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Fuzzy Hyper p-ideals of Hyper BCK-algebras

M. Aslam Malik[a], Muhammad Touqeer[b]

[a]*Department of Mathematics, University of the Punjab, Quaid-e-Azam Campus, Lahore-54590, Pakistan*

[b]*Department of Mathematics, University of the Punjab, Quaid-e-Azam Campus, Lahore-54590, Pakistan*

Abstract. The paper is a reflection of “fuzzy sets” applied to “hyper p-ideals” and their comparison with simple “fuzzy hyper BCK-ideals”. The idea of “fuzzy (weak, strong) hyper p-ideals” is presented and characterization of these ideals is conferred using different concepts like that of “level subsets, hyper homomorphic pre-image” etc. The connections between “fuzzy (weak, strong) hyper p-ideals” are discussed and “the strongest fuzzy relation” on a “hyper BCK-algebra” is conferred.

1. Introduction

The “hyper structure theory” was presented by Marty [16], in 1934, at the “8th Congress of Scandinavian Mathematicians”. Now a days hyperstructures are widely used in both pure and applied mathematics. During the exploration of properties of set difference, Imai and Iseki in 1966 bring together a set of axioms commonly known as BCK-algebras. Komori [14] in 1983, introduced a new class of algebras called BCC-algebras or BIK$^+$-algebras. Dudek et al. [5, 8] discussed the properties of branches, ideals and atoms in weak BCC-algebras. Dudek [4] introduced the concept of solid weak BCC-algebras and further, he and Thomys [6] generalized the concept of BCC-algebras. Borzooei et al. [2] discussed the applications of hyperstructures in BCC-algebras. Later in 2000, this theory was applied to BCK-algebras by Jun et al. [13]. Jun et al. [12], deliberated the properties of “fuzzy strong hyper BCK-ideals”. The most apposite theory of “fuzzy sets” which is a tool for handling with uncertainties was presented by Zadeh [17] in 1965. Dudek et al. [7], “applied the fuzzy sets to BCC-algebras”. Moreover in 2001, “Jun and Xin [10] applied the fuzzy set

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Email addresses: malikpu@yahoo.com (M. Aslam Malik), touqeer-fareed@yahoo.com (Muhammad Touqeer)

theory to hyper BCK-algebras". This paper confers, "the concept of fuzzification of (weak, strong) hyper p-ideals in hyper BCK-algebras" and associated properties.

2. Preliminaries

"If H is a non-empty set with the hyperoperation 'o' from $H \times H$ into $P^*(H)$ the collection of all non-empty subsets of H, then for any subsets A and B of H by AoB we denote the set $\bigcup\{a \circ b \mid a \in A, b \in B\}$". "If $A = \{a\}$, then instead of {a}oB we write aoB".

Definition 2.1. [13] *"Hyper BCK-algebra is a non-empty set H equipped with a hyperoperation "$\circ$" and a constant* 0 *fulfilling the following conditions:*
($HK1$) $(u \circ w) \circ (v \circ w) \ll u \circ v$
($HK2$) $(u \circ v) \circ w = (u \circ w) \circ v$
($HK3$) $u \circ H \ll \{u\}$
($HK4$) $u \ll v$ *and* $v \ll u$ *imply* $u = v$
for any $u, v, w \in H$. *Here* $u \ll v$ *is defined by* $0 \in u \circ v$ *and for any* $G,\ I \subseteq H$, $G \ll I$ *is defined as* $\forall\ a \in G, \exists\ b \in I$ *such that* $a \ll b$. *The relation "$\ll$" is called the hyper order in H".*

Proposition 2.2. [13] *"For a hyper BCK-algebra H, the following properties are obvious:*
(i) $u \circ 0 = \{u\}$
(ii) $u \circ v \ll u$
(iii) $0 \circ G = \{0\}$
(iv) $v \ll w$ *implies* $u \circ w \ll u \circ v$
(v) $G \subseteq I$ *implies* $G \ll I$
for any $u, v, w \in H$ *and for non-empty subsets G and I of H".*

Moreover for the basic study relevant to "hyper BCK-subalgebras and (weak, strong, reflexive) hyper BCK-ideals", please see [13]. From now onwards, H will represent a "hyper BCK-algebra".

