\langle 0,\gamma\rangle}\\
&\otimes\omega_{\gamma,\delta}((R^{3,\gamma\delta})_{(3)})^{(2,\delta)}(R^{4,\gamma\delta})_{(2,\delta)}
\end{aligned}$$

and

$$\begin{aligned}
&(\bar{\Delta}_{\alpha,\beta} \otimes \bar{\Delta}_{\gamma,\delta})(R^{1,\alpha\beta} \otimes R^{2,\alpha\beta} \otimes R^{3,\gamma\delta} \otimes R^{4,\gamma\delta})\\
= \quad &R^{1,\alpha}\tilde{R}^{1,\alpha} \otimes R^{2,\alpha}\tilde{R}^{2,\alpha} \otimes r^{1,\beta}\tilde{r}^{1,\beta} \otimes r^{2,\beta}\tilde{r}^{2,\beta} \otimes \tilde{R}^{3,\beta\gamma\beta^{-1}}\tilde{r}^{3,\gamma}\\
&\otimes\varphi_{\beta^{-1}}(\tilde{R}^{4,\beta\gamma\beta^{-1}})\tilde{r}^{4,\gamma} \otimes R^{3,\beta\gamma\beta^{-1}}r^{3,\delta} \otimes \varphi_{\beta^{-1}}(R^{4,\beta\delta\beta^{-1}})r^{4,\delta}.
\end{aligned}$$

Applying $id\otimes\varepsilon_e\otimes\varepsilon_A\otimes id\otimes id\otimes\varepsilon_e\otimes\varepsilon_A\otimes id$ to the equalities above for $\alpha=e$ and $\gamma=e$, we can obtain

$$\begin{aligned}
&R^{1,\beta}\otimes R^{2,\beta}\otimes R^{3,\delta}\otimes R^{4,\delta}\\
=\;&R^{1,e}\varepsilon_e(R^{2,e})\tilde{R}^{1,e}\varepsilon_e(\tilde{R}^{2,e})\otimes\varepsilon_A(r^{1,\beta})r^{2,\beta}\varepsilon_A(\tilde{r}^{1,\beta})\tilde{r}^{2,\beta}\\
&\otimes\tilde{R}^{3,e}\varepsilon_e(\tilde{R}^{4,e})\tilde{r}^{3,e}\varepsilon_e(\tilde{r}^{4,e})\otimes\varepsilon_A(R^{3,\beta\delta\beta^{-1}})\varphi_{\beta^{-1}}(R^{4,\beta\delta\beta^{-1}})\varepsilon_A(r^{3,\delta})r^{4,\delta}\\
=\;&U^{1,\beta\delta\beta^{-1}}P^1\otimes Q^{1,\beta}V^{1,\beta}\otimes P^2V^{2,\beta}\otimes\varphi_{\beta^{-1}}(U^{2,\beta\delta\beta^{-1}})Q^{2,\delta}\\
\overset{(A3)}{=}\;&U^{1,\delta}P^1\otimes Q^{1,\beta}V^{1,\beta}\otimes P^2V^{2,\beta}\otimes U^{2,\delta}Q^{2,\delta}.
\end{aligned}$$

This is exactly what we have to show. □

**Proposition 5.7.** *Suppose that $(A\rtimes^\pi H,R)$ has a quasitriangular structure with*

$$R_{\alpha,\beta}=U^{1,\beta}P^1\otimes Q^{1,\alpha}V^{1,\alpha}\otimes P^2V^{2,\alpha}\otimes U^{2,\beta}Q^{2,\beta}$$

*for all $\alpha,\beta\in\pi$. Then, for all $\alpha,\beta\in\pi$, $a\in A$ and $h\in H_\alpha$ we have*

*(C1)* $a_{[0,\alpha]}U^{1,\alpha}\otimes a_{[-1,\alpha]}U^{2,\alpha}=U^{1,\alpha}a\otimes U^{2,\alpha}$,

*(C2)* $V^{1,\alpha}\otimes aV^{2,\alpha}=V^{1,\alpha}a_{[-1,\alpha]}\otimes V^{2,\alpha}a_{[0,\alpha]}$,

*(C3)* $h_{\langle 0,\alpha\rangle}U^{1,\alpha}\otimes h_{\langle -1,\alpha\rangle}U^{2,\alpha}=U^{1,\alpha}\otimes U^{2,\alpha}h$,

*(C4)* $hV^{1,\alpha}\otimes V^{2,\alpha}=V^{1,\alpha}h_{\langle -1,\alpha\rangle}\otimes V^{2,\alpha}h_{\langle 0,\alpha\rangle}$,

*(C5)* $\omega_{\alpha\beta\alpha^{-1},\alpha}(a)^{(2,\alpha)}Q^{1,\alpha}\otimes\varphi_{\alpha^{-1}}(\omega_{\alpha\beta\alpha^{-1},\alpha}(a)^{(1,\alpha\beta\alpha^{-1})})Q^{2,\beta}$
$=Q^{1,\alpha}\omega_{\alpha,\beta}(a)^{(1,\alpha)}\otimes Q^{2,\beta}\omega_{\alpha,\beta}(a)^{(2,\beta)}$,

*(C6)* $(P^1)_{[-1,\alpha]}(V^{1,\alpha})_{\langle -1,\alpha\rangle}\otimes(P^1)_{[0,\alpha]}(V^{1,\alpha})_{\langle 0,\alpha\rangle}\otimes P^2V^{2,\alpha}$
$=V^{1,\alpha}\otimes P^1\otimes V^{2,\alpha}P^2$,

*(C7)* $(U^{1,\beta})_{[-1,\alpha]}(Q^{1,\alpha})_{\langle -1,\alpha\rangle}\otimes(U^{1,\beta})_{[0,\alpha]}(Q^{1,\alpha})_{\langle 0,\alpha\rangle}\otimes U^{2,\beta}Q^{2,\beta}$
$=Q^{1,\alpha}\otimes U^{1,\beta}\otimes Q^{2,\beta}U^{2,\beta}$,