Lemma 2.3. [12, 13] *For any H,*
(i) *"any strong hyper BCK-ideal of H is a hyper BCK-ideal of H".*
(ii) *"any hyper BCK-ideal of H is a weak hyper BCK-ideal of H".*

Lemma 2.4. [12] *"For any reflexive hyper BCK-ideal I of H, if* $u \circ v \cap I \neq \emptyset$ *then* $u \circ v \ll I, \forall\ u, v \in H$*".*

Proposition 2.5. [11] *"If G is a subset of H and I is any hyper BCK-ideal of H, such that,* $G \ll I$ *then* $G \subseteq I$*".*

Definition 2.6. *For a "hyper BCK-algebra" H, a non-empty subset* $I \subseteq H$*, containing* 0 *is known as*
- *a "weak hyper p-ideal" of H if*

$$(a \circ c) \circ (b \circ c) \subseteq I \text{ and } b \in I \text{ imply } a \in I.$$

- *a "hyper p-ideal" of H if*

$$(a \circ c) \circ (b \circ c) \ll I \text{ and } b \in I \text{ imply } a \in I.$$

- *a "strong hyper p-ideal" of H if*

$$(a \circ c) \circ (b \circ c) \bigcap I \neq \emptyset \text{ and } b \in I \text{ imply } a \in I.$$

Theorem 2.7. *Every "(strong, weak) hyper p-ideal" is a "(strong, weak) hyper BCK-ideal".*

Proof. Let I be a "hyper p-ideal of H". Then, for any $i, j, k \in H$,

$$(i \circ k) \circ (j \circ k) \ll I \text{ and } j \in I \text{ imply } i \in I.$$

Putting $k = 0$ we get

$$(i \circ 0) \circ (j \circ 0) \ll I \text{ and } j \in I \text{ imply } i \in I.$$

Therefore, $(i \circ j) \ll I$ and $j \in I \Rightarrow i \in I$. Hence proved. □

Generally, every "(strong, weak) hyper BCK-ideal" is not a "(strong, weak) hyper p-ideal". It can be observed with the help of examples given below:

Example 2.8. *"Let $H = \{0, a, b\}$. We Contemplate the following table:*

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{0, a\}$
b	$\{b\}$	$\{b\}$	$\{0, a\}$

Then H is a hyper BCK-algebra". Take $I = \{0, a\}$. Then I is a "weak hyper BCK-ideal", however, not a "weak hyper p-ideal of H" as
$(b \circ b) \circ (0 \circ b) = \{0, a\} \subseteq I$ *and* $0 \in I$ *but* $b \notin I$.

Example 2.9. *"Let $H = \{0, a, b\}$. We Contemplate the following table:*

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{a\}$
b	$\{b\}$	$\{b\}$	$\{0, b\}$

Then H is a hyper BCK-algebra". Take $I = \{0, b\}$. Then, I is a "hyper BCK-ideal" but not a "hyper p-ideal" as
$(a \circ a) \circ (0 \circ a) = \{0\} \ll I$, $0 \in I$ *but* $a \notin I$.
Here $I = \{0, b\}$ is also a "strong hyper BCK-ideal" however, it is not a "strong hyper p-ideal of H" as
$(a \circ a) \circ (0 \circ a) = \{0\} \bigcap I \neq \emptyset$ *and* $0 \in I$ *but* $a \notin I$.

Theorem 2.10. *For any "hyper BCK-algebra",*
(*i*) *"any hyper p-ideal is also a weak hyper p-ideal".*
(*ii*) *"any strong hyper p-ideal is also a hyper p-ideal".*

Proof. (*i*) Let, I is a "hyper p-ideal of H".
Let, $(i \circ k) \circ (j \circ k) \subseteq I$ and $j \in I$. Then, $(i \circ k) \circ (j \circ k) \subseteq I$ implies $(i \circ k) \circ (j \circ k) \ll I$ (by Proposition 2.2(v)), which along with $j \in I$ implies $i \in I$, which is our required condition.
(*ii*) Let, I is a "strong hyper p-ideal of H". Let, $(i \circ k) \circ (j \circ k) \ll I$ and $j \in I$. Then, $\forall\, \alpha \in (i \circ k) \circ (j \circ k)$, $\exists\, \beta \in I$ such that $\alpha \ll \beta$. Thus $0 \in \alpha \circ \beta$ and $(\alpha \circ \beta) \cap I \neq \emptyset$, which along with $\beta \in I$ implies $\alpha \in I$, that is $(i \circ k) \circ (j \circ k) \subseteq I$. Thus $(i \circ k) \circ (j \circ k) \cap I \neq \emptyset$, which along with $j \in I$ implies $i \in I$, which is our required condition. □

Generally, the converse of above thoerem doesn't hold. It can be observed by the following examples:

Example 2.11. *"Let $H = \{0, a, b\}$. We Contemplate the following table:*

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{0, a\}$
b	$\{b\}$	$\{b\}$	$\{0, a, b\}$

Then H is a hyper BCK-algebra". Take $I = \{0, b\}$. Clearly, I is a "weak hyper p-ideal of H". But for $(a \circ a) \circ (0 \circ a) = \{0, a\} \ll I$ and $0 \in I$, $a \notin I$, so I isn't a "hyper p-ideal".

Example 2.12. *"We cogitate the table given below which explains the hyper BCK-algebra $H = \{0, a, b\}$:*

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{0, a\}$
b	$\{b\}$	$\{a, b\}$	$\{0, a, b\}$

Take $I = \{0, a\}$". Clearly, I is a "hyper p-ideal" but not a "strong hyper p-ideal of H" as, $(b \circ 0) \circ (a \circ 0) \cap I = \{a, b\} \cap I \neq \emptyset$ and $a \in I$ but $b \notin I$.

For detail study of "fuzzy (weak, strong) hyper BCK-ideals", one must consult [10].

Theorem 2.13. [10] *For any H,*
(*i*) *"any fuzzy hyper BCK-ideal of H is a fuzzy weak hyper BCK-ideal of H".*
(*ii*) *"any fuzzy strong hyper BCK-ideal of H is a fuzzy hyper BCK-ideal of H".*

3. Fuzzy Hyper *p*-ideals

Now we present the idea of "fuzzy (weak, strong) hyper *p*-ideals" and confer associated properties.

Definition 3.1. *For a "hyper BCK-algebra" H , a "fuzzy set" ϖ in H is called a*

- *"fuzzy weak hyper p-ideal of H" if, for any $a, b, c \in H$*
$$\varpi(0) \geq \varpi(a) \geq \min\ \{\inf_{x \in (a \circ c) \circ (b \circ c)}\ \varpi(x),\ \varpi(b)\}$$

- *"fuzzy hyper p-ideal of H" if, $a \ll b$ implies $\varpi(a) \geq \varpi(b)$ and for any $a, b, c \in H$,*
$$\varpi(a) \geq \min\ \{\inf_{x \in (a \circ c) \circ (b \circ c)}\ \varpi(x),\ \varpi(b)\}$$

- *"fuzzy strong hyper p-ideal of H" if, $\forall\ a, b, c \in H$,*
$$\inf_{x \in a \circ a}\ \varpi(x) \geq \varpi(a) \geq \min\ \{\sup_{y \in (a \circ c) \circ (b \circ c)}\ \varpi(y),\ \varpi(b)\}$$

Theorem 3.2. *Any "fuzzy (weak, strong) hyper p-ideal" is a "fuzzy (weak, strong) hyper BCK-ideal".*

Proof. Let, ϖ is a "fuzzy hyper p-ideal of H". Then, $\forall\ i, j, k \in H$ we get,

$$\varpi(i) \geq \min\ \{\inf_{a\in(i\circ k)\circ(j\circ k)}\ \varpi(a),\ \varpi(j)\}$$

Putting $k = 0$ we get,

$$\varpi(i) \geq \min\ \{\inf_{a\in(i\circ 0)\circ(j\circ 0)}\ \varpi(a),\ \varpi(j)\}$$

which gives,

$$\varpi(i) \geq \min\ \{\inf_{a\in i\circ j}\ \varpi(a),\ \varpi(j)\}$$

Hence proved. □

Generally, the converse of above thoerem doesn't hold. Consider the "hyper BCK-algebra $H = \{0, a, b\}$" defined by the table, given in Example (2.9). Define a "fuzzy set ϖ in H" by:

$$\varpi(0) = 1, \varpi(a) = 0.6, \varpi(b) = 0$$

It is easy to substantiate that ϖ is a "fuzzy weak hyper BCK-ideal" but not a "fuzzy weak hyper p-ideal of H" as

$$\varpi(a) = 0.6 < 1 = \min\ \{\inf_{a\in(a\circ a)\circ(0\circ a)}\ \varpi(a),\ \varpi(0)\}$$