*(C8)* $Q^{1,\alpha}V^{1,\alpha}\otimes(V^{2,\alpha})_{[-1,\beta]}(Q^{2,\beta})_{\langle -1,\beta\rangle}\otimes(V^{2,\alpha})_{[0,\beta]}(Q^{2,\beta})_{\langle 0,\beta\rangle}$
$=V^{1,\alpha}Q^{1,\alpha}\otimes Q^{2,\beta}\otimes V^{2,\alpha}$,

*(C9)* $U^{1,\alpha}P^1\otimes P^2_{[-1,\alpha]}(U^{2,\alpha})_{\langle -1,\alpha\rangle}\otimes P^2_{[0,\alpha]}(U^{2,\alpha})_{\langle 0,\alpha\rangle}=P^1U^{1,\alpha}\otimes U^{2,\alpha}\otimes P^2$.

*Proof.* Since $R$ satisfies (Q1), then for all $a\in A$, $h\in H_{\alpha\beta}$ with $\alpha,\beta\in\pi$, we have

$$\begin{aligned}
&a_{(2)[0,\alpha\beta\alpha^{-1}]}[\omega_{\alpha\beta\alpha^{-1},\alpha}(a_{(3)})^{(1,\alpha\beta\alpha^{-1})}]_{\langle 0,\alpha\beta\alpha^{-1}\rangle}h_{(1,\alpha\beta\alpha^{-1})\langle 0,\alpha\beta\alpha^{-1}\rangle}U^{1,\beta}P^1\otimes\omega_{\alpha\beta\alpha^{-1},\alpha}(a_{(3)})^{(2,\alpha)}h_{(2,\alpha)}Q^{1,\alpha}V^{1,\alpha}\\
&\otimes a_{(1)}P^2V^{2,\alpha}\otimes\varphi_{\alpha^{-1}}(a_{(2)[-1,\alpha\beta\alpha^{-1}]})\varphi_{\alpha^{-1}}([\omega_{\alpha\beta\alpha^{-1},\alpha}(a_{(3)})^{(1,\alpha\beta\alpha^{-1})}]_{\langle -1,\alpha\beta\alpha^{-1}\rangle})\varphi_{\alpha^{-1}}(h_{(1,\alpha\beta\alpha^{-1})\langle -1,\alpha\beta\alpha^{-1}\rangle})U^{2,\beta}Q^{2,\beta}\\
&=U^{1,\beta}P^1a_{(1)}\otimes Q^{1,\alpha}V^{1,\alpha}a_{(2)[-1,\alpha]}[\omega_{\alpha,\beta}(a_{(3)})^{(1,\alpha)}]_{\langle -1\rangle}h_{(1,\alpha)\langle -1,\alpha\rangle}\otimes\\
&\quad P^2V^{2,\alpha}a_{(2)[0,\alpha]}[\omega_{\alpha,\beta}(a_{(3)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle}h_{(1,\alpha)\langle 0,\alpha\rangle}\otimes U^{2,\beta}Q^{2,\beta}\omega_{\alpha,\beta}(a_{(3)})^{(2,\beta)}h_{(2,\beta)}.
\end{aligned}\tag{5.1}$$

Moreover, since $R$ also fulfills (Q2), we have

$$\begin{aligned}
&(U^{1,\gamma})_{(1)}P^1_{(1)}\otimes(U^{1,\gamma})_{(2)[-1,\alpha]}P^1_{(2)[-1,\alpha]}[\omega_{\alpha,\beta}(U^{1,\gamma}_{(3)}P^1_{(3)})^{(1,\alpha)}]_{\langle -1,\alpha\rangle}\\
&Q^{1,\alpha\beta}_{(1,\alpha)\langle -1,\alpha\rangle}(V^{1,\alpha\beta})_{(1,\alpha)\langle -1,\alpha\rangle}\otimes U^{1,\gamma}_{(2)[0,\alpha]}P^1_{(2)[0,\alpha]}[\omega_{\alpha,\beta}(U^{1,\gamma}_{(3)}P^1_{(3)})^{(1,\alpha)}]_{\langle 0,\alpha\rangle}\\
&Q^{1,\alpha\beta}_{(1,\alpha)\langle 0,\alpha\rangle}V^{1,\alpha\beta}_{(1,\alpha)\langle 0,\alpha\rangle}\otimes\omega_{\alpha,\beta}(U^{1,\gamma}_{(3)}P^1_{(3)})^{(2,\beta)}Q^{1,\alpha\beta}_{(2)}V^{1,\alpha\beta}_{(2,\beta)}\otimes P^2V^{2,\alpha\beta}\otimes U^{2,\gamma}Q^{2,\gamma}\\
&=U^{1,\beta\gamma\beta^{-1}}P^1\otimes Q^{1,\alpha}V^{1,\alpha}\otimes u^{1,\gamma}p^1\otimes q^{1,\beta}v^{1,\beta}\otimes P^2V^{2,\alpha}p^2v^{2,\beta}\otimes\varphi_{\beta^{-1}}(U^{2,\beta\gamma\beta^{-1}}Q^{2,\beta\gamma\beta^{-1}})u^{2,\gamma}q^{2,\gamma}.
\end{aligned}\tag{5.2}$$

Furthermore, by (Q3), we have

$$\begin{aligned}
&U^{1,\beta\gamma}P^1\otimes Q^{1,\alpha}V^{1,\alpha}\otimes P^2_{(1)}V^{2,\alpha}_{(1)}\\
&\otimes P^2_{(2)[-1,\beta]}V^{2,\alpha}_{(2)[-1,\beta]}[\omega_{\beta,\gamma}(P^2_{(3)}V^{2,\alpha}_{(3)})^{(1,\beta)}]_{\langle -1,\beta\rangle}U^{2,\beta\gamma}_{(1,\beta)\langle -1,\beta\rangle}Q^{2,\beta\gamma}_{(1,\beta)\langle -1,\beta\rangle}\\
&\otimes P^2_{(2)[0,\beta]}V^{2,\alpha}_{(2)[0,\beta]}[\omega_{\beta,\gamma}(P^2_{(3)}V^{2,\alpha}_{(3)})^{(1,\beta)}]_{\langle 0,\beta\rangle}U^{2,\beta\gamma}_{(1,\beta)\langle 0,\beta\rangle}Q^{2,\beta\gamma}_{(1,\beta)\langle 0,\beta\rangle}\otimes\omega_{\beta,\gamma}(P^2_{(3)}V^{2,\alpha}_{(3)})^{(2,\gamma)}U^{2,\beta\gamma}_{(2,\gamma)}Q^{2,\beta\gamma}_{(2,\gamma)}\\
&=U^{1,\gamma}P^1u^{1,\beta}p^1\otimes Q^{1,\alpha}V^{1,\alpha}q^{1,\alpha}v^{1,\alpha}\otimes p^2v^{2,\alpha}\otimes u^{2,\beta}q^{2,\beta}\otimes P^2V^{2,\alpha}\otimes U^{2,\gamma}Q^{2,\gamma}.
\end{aligned}\tag{5.3}$$

By considering $h = 1_{\alpha\beta}$ in (5.1) and applying $id \otimes \varepsilon_e \otimes \varepsilon_A \otimes id$ and $\varepsilon_A \otimes id \otimes id \otimes \varepsilon_e$ to both sides of the resulting equation respectively, we get the relations (C1) and (C2). If we let $a = 1_A$ in (5.1) and apply $id \otimes \varepsilon_e \otimes \varepsilon_A \otimes id$ and $\varepsilon_A \otimes id \otimes id \otimes \varepsilon_e$ to both sides, we get the relations (C3) and (C4). Applying $\varepsilon_A \otimes id_H \otimes \varepsilon_A \otimes id_H$ to (5.1) for $h = 1_{\alpha\beta}$ yields (C5). We apply $\varepsilon_A \otimes id \otimes id \otimes \varepsilon_e \otimes id \otimes \varepsilon_e$ and $\varepsilon_A \otimes id \otimes id \otimes \varepsilon_e \otimes \varepsilon_A \otimes id$ to (5.3) and obtain relations (C6) and (C7). (C8) and (C9) can be easily obtained by applying $\varepsilon_A \otimes id \otimes \varepsilon_A \otimes id \otimes id \otimes \varepsilon_H$ and $id \otimes \varepsilon_e \otimes \varepsilon_A \otimes id \otimes id \otimes \varepsilon_e$ to (5.2). □

By Propositions (5.5)-(5.7), we have the main result of this section.

**Theorem 5.8.** *Let $A$ be a Hopf algebra and $\Omega(A) = (H, \rho, \varrho, \omega)$ a crossed semi-Hopf $\pi$-coalgebra coextending structure of $A$ with the crossing $\varphi$. The following statements are equivalent:*

*(a) $A \rtimes^{\pi} H$ has a quasitriangular structure $R$, where*

$$R_{\alpha,\beta} = U^{1,\beta}P^1 \otimes Q^{1,\alpha}V^{1,\alpha} \otimes P^2V^{2,\alpha} \otimes U^{2,\beta}Q^{2,\beta}$$

*for all $\alpha, \beta \in \pi$,*

*(b) There exist a set of quadruples $(P, Q, U, V)$, where $P = P^1 \otimes P^2 \in A \otimes A$, $U = \{U_\alpha = U^{1,\alpha} \otimes U^{2,\alpha} \in A \otimes H_\alpha\}_{\alpha\in\pi}$, $V = \{V_\alpha = V^{1,\alpha} \otimes V^{2,\alpha} \in H_\alpha \otimes A\}_{\alpha\in\pi}$ and $Q = \{Q_{\alpha,\beta} = Q^{1,\alpha} \otimes Q^{2,\beta} \in H_\alpha \otimes H_\beta\}_{\alpha,\beta\in\pi}$ such that $(A, P)$ is a quasitriangular Hopf algebra, $(A, H, U)$ is a generalized right dual compatible $\pi$-pair and $(H, A, V)$ is a generalized left dual compatible $\pi$-pair and $(H, Q)$ is a weak quasitriangular structure and the compatibilities conditions (C1)-(C9) hold.*

## 6. Applications

In this section, we shall consider applications of our main results in the previous sections, which lead to the group versions of bicrossed coproduct and crossed coproduct.

### 6.1. Group Bicrossed Coproducts

By considering $\omega(a) = \varepsilon_A(a)1_H \otimes 1_H$, we can get a new coproduct named *$\pi$-bicrossed coproduct* which will be denoted by $A \bowtie^{\pi} H$, from Theorem 3.3, we have

**Theorem 6.1.** *Let $H$ be a crossed Hopf $\pi$-coalgebra and $A$ a bialgebra. If $H$ is a right $\pi$-$A$-comodule algebra and $A$ is a left $\pi$-$H$-comodule algebra. Then the following statements are equivalent:*

*(i) The $\pi$-bicrossed coproduct $A \bowtie^{\pi} H$ is a semi-Hopf $\pi$-coalgebra via the coproduct and the counit given by*

$$\bar{\Delta}_{\alpha,\beta}(a \bowtie h) = a_{(1)} \bowtie a_{(2)[-1,\alpha]}h_{(1,\alpha)\langle -1,\alpha\rangle} \otimes a_{(2)[0,\alpha]}h_{(1,\alpha)\langle 0,\alpha\rangle} \bowtie h_{(2,\beta)},$$

$$\bar{\varepsilon}_e(a \bowtie g) = \varepsilon_A(a)\varepsilon_e(g)$$

*for all $a \in A$ and $h \in H_{\alpha\beta}$, $g \in H_e$, and the usual tensor product algebra structure.*