Now, again consider the "hyper BCK-algebra $H = \{0, a, b\}$" defined by the table given in Example (2.9) and define a "fuzzy set ϖ in H" by:

$$\varpi(0) = 0.8,\ \varpi(a) = 0.5,\ \varpi(b) = 0.3$$

Clearly ϖ is a "fuzzy hyper BCK-ideal" but not a "fuzzy hyper p-ideal" of H since

$$\varpi(a) = 0.5 < 0.8 = \min\ \{\inf_{a\in(a\circ a)\circ(0\circ a)}\ \varpi(a),\ \varpi(0)\}$$

Example 3.3. *"Let $H = \{0, a, b, c\}$ be a hyper BCK-algebra defined by the table given below:*

$*$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{0, a\}$	$\{0, a\}$
b	$\{b\}$	$\{b\}$	$\{0, a\}$	$\{0, a\}$
c	$\{c\}$	$\{c\}$	$\{c\}$	$\{0, a\}$

Define a fuzzy set ϖ in H by":

$$\varpi(0) = \varpi(a) = 1,\ \varpi(b) = \tfrac{1}{2},\ \varpi(c) = \tfrac{1}{3}$$.

Clearly, ϖ is a "fuzzy strong hyper BCK-ideal" of but not a "fuzzy strong hyper p-ideal" of H since

$$\varpi(b) = \tfrac{1}{2} < 1 = \min\ \{\sup_{a\in(b\circ b)\circ(a\circ b)}\ \varpi(a),\ \varpi(a)\}$$

Theorem 3.4. *For any "hyper BCK-algebra",*
(i) *"Any fuzzy hyper p-ideal is a fuzzy weak hyper p-ideal".*
(ii) *"Any fuzzy Strong hyper p-ideal is a fuzzy hyper p-ideal".*

Proof. (i) Let, ϖ be a "fuzzy hyper p-ideal of H". Since, "every fuzzy hyper p-ideal is a fuzzy hyper BCK-ideal" (by Theorem 3.2) and "every fuzzy hyper BCK-ideal is a fuzzy weak hyper BCK-ideal" (by Theorem 2.13(i)), therefore ϖ is also a "fuzzy weak hyper BCK-ideal of H". Hence ϖ satisfies $\varpi(0) \geq \varpi(i)$, for all $i \in H$. Also being a "fuzzy hyper p-ideal" ϖ satisfies:

$$\varpi(i) \geq \min\ \{\inf_{x\in(i\circ k)\circ(j\circ k)}\ \varpi(x),\ \varpi(j)\}$$

$\forall\ i, j, k \in H$. Hence ϖ is a "fuzzy weak hyper p-ideal of H".

(ii) Let, ϖ is a "fuzzy strong hyper p-ideal of H". Since, "every fuzzy strong hyper p-ideal is a fuzzy strong hyper BCK-ideal" (by Theorem 3.2) and "every fuzzy strong hyper BCK-ideal is a fuzzy hyper BCK-ideal" (by Theorem 2.13(ii)), therefore ϖ is also a "fuzzy hyper BCK-ideal" of H. Hence for any $i, j \in H$, if $i \ll j$ then $\varpi(i) \geq \varpi(j)$.
Also being a "fuzzy strong hyper p-ideal", ϖ satisfies for any $i, j, k \in H$

$$\varpi(i) \geq \min \{\sup_{x\in(i\circ k)\circ(j\circ k)} \varpi(x),\ \varpi(j)\}$$

Since $\sup_{x\in(i\circ k)\circ(j\circ k)} \varpi(x) \geq \varpi(y)$, $\forall\, y \in (i \circ k) \circ (j \circ k)$, therefore we get,
$\varpi(i) \geq \min \{\sup_{x\in(i\circ k)\circ(j\circ k)} \varpi(x),\ \varpi(j)\} \geq \min \{\varpi(y),\ \varpi(j)\}$,
for all $y \in (i \circ k) \circ (j \circ k)$
Since $\varpi(y) \geq \inf_{z\in(i\circ k)\circ(j\circ k)} \varpi(k)$, $\forall\, y \in (i \circ k) \circ (j \circ k)$, therefore we have,
$\varpi(i) \geq \min \{\varpi(y),\ \varpi(j)\} \geq \min \{\inf_{z\in(i\circ k)\circ(j\circ k)} \varpi(z),\ \varpi(j)\}$, that is

$$\varpi(i) \geq \min \{\inf_{z\in(i\circ k)\circ(j\circ k)} \varpi(z),\ \varpi(j)\}$$