*(ii) The following conditions are satisfied: for all $a, b \in A$ and $h, g \in H$,*

*(a)* $b_{[-1,\alpha]}h_{\langle -1,\alpha\rangle} \otimes b_{[0,\alpha]}h_{\langle 0,\alpha\rangle} = h_{\langle -1,\alpha\rangle}b_{[-1,\alpha]} \otimes h_{\langle 0,\alpha\rangle}b_{[0,\alpha]}$,

*(b)* $h_{\langle -1,\alpha\beta\rangle(1,\alpha)} \otimes h_{\langle -1,\alpha\beta\rangle(2,\beta)} \otimes h_{\langle 0,\alpha\beta\rangle} = h_{(1,\alpha)\langle -1,\alpha\rangle} \otimes h_{(1,\alpha)\langle 0,\alpha\rangle[-1,\beta]}h_{(2,\beta)\langle -1,\beta\rangle} \otimes h_{(1,\alpha)\langle 0,\alpha\rangle[0,\beta]}h_{(2,\beta)\langle 0,\beta\rangle}$,

*(c)* $a_{[-1,\alpha]} \otimes a_{[0,\alpha](1)} \otimes a_{[0,\alpha](2)} = a_{(1)[-1,\alpha]}a_{(2)[-1,\alpha]\langle -1,\alpha\rangle} \otimes a_{(1)[0,\alpha]}a_{(2)[-1,\alpha]\langle 0,\alpha\rangle} \otimes a_{(2)[0,\alpha]}$.

*If $H$ is a Hopf $\pi$-coalgebra and $A$ a Hopf algebra with the antipodes $S = \{S_\alpha\}$ and $S_A$, respectively, then the $\pi$-bicrossed coproduct is also a Hopf $\pi$-coalgebra with the antipode given by*

$$\bar{S}_\alpha(a \bowtie h) = S_A(a_{[0,\alpha]}h_{\langle 0,\alpha\rangle}) \bowtie S_\alpha(a_{[-1,\alpha]}h_{\langle -1,\alpha\rangle})$$

*for all $a \in A$ and $h \in H_\alpha$ with $\alpha \in \pi$.*

The following result that characterizes the quasitriangular structures on a $\pi$-bicrossed coproduct can be obtained from Theorem 5.8 by considering $\omega_{\alpha,\beta}(a) = \varepsilon_A(a)1_\alpha \otimes 1_\beta$.

**Theorem 6.2.** *Let $H$ be a crossed Hopf $\pi$-coalgebra and $A$ a bialgebra. If $H$ is a right $\pi$-$A$-comodule algebra and $A$ is a left $\pi$-$H$-comodule algebra. The following statements are equivalent:*

(a) *$A \bowtie^{\pi} H$ has a quasitriangular structure $R$, where*

$$R_{\alpha,\beta} = U^{1,\beta}P^1 \otimes Q^{1,\alpha}V^{1,\alpha} \otimes P^2V^{2,\alpha} \otimes U^{2,\beta}Q^{2,\beta}$$

*for all $\alpha, \beta \in \pi$,*

(b) *There exist a set of quadruples $(P, Q, U, V)$, where $P = P^1 \otimes P^2 \in A \otimes A$, $U = \{U_\alpha = U^{1,\alpha} \otimes U^{2,\alpha} \in A \otimes H_\alpha\}_{\alpha\in\pi}$, $V = \{V_\alpha = V^{1,\alpha} \otimes V^{2,\alpha} \in H_\alpha \otimes A\}_{\alpha\in\pi}$ and $Q = \{Q_{\alpha,\beta} = Q^{1,\alpha} \otimes Q^{2,\beta} \in H_\alpha \otimes H_\beta\}_{\alpha,\beta\in\pi}$ such that $(A, P)$ is a quasitriangular Hopf algebra, $(A, H, U)$ is a right dual compatible $\pi$-pair and $(H, A, V)$ is a left dual compatible $\pi$-pair and $(H, Q)$ is a quasitriangular structure and the compatibilities conditions hold:*

*(D1)* $a_{[0,\alpha]}U^{1,\alpha} \otimes a_{[-1,\alpha]}U^{2,\alpha} = U^{1,\alpha}a \otimes U^{2,\alpha}$,
*(D2)* $V^{1,\alpha} \otimes aV^{2,\alpha} = V^{1,\alpha}a_{[-1,\alpha]} \otimes V^{2,\alpha}a_{[0,\alpha]}$,
*(D3)* $h_{\langle 0,\alpha\rangle}U^{1,\alpha} \otimes h_{\langle -1,\alpha\rangle}U^{2,\alpha} = U^{1,\alpha} \otimes U^{2,\alpha}h$,
*(D4)* $hV^{1,\alpha} \otimes V^{2,\alpha} = V^{1,\alpha}h_{\langle -1,\alpha\rangle} \otimes V^{2,\alpha}h_{\langle 0,\alpha\rangle}$,
*(D5)* $(P^1)_{[-1,\alpha]}(V^{1,\alpha})_{\langle -1,\alpha\rangle} \otimes (P^1)_{[0,\alpha]}(V^{1,\alpha})_{\langle 0,\alpha\rangle} \otimes P^2V^{2,\alpha} = V^{1,\alpha} \otimes P^1 \otimes V^{2,\alpha}P^2$,
*(D6)* $(U^{1,\beta})_{[-1,\alpha]}(Q^{1,\alpha})_{\langle -1,\alpha\rangle} \otimes (U^{1,\beta})_{[0,\alpha]}(Q^{1,\alpha})_{\langle 0,\alpha\rangle} \otimes U^{2,\beta}Q^{2,\beta} = Q^{1,\alpha} \otimes U^{1,\beta} \otimes Q^{2,\beta}U^{2,\beta}$,
*(D7)* $Q^{1,\alpha}V^{1,\alpha} \otimes (V^{2,\alpha})_{[-1,\beta]}(Q^{2,\beta})_{\langle -1,\beta\rangle} \otimes (V^{2,\alpha})_{[0,\beta]}(Q^{2,\beta})_{\langle 0,\beta\rangle} = V^{1,\alpha}Q^{1,\alpha} \otimes Q^{2,\beta} \otimes V^{2,\alpha}$,
*(D8)* $U^{1,\alpha}P^1 \otimes P^2_{[-1,\alpha]}(U^{2,\alpha})_{\langle -1,\alpha\rangle} \otimes P^2_{[0,\alpha]}(U^{2,\alpha})_{\langle 0,\alpha\rangle} = P^1U^{1,\alpha} \otimes U^{2,\alpha} \otimes P^2$

*for all $a \in A$ and $h \in H_\alpha$.*

Let $A$ be a Hopf algebra and $H$ a crossed Hopf $\pi$-coalgebra and $T = \{T_\alpha = T^{1,\alpha} \otimes T^{2,\alpha} \in H_\alpha \otimes A\}_{\alpha\in\pi}$ a left dual compatible pair with the inverse $T^{-1} = \{T_\alpha^{-1} = (T^{-1})^{1,\alpha} \otimes (T^{-1})^{2,\alpha} \in H_\alpha \otimes A\}_{\alpha\in\pi}$. Define the following coactions:

$$\rho_\alpha : H_\alpha \to H_\alpha \otimes A,\ \rho_\alpha(h) = (T^{-1})^{1,\alpha}hT^{1,\alpha} \otimes (T^{-1})^{2,\alpha}T^{2,\alpha}$$

$$\varrho_\alpha : A \to H_\alpha \otimes A,\ \rho_\alpha(a) = (T^{-1})^{1,\alpha}T^{1,\alpha} \otimes (T^{-1})^{2,\alpha}aT^{2,\alpha}.$$

It is easily checked that $H$ is a right $\pi$-$A$-comodule algebra and $A$ is a left $\pi$-$H$-comodule algebra. Then the corresponding $\pi$-bicrossed coproduct is denoted by $A \rtimes_T^{\pi} H$. As a special case of Theorem 6.2, we get

**Corollary 6.3.** *Let $A$ be a Hopf algebra and $H$ a crossed Hopf $\pi$-coalgebra and $T = \{T_\alpha = T^{1,\alpha} \otimes T^{2,\alpha} \in H_\alpha \otimes A\}_{\alpha\in\pi}$ a left dual compatible pair. The following statements are equivalent:*

(a) *$A \rtimes_T^{\pi} H$ has a quasitriangular structure, where*

$$R_{\alpha,\beta} = U^{1,\beta}P^1 \otimes Q^{1,\alpha}V^{1,\alpha} \otimes P^2V^{2,\alpha} \otimes U^{2,\beta}Q^{2,\beta}$$

*for all $\alpha, \beta \in \pi$,*

(b) *There exist a set of quadruples $(P, Q, U, V)$, where $P = P^1 \otimes P^2 \in A \otimes A$, $U = \{U_\alpha = U^{1,\alpha} \otimes U^{2,\alpha} \in A \otimes H_\alpha\}_{\alpha\in\pi}$, $V = \{V_\alpha = V^{1,\alpha} \otimes V^{2,\alpha} \in H_\alpha \otimes A\}_{\alpha\in\pi}$ and $Q = \{Q_{\alpha,\beta} = Q^{1,\alpha} \otimes Q^{2,\beta} \in H_\alpha \otimes H_\beta\}_{\alpha,\beta\in\pi}$ such that $(A, P)$ is a quasitriangular Hopf algebra, $(A, H, U)$ is a right dual compatible $\pi$-pair and $(H, A, V)$ is a left dual compatible $\pi$-pair and $(H, Q)$ is a quasitriangular structure and the compatibilities conditions hold:*

*(E1)* $(T^{-1})^{2,\alpha}aT^{2,\alpha}U^{1,\alpha} \otimes (T^{-1})^{1,\alpha}T^{1,\alpha}U^{2,\alpha} = U^1a \otimes U^{2,\alpha}$,
*(E2)* $V^{1,\alpha} \otimes aV^{2,\alpha} = V^{1,\alpha}(T^{-1})^{1,\alpha}T^{1,\alpha} \otimes V^{2,\alpha}(T^{-1})^{2,\alpha}aT^{2,\alpha}$,
*(E3)* $(T^{-1})^{2,\alpha}T^{2,\alpha}U^{1,\alpha} \otimes (T^{-1})^{1,\alpha}hT^{1,\alpha}U^{2,\alpha} = U^{1,\alpha} \otimes U^{2,\alpha}h$,
*(E4)* $hV^{1,\alpha} \otimes V^{2,\alpha} = V^{1,\alpha}(T^{-1})^{1,\alpha}hT^{1,\alpha} \otimes V^{2,\alpha}(T^{-1})^{2,\alpha}T^{2,\alpha}$,
*(E5)* $(T^{-1})^{1,\alpha}V^{1,\alpha}T^{1,\alpha} \otimes (T^{-1})^{2,\alpha}P^1T^{2,\alpha} \otimes P^2V^{2,\alpha} = V^{1,\alpha} \otimes P^1 \otimes V^{2,\alpha}P^2$,
*(E6)* $(T^{-1})^{1,\alpha}Q^{1,\alpha}T^{1,\alpha} \otimes (T^{-1})^{2,\alpha}U^{1,\beta}T^{2,\alpha} \otimes U^{2,\beta}Q^{2,\beta} = Q^{1,\alpha} \otimes U^{1,\beta} \otimes Q^{2,\beta}U^{2,\beta}$,
*(E7)* $Q^{1,\alpha}V^{1,\alpha} \otimes (T^{-1})^{1,\beta}Q^{2,\beta}T^{1,\beta} \otimes (T^{-1})^{2,\beta}V^{2,\alpha}T^{2,\beta} = V^{1,\alpha}Q^{1,\alpha} \otimes Q^{2,\beta} \otimes V^{2,\alpha}$,
*(E8)* $U^{1,\alpha}P^1 \otimes (T^{-1})^{1,\alpha}U^{2,\alpha}T^{1,\alpha} \otimes (T^{-1})^{2,\alpha}P^2T^{2,\alpha} = P^1U^{1,\alpha} \otimes U^{2,\alpha} \otimes P^2$.