Hence proved. □

Generally, the converse of above thoerem doesn't hold. Consider the "hyper BCK-algebra $H = \{0, a, b\}$" defined by the table given in Example (2.11). Define a "fuzzy set ϖ in H" by:

$$\varpi(0) = 1,\ \varpi(a) = 0.6,\ \varpi(b) = 0.9$$

Then ϖ is a "fuzzy weak hyper p-ideal" but not a "fuzzy hyper p-ideal of H" as:

$$a \leq b \text{ but } \varpi(a) = 0.6 < 0.9 = \varpi(b)$$

Example 3.5. *"Consider a hyper BCK-algebra $H = \{0, a, b\}$ defined by the following table:*

$\circ$	$\{0\}$	$\{a\}$	$\{b\}$
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{a\}$
b	$\{b\}$	$\{b\}$	$\{0, b\}$

Define a fuzzy set ϖ in H by":

$$\varpi(0) = \varpi(a) = 1,\ \varpi(b) = \tfrac{1}{2}$$

Then ϖ is a "fuzzy hyper p-ideal" but it is not a "fuzzy strong hyper p-ideal of H" as:

$$\varpi(b) = \tfrac{1}{2} < 1 = \min \{\sup_{x\in(b\circ b)\circ(a\circ b)} \varpi(x),\ \varpi(a)\}$$

Theorem 3.6. *A "fuzzy set ϖ in H", ϖ is a "fuzzy (weak, strong) hyper p-ideal of H" iff $\forall\, t \in [0, 1]$, $\varpi_t \neq \emptyset$ is a "(weak, strong) hyper p-ideal of H".*

Proof. Let, ϖ is a "fuzzy hyper p-ideal of H". Since $\varpi_t \neq \emptyset$, so for any $i \in \varpi_t$, $\varpi(i) \geq t$. "Since every fuzzy hyper p-ideal is also a fuzzy weak hyper p-ideal" (by Theorem 3.4(i)), so ϖ is also a "fuzzy weak hyper p-ideal of H". Thus $\varpi(0) \geq \varpi(i) \geq t$, for all $i \in H$, which implies $0 \in \varpi_t$.
Let $(i \circ k) \circ (j \circ k) \ll \varpi_t$ and $j \in \varpi_t$, then $\forall\, x \in (i \circ k) \circ (j \circ k)$, $\exists\, y \in \varpi_t$ such that $x \ll y$. So $\varpi(x) \geq \varpi(y) \geq t$, $\forall\, x \in (i \circ k) \circ (j \circ k)$. Thus $\inf_{x\in(i\circ k)\circ(j\circ k)} \varpi(x) \geq t$. Also $\varpi(j) \geq t$, as $j \in \varpi_t$. Therefore

$$\varpi(i) \geq \min \{\inf_{x\in(i\circ k)\circ(j\circ k)} \varpi(x),\ \varpi(j)\} \geq \min \{t, t\} = t$$

$\Rightarrow i \in \varpi_t$. Hence ϖ_t is "hyper p-ideal" of H.
Conversely, Let, "$\varpi_t \neq \emptyset$ is a "hyper p-ideal of H", $\forall\, t \in [0, 1]$". Let $i \ll j$ for some $i, j \in H$ and put $\varpi(j) = t$. Then $j \in \varpi_t$. So $i \ll j \in \varpi_t \Rightarrow i \ll \varpi_t$. "Being a hyper p-ideal, ϖ_t is also a hyper BCK-ideal of H" (By Theorem

(2.7)) therefore by Proposition 2.5, $i \in \varpi_t$. Hence $\varpi(i) \geq t = \varpi(j)$. That is $i \ll j \Rightarrow \varpi(i) \geq \varpi(j)$, for all $i, j \in H$. Moreover, for any $i, j, k \in H$, let $d = \min\ \{\inf_{z\in(i\circ k)\circ(j\circ k)}\ \varpi(z),\ \varpi(j)\}$. Then $\varpi(j) \geq d \Rightarrow j \in \varpi_d$ and for all $e \in (i \circ k) \circ (j \circ k)$, $\varpi(e) \geq \inf_{z\in(i\circ k)\circ(j\circ k)}\ \varpi(z) \geq d$, which implies $e \in \varpi_d$. Thus $(i \circ k) \circ (j \circ k) \subseteq \varpi_d$. By Proposition 2.2(v), $(i \circ k) \circ (j \circ k) \subseteq \varpi_d \Rightarrow (i \circ k) \circ (j \circ k) \ll \varpi_d$, which along with $j \in \varpi_d$ implies $i \in \varpi_d$. Hence we get

$$\varpi(i) \geq d = \min\ \{\inf_{z\in(i\circ k)\circ(j\circ k)}\ \varpi(z),\ \varpi(j)\}.$$