*for all $\alpha, \beta \in \pi$, $a \in A$ and $h \in H_\alpha$.*

**Corollary 6.4.** *Let $(A, P)$ be a coquasitriangular Hopf algebra and $H$ a quasitriangular Hopf $\pi$-coalgebras and $T = \{T_\alpha = T^{1,\alpha} \otimes T^{2,\alpha} \in H_\alpha \otimes A\}_{\alpha\in\pi}$ a left dual compatible pair. Then $A \rtimes^\pi_T H$ is a quasitriangular Hopf $\pi$-coalgebra with the quasitriangular structure given by*

$$R_{\alpha,\beta} = \tilde{T}^{2,\beta^{-1}} P^1 \otimes Q^{1,\alpha} T^{1,\alpha} \otimes P^2 T^{2,\alpha} \otimes S_{\beta^{-1}}(\tilde{T}^{1,\beta^{-1}}) Q^{2,\beta}.$$

As a consequence, we shall derive the necessary and sufficient conditions for $A \rtimes^\pi_T H$ to be a quasitriangular Hopf $\pi$-coalgebra.

**Corollary 6.5.** *Let $(A, P)$ be a Hopf algebra and $H$ a crossed Hopf $\pi$-coalgebras and $T = \{T_\alpha = T^{1,\alpha}\otimes T^{2,\alpha} \in H_\alpha \otimes A\}_{\alpha\in\pi}$ a left dual compatible $\pi$-pair. Then $A \rtimes^\pi_T H$ is a quasitriangular Hopf $\pi$-coalgebra if and only if $A$ and $H$ are quasitriangular.*

**Example 6.6.** *Let $\mathcal{GL}_2(k)$ be the group of invertible $2\times 2$-matrices with coefficients in $k$. Recall from Virelizier in [9] that a crossed Hopf $\mathcal{GL}_2(k)$-coalgebra structure of the family of $k$-algebras $H = \{H_\alpha\}_{\alpha\in\mathcal{GL}_2(k)}$ is given by*

$$\begin{cases} \Delta_{\alpha,\beta}(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S_\alpha(g) = g, \\ \Delta_{\alpha,\beta}(x_i) = 1 \otimes x_i + \sum_{k=1}^2 b_{ki} x_k \otimes g, \quad \varepsilon(x_i) = 0, \quad S_\alpha(x_i) = \sum_{k=1}^2 a_{k,i} g x_k, \\ \Delta_{\alpha,\beta}(y_i) = y_i \otimes 1 + g \otimes y_i, \quad \varepsilon(y_i) = 0, \quad S_\alpha(y_i) = -g y_i, \\ \varphi_\alpha(g) = g, \varphi_\alpha(x_i) = \sum_{k=1}^2 a_{k,i} x_k, \ \ \varphi_\alpha(y_i) = \sum_{k=1}^2 \tilde{a}_{i,k} y_k, \end{cases}$$

*for any $\alpha = (a_{ij}), \beta = (b_{ij}) \in \mathcal{GL}_2(k)$ and $1 \le i \le 2$, where $(\tilde{a}_{i,i}) = \alpha^{-1}$ and $H_\alpha$ is the $k$-algebra generated by $g, x_1, x_2, y_1, y_2$ subject to the following relation:*

$$\begin{cases} g^2 = 1, \quad x_1^2 = x_2^2 = 0, \quad g x_i = -x_i g, \quad x_1 x_2 = -x_2 x_1, \\ y_1^2 = y_2^2 = 0, \quad g y_i = -y_i g, \quad y_1 y_2 = -y_2 y_1, \\ \qquad x_i y_j - y_j x_i = (a_{ji} - \delta_{ij}) g, \end{cases}$$

*for $\alpha = (a_{ij}) \in \mathcal{GL}_2(k)$ and $1 \le i, j \le 2$. The Hopf algebra $H = \{H_\alpha\}_{\alpha\in\mathcal{GL}_2(k)}$ is a quasitriangular Hopf $\pi$-coalgebra, for any $\alpha = (a_{ij}), \beta = (b_{ij}) \in \mathcal{GL}_2(k)$,*

$$\begin{aligned} Q_{\alpha,\beta} =& \frac{1}{2}(x_1 \otimes y_1 + x_1 \otimes g y_1 + g x_1 \otimes y_1 - g x_1 \otimes g y_1 \\ &+ x_2 \otimes y_2 + x_2 \otimes g y_2 + g x_2 \otimes y_2 - g x_2 \otimes g y_2 \\ &+ x_1 x_2 \otimes y_1 y_2 + x_1 x_2 \otimes g y_1 y_2 + g x_1 x_2 \otimes y_1 y_2 - g x_1 x_2 \otimes g y_1 y_2). \end{aligned}$$

*Let $A$ be the group algebra $kZ_2$ with the obvious Hopf algebra structure and let $\sigma$ be a generator of $Z_2$ in multiplicative notation. We have a quasitriangular structure $P$ given by*

$$P = \frac{1}{2}(1_A \otimes 1_A + 1_A \otimes \sigma + \sigma \otimes 1_A - \sigma \otimes \sigma).$$

*Let $T = \{T_\alpha = \frac{1}{2}(1 \otimes 1_A + 1 \otimes \sigma + g \otimes 1_A - g \otimes \sigma) \in H_\alpha \otimes A\}$. Observe that $T$ is a left dual compatible $\pi$-pair. From Corollary 6.5, $A \rtimes^\pi_T H$ is a quasitriangular Hopf $\pi$-coalgebra with the structure $R$ given by*

$$\begin{aligned} R_{\alpha,\beta} =& \frac{1}{16}(1_A \otimes 1 \otimes 1_A \otimes 1 + \sigma \otimes 1 \otimes 1_A \otimes 1 + 1_A \otimes 1 \otimes 1_A \otimes g - \sigma \otimes 1 \otimes 1_A \otimes g) \\ &\times (1_A \otimes 1 \otimes 1_A \otimes 1 + 1_A \otimes 1 \otimes \sigma \otimes 1 + \sigma \otimes 1 \otimes 1_A \otimes 1 - \sigma \otimes 1 \otimes \sigma \otimes 1) \\ &\times (1_A \otimes x_1 \otimes 1_A \otimes y_1 + 1_A \otimes x_1 \otimes 1_A \otimes g y_1 \\ &+ 1_A \otimes g x_1 \otimes 1_A \otimes y_1 - 1_A \otimes g x_1 \otimes 1_A \otimes g y_1 \\ &+ 1_A \otimes x_2 \otimes 1_A \otimes y_2 + 1_A \otimes x_2 \otimes 1_A \otimes g y_2 \\ &+ 1_A \otimes g x_2 \otimes 1_A \otimes y_2 - 1_A \otimes g x_2 \otimes 1_A \otimes g y_2 \\ &+ 1_A \otimes x_1 x_2 \otimes 1_A \otimes y_1 y_2 + 1_A \otimes x_1 x_2 \otimes 1_A \otimes g y_1 y_2 \\ &+ 1_A \otimes g x_1 x_2 \otimes 1_A \otimes y_1 y_2 - 1_A \otimes g x_1 x_2 \otimes 1_A \otimes g y_1 y_2) \\ &\times (1_A \otimes 1 \otimes 1_A \otimes 1 + 1_A \otimes 1 \otimes \sigma \otimes 1 + 1_A \otimes g \otimes 1_A \otimes 1 - 1_A \otimes g \otimes \sigma \otimes 1). \end{aligned}$$

*6.2. $\pi$-crossed Coproducts*

Let $A$ be a Hopf algebra and $H$ a crossed Hopf $\pi$-coalgebra. Assume that $A$ is a left $\pi$-$H$-comodule algebra with the coaction structure $\varrho$. If there exists a family of algebra maps $\omega = \{\omega_{\alpha,\beta} : A \to H_\alpha \otimes H_\beta\}_{\alpha,\beta\in\pi}$ such that the following conditions hold:

$$\varepsilon_e(\omega_{e,\alpha}(a)^{(1,e)})\omega_{e,\alpha}(a)^{(2,\alpha)} = \varepsilon_A(a)1_\alpha = \omega_{\alpha,e}(a)^{(1,\alpha)}\varepsilon_e(\omega_{\alpha,e}(a)^{(2,e)})$$

for all $\alpha, \beta \in \pi$ and $a \in A$. Then we have a coextenting $\pi$-datum $\Omega(A) = (H, \rho, \varrho, \omega)$, where $\rho = \{\rho_\alpha\}_{\alpha\in\pi}$ is given by $\rho_\alpha(h) = h \otimes 1_A$ for all $h \in H_\alpha$. The special $\pi$-unified coproduct is called $\pi$-crossed coproduct. By Theorem 3.3, we have

**Theorem 6.7.** *Under the above assumption, the following statements are equivalent:*

*(i) $\pi$-crossed coproduct $A \rtimes^\pi H$ is a semi-Hopf $\pi$-coalgebra via the coproduct and the counit given by*

$$\bar{\Delta}_{\alpha,\beta}(a \rtimes h) = a_{(1)} \rtimes a_{(2)[-1,\alpha]}\omega_{\alpha,\beta}(a_{(3)})^{(1,\alpha)}h_{(1,\alpha)} \otimes a_{(2)[0,\alpha]} \rtimes \omega_{\alpha,\beta}(a_{(3)})^{(2,\beta)}h_{(2,\beta)}.$$

$$\bar{\varepsilon}_e(a \rtimes g) = \varepsilon_A(a)\varepsilon_e(g)$$

*for all $a \in A$ and $h \in H_{\alpha\beta}$, $g \in H_e$ and the usual tensor product algebra structure.*

*(ii) The following conditions hold: for all $\alpha, \beta, \gamma \in \pi$, $a, b \in A$,*

(a) $\omega_{\alpha,\beta}(b)^{(1,\alpha)}h_{(1,\alpha)} \otimes \omega_{\alpha,\beta}(b)^{(2,\beta)}h_{(2,\beta)} = h_{(1,\alpha)}\omega_{\alpha,\beta}(b)^{(1,\alpha)} \otimes h_{(2,\beta)}\omega_{\alpha,\beta}(b)^{(2,\beta)}$, *all $h \in H_{\alpha\beta}$,*

(b) $b_{[-1,\alpha]}h \otimes b_{[0,\alpha]} = hb_{[-1,\alpha]} \otimes b_{[0,\alpha]}$, *all $h \in H_\alpha$,*

(c) $a_{[-1,\alpha]} \otimes a_{[0,\alpha](1)} \otimes a_{[0,\alpha](2)} = a_{(1)[-1,\alpha]}a_{(2)[-1,\alpha]} \otimes a_{(1)[0,\alpha]} \otimes a_{(2)[0,\alpha]}$,