Hence proved. □

Theorem 3.7. *If ϖ is a "fuzzy (weak, strong) hyper p-ideal of H" then, $A = \{i \in H \mid \varpi(i) = \varpi(0)\}$ is a "(weak, strong) hyper p-ideal of H".*

Proof. Let, ϖ is a "fuzzy strong hyper p-ideal of H". Clearly, $0 \in A$. Let $(i \circ k) \circ (j \circ k) \cap A \neq \emptyset$ and $j \in A$ for some $i, j, k \in H$. Then $\exists\, i_\circ \in (i \circ k) \circ (j \circ k) \cap A$ such that $\varpi(i_\circ) = \varpi(0)$. Also $\varpi(j) = \varpi(0)$. Then

$$\begin{aligned} \varpi(i) &\geq \min\ \{\sup_{x\in(i\circ k)\circ(j\circ k)}\ \varpi(x),\ \varpi(j)\} \geq \min\ \{\varpi(i_\circ),\ \varpi(j)\} \\ &= \min\ \{\varpi(0),\ \varpi(0)\} = \varpi(0) \\ &\Rightarrow \varpi(i) \geq \varpi(0) \end{aligned}$$

"Being a fuzzy strong hyper p-ideal, ϖ is also a fuzzy weak hyper p-ideal of H" (by Theorem 3.4), so it satisfies $\varpi(0) \geq \varpi(i)$, $\forall\, i \in H$. Therefore $\varpi(0) = \varpi(i)$ and so $i \in A$. Hence proved. □

Likewise, as done above, we can Corroborate the result for the other two cases. The "transfer principle" for "fuzzy sets" described in [15] suggest the following result.

Theorem 3.8. *Let ϖ be a "fuzzy set in H" defined by:*

$$\varpi(a) = \begin{cases} t & if\ a \in A \\ 0 & if\ a \notin A \end{cases}$$

$\forall\, a \in H$, where, $A \subseteq H$ and $t \in (0, 1]$. Then, "A is a (weak, strong) hyper p-ideal iff ϖ is a fuzzy (weak, strong) hyper p-ideal".

Proof. Let, A is a "strong hyper p-ideal of H". Then for any $i, j, k \in H$ if $(i \circ k) \circ (j \circ k) \cap A \neq \emptyset$ and $j \in A$ then $i \in A$. Thus we have

$$\varpi(i) = t = \min\ \{\sup_{x\in(i\circ k)\circ(j\circ k)}\ \varpi(x),\ \varpi(j)\}$$

If $(i \circ k) \circ (j \circ k) \cap A = \emptyset$ and $j \notin A$ then $\varpi(y) = 0$, $\forall\, y \in (i \circ k) \circ (j \circ k)$ and $\varpi(j) = 0$, therefore

$$\min\ \{\sup_{x\in(i\circ k)\circ(j\circ k)}\ \varpi(x),\ \varpi(j)\} = 0 \leq \varpi(i)$$

If $(i \circ k) \circ (j \circ k) \cap A = \emptyset$ and $j \in A$, OR, $(i \circ k) \circ (j \circ k) \cap A \neq \emptyset$ and $j \notin A$, Then in both of these cases we have:

$$\min\ \{\sup_{x\in(i\circ k)\circ(j\circ k)}\ \varpi(x),\ \varpi(j)\} = 0 \leq \varpi(i)$$

Now by Proposition 2.2(ii), "we have $i \circ i \leq i$, $\forall\, i \in H$". Then, $\forall\, z \in i \circ i$, $z \ll i$.
"Being a strong hyper p-ideal of H, $A = \varpi_t$ is a hyper p-ideal of H" (by Theorem 2.10(ii)) and hence ϖ is a "fuzzy hyper p-ideal" of H (by Theorem 3.6). Therefore

$$\begin{aligned} z \ll i &\Rightarrow \varpi(z) \geq \varpi(i), \text{ for all } z \in i \circ i \\ &\Rightarrow \inf_{z\in i\circ i}\ \varpi(z) \geq \varpi(i),\ \forall\, i \in H \end{aligned}$$

Hence ϖ is a "fuzzy strong hyper p-ideal" of H.
Conversely, Let ϖ is a "fuzzy strong hyper p-ideal of H". Then, by Theorem 3.6, "$\forall\, t \in (0, 1]$, $\varpi_t = A$ is a strong hyper p-ideal of H". Correspodingly, we can verify the result for the other two types of ideals. □

Theorem 3.9. *The family of "fuzzy strong hyper p-ideals" is a "completely distributive lattice with respect to join and meet".*

Proof. Let $\{\varpi_i \mid i \in I\}$ be a family of "fuzzy strong hyper p-ideals of H". "Since $[0,1]$ is a completely distributive lattice with respect to the usual ordering in $[0,1]$", it is sufficient to corroborate that, $\vee_{i\in I}\,\varpi_i$ and $\wedge_{i\in I}\,\varpi_i$ are "fuzzy strong hyper p-ideals of H".

For any $a \in H$ we have,

$$\inf_{x\in a\circ a}\ ((\vee_{i\in I}\,\varpi_i)(x)) = \inf_{x\in a\circ a}\ (\sup_{i\in I}\ \varpi_i(x))$$
$$= \sup_{i\in I}\ (\inf_{x\in a\circ a}\ \varpi_i(x))) \geq \sup_{i\in I}\ \varpi_i(a) = (\vee_{i\in I}\,\varpi_i)(a)$$
$$\Rightarrow \inf_{x\in a\circ a}\ ((\vee_{i\in I}\,\varpi_i)(x)) \geq (\vee_{i\in I}\,\varpi_i)(a)$$

Moreover, for any $a, b, c \in H$, we have

$$(\vee_{i\in I}\,\varpi_i)(a) = \sup_{i\in I}\ \varpi_i(a) \geq \sup_{i\in I}\ [\min\ \{\sup_{y\in(a\circ c)\circ(b\circ c)}\ \varpi_i(y),\ \varpi_i(b)\}]$$
$$= \min\ \{\sup_{i\in I}\ (\sup_{y\in(a\circ c)\circ(b\circ c)}\ \varpi_i(y)),\ \sup_{i\in I}\ (\varpi_i(b))\}$$
$$= \min\ \{\sup_{y\in(a\circ c)\circ(b\circ c)}\ (\sup_{i\in I}\ \varpi_i(y)),\ \sup_{i\in I}\ (\varpi_i(b))\}$$
$$= \min\ \{\sup_{y\in(a\circ c)\circ(b\circ c)}\ ((\vee_{i\in I}\,\varpi_i)(y)),\ (\vee_{i\in I}\,\varpi_i)(b)\}$$
$$\Rightarrow (\vee_{i\in I}\,\varpi_i)(a) \geq \min\ \{\sup_{y\in(a\circ c)\circ(b\circ c)}\ ((\vee_{i\in I}\,\varpi_i)(y)),\ (\vee_{i\in I}\,\varpi_i)(b)\}$$

Hence $\vee_{i\in I}\,\varpi_i$ is a "fuzzy strong hyper p-ideal" of H.

Now, we prove that $\wedge_{i\in I}\,\varpi_i$ is a "fuzzy strong hyper p-ideal of H".

For any $a \in H$ we have,

$$\inf_{x\in a\circ a}\ ((\wedge_{i\in I}\,\varpi_i)(x)) = \inf_{x\in a\circ a}\ (\inf_{i\in I}\ \varpi_i(x))$$
$$= \inf_{i\in I}\ (\inf_{x\in a\circ a}\ \varpi_i(x))) \geq \inf_{i\in I}\ \varpi_i(a) = (\wedge_{i\in I}\,\varpi_i)(a)$$
$$\Rightarrow \inf_{x\in a\circ a}\ ((\wedge_{i\in I}\,\varpi_i)(x)) \geq (\wedge_{i\in I}\,\varpi_i)(a)$$