(d) $\omega_{\alpha,\beta}(a_{(1)})^{(1,\alpha)}a_{(2)[-1,\alpha\beta](1,\alpha)} \otimes \omega_{\alpha,\beta}(a_{(1)})^{(2,\beta)}a_{(2)[-1,\alpha\beta](2,\beta)} \otimes a_{(2)[0,\alpha\beta]}$
$= a_{(1)[-1,\alpha]}\omega_{\alpha,\beta}(a_{(2)})^{(1,\alpha)} \otimes a_{(1)[0,\alpha][-1,\beta]}\omega_{\alpha,\beta}(a_{(2)})^{(2,\beta)} \otimes a_{(1)[0,\alpha][0,\beta]}$,

(e) $\omega_{\alpha,\beta}(a_{(1)})^{(1,\alpha)}[\omega_{\alpha\beta,\gamma}(a_{(2)})^{(1,\alpha\beta)}]_{(1,\alpha)} \otimes \omega_{\alpha,\beta}(a_{(1)})^{(2,\beta)}[\omega_{\alpha\beta,\gamma}(a_{(2)})^{(1,\alpha\beta)}]_{(2,\beta)} \otimes \omega_{\alpha\beta,\gamma}(a_{(2)})^{(2,\gamma)}$
$= a_{(1)[-1,\alpha]}\omega_{\alpha,\beta\gamma}(a_{(2)})^{(1,\alpha)} \otimes \omega_{\beta,\gamma}(a_{(1)[0,\alpha]})^{(1,\beta)}[\omega_{\alpha,\beta\gamma}(a_{(2)})^{(2,\beta\gamma)}]_{(1,\beta)}$
$\otimes \omega_{\beta,\gamma}(a_{(1)[0,\alpha]})^{(2,\gamma)}[\omega_{\alpha,\beta\gamma}(a_{(2)})^{(2,\beta\gamma)}]_{(2,\gamma)}$.

**Theorem 6.8.** *The following statements are equivalent:*

*(a) $A \rtimes^\pi H$ has a quasitriangular structure $R$, where*

$$R_{\alpha,\beta} = U^{1,\beta}P^1 \otimes Q^{1,\alpha}V^{1,\alpha} \otimes P^2V^{2,\alpha} \otimes U^{2,\beta}Q^{2,\beta}$$

*for all $\alpha, \beta \in \pi$,*

*(b) There exist a set of quadruples $(P, Q, U, V)$, where $P = P^1 \otimes P^2 \in A \otimes A$, $U = \{U_\alpha = U^{1,\alpha} \otimes U^{2,\alpha} \in A \otimes H_\alpha\}_{\alpha\in\pi}$, $V = \{V_\alpha = V^{1,\alpha} \otimes V^{2,\alpha} \in H_\alpha \otimes A\}_{\alpha\in\pi}$ and $Q = \{Q_{\alpha,\beta} = Q^{1,\alpha} \otimes Q^{2,\beta} \in H_\alpha \otimes H_\beta\}_{\alpha,\beta\in\pi}$ such that $(A, P)$ is a quasitriangular Hopf algebra, $(A, H, U)$ is a generalized right dual compatible $\pi$-pair and $(H, A, V)$ is a generalized left dual compatible $\pi$-pair and $(H, Q)$ is a weak quasitriangular structure and the compatibilities conditions hold: for all $\alpha, \beta \in \pi$, $a \in A$ and $h \in H_{\alpha\beta}$,*

*(F1)* $a_{[0,\alpha]}U^{1,\alpha} \otimes a_{[-1,\alpha]}U^{2,\alpha} = U^{1,\alpha}a \otimes U^{2,\alpha}$,

*(F2)* $V^{1,\alpha} \otimes aV^{2,\alpha} = V^{1,\alpha}a_{[-1,\alpha]} \otimes V^{2,\alpha}a_{[0,\alpha]}$,

*(F3)* $U^{1,\alpha} \otimes hU^{2,\alpha} = U^{1,\alpha} \otimes U^{2,\alpha}h$,

*(F4)* $hV^{1,\alpha} \otimes V^{2,\alpha} = V^{1,\alpha}h \otimes V^{2,\alpha}$,

*(F5)* $\omega_{\alpha\beta\alpha^{-1},\alpha}(a)^{(2,\alpha)}Q^{1,\alpha} \otimes \varphi_{\alpha^{-1}}(\omega_{\alpha\beta\alpha^{-1},\alpha}(a)^{(1,\alpha\beta\alpha^{-1})})Q^{2,\beta} = Q^{1,\alpha}\omega_{\alpha,\beta}(a)^{(1,\alpha)} \otimes Q^{2,\beta}\omega_{\alpha,\beta}(a)^{(2,\beta)}$,

*(F6)* $(P^1)_{[-1,\alpha]}V^{1,\alpha} \otimes (P^1)_{[0,\alpha]} \otimes P^2V^{2,\alpha} = V^{1,\alpha} \otimes P^1 \otimes V^{2,\alpha}P^2$,

*(F7)* $(U^{1,\beta})_{[-1,\alpha]}Q^{1,\alpha} \otimes (U^{1,\beta})_{[0,\alpha]} \otimes U^{2,\beta}Q^{2,\beta} = Q^{1,\alpha} \otimes U^{1,\beta} \otimes Q^{2,\beta}U^{2,\beta}$,

*(F8)* $Q^{1,\alpha}V^{1,\alpha} \otimes (V^{2,\alpha})_{[-1,\beta]}Q^{2,\beta} \otimes (V^{2,\alpha})_{[0,\beta]} = V^{1,\alpha}Q^{1,\alpha} \otimes Q^{2,\beta} \otimes V^{2,\alpha}$,

*(F9)* $U^{1,\alpha}P^1 \otimes P^2_{[-1,\alpha]}U^{2,\alpha} \otimes P^2_{[0,\alpha]} = P^1U^{1,\alpha} \otimes U^{2,\alpha} \otimes P^2$.

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