Moreover, for any $a, b, c \in H$, we have

$$(\wedge_{i\in I}\,\varpi_i)(a) = \inf_{i\in I}\ \varpi_i(a) \geq \inf_{i\in I}\ [\min\ \{\sup_{y\in(a\circ c)\circ(b\circ c)}\ \varpi_i(y),\ \varpi_i(b)\}]$$
$$= \min\ \{\inf_{i\in I}\ (\sup_{y\in(a\circ c)\circ(b\circ c)}\ \varpi_i(y)),\ \inf_{i\in I}\ (\varpi_i(b))\}$$
$$= \min\ \{\sup_{y\in(a\circ c)\circ(b\circ c)}\ (\inf_{i\in I}\ \varpi_i(y)),\ \inf_{i\in I}\ (\varpi_i(b))\}$$
$$= \min\ \{\sup_{y\in(a\circ c)\circ(b\circ c)}((\wedge_{i\in I}\,\varpi_i)(y)),\ (\wedge_{i\in I}\,\varpi_i)(b)\}$$
$$\Rightarrow (\wedge_{i\in I}\,\varpi_i)(a) \geq \min\ \{\sup_{y\in(a\circ c)\circ(b\circ c)}\ ((\wedge_{i\in I}\,\varpi_i)(y)),\ (\wedge_{i\in I}\,\varpi_i)(b)\}$$

Hence $\wedge_{i\in I}\,\varpi_i$ is a "fuzzy strong hyper p-ideal of H".

Hence proved. □

Correspondingly, as done above, we can Corroborate the result for the other two cases. For the definition of "the stronges fuzzy relation on H", one must see [1].

Theorem 3.10. *Let ϖ be a "fuzzy set" and let λ_ϖ be "the strongest fuzzy relation on H". ϖ is a "fuzzy strong hyper p-ideal iff λ_ϖ is a fuzzy strong hyper p-ideal of $H \times H$".*

Proof. Let, ϖ is a "fuzzy strong hyper p-ideal of H". Consider

$$\inf_{(x,y)\in(i_1,i_2)\circ(i_1,i_2)}\ \lambda_\varpi(x,y) = \inf_{(x,y)\in(i_1\circ i_1, i_2\circ i_2)}\ [\min\ \{\varpi(x),\ \varpi(y)\}]$$
$$= \min\ \{\inf_{x\in i_1\circ i_1}\ \varpi(x),\ \inf_{y\in i_2\circ i_2}\ \varpi(y)\} \geq \min\ \{\varpi(i_1),\ \varpi(i_2)\} = \lambda_\varpi(i_1, i_2)$$
$$\Rightarrow \inf_{(x,y)\in(i_1,i_2)\circ(i_1,i_2)}\ \lambda_\varpi(x,y) \geq \lambda_\varpi(i_1, i_2),\ \forall\ (i_1, i_2) \in H \times H$$

Now, for any (i_1, i_2), (j_1, j_2), (k_1, k_2) in $H \times H$, consider

$$\lambda_{\varpi}(i_1, i_2) = \min\ \{\varpi(i_1),\ \varpi(i_2)\} \geq$$

$$\min\ [\min\ \{\sup_{z\in(i_1\circ k_1)\circ(j_1\circ k_1)}\ \varpi(z),\ \varpi(j_1)\},\ \min\ \{\sup_{d\in(i_2\circ k_2)\circ(j_2\circ k_2)}\ \varpi(d)$$
$, \varpi(j_2)\}]$

$$= \min\ [\min\ \{\sup_{z\in(i_1\circ k_1)\circ(j_1\circ k_1)}\ \varpi(z),\ \sup_{d\in(i_2\circ k_2)\circ(j_2\circ k_2)}\ \varpi(d)\},\ \min\ \{\varpi(j_1)$$
$, \varpi(j_2)\}]$

$$= \min\ [\min\ \{\sup\ (\varpi(z),\ \varpi(d))\},\ \lambda_{\varpi}(j_1, j_2)]$$

where

$$z \in (i_1 \circ k_1) \circ (j_1 \circ k_1) \text{ and } d \in (i_2 \circ k_2) \circ (j_2 \circ k_2)$$
$$\Rightarrow \lambda_{\varpi}(i_1, i_2) \geq \min\ [\sup\ \{\min\ (\varpi(z), \varpi(d))\},\ \lambda_{\varpi}(j_1, j_2)]$$

where

$$z \in (i_1 \circ k_1) \circ (j_1 \circ k_1),\ d \in (i_2 \circ k_2) \circ (j_2 \circ k_2)$$
$$\Rightarrow \lambda_{\varpi}(i_1, i_2) \geq \min\ \{\sup\ \lambda_{\varpi}(z, d),\ \lambda_{\varpi}(j_1, j_2)\}$$

where

$$(z, d) \in ((i_1 \circ k_1) \circ (j_1 \circ k_1),\ (i_2 \circ k_2) \circ (j_2 \circ k_2))$$
$= ((i_1, i_2) \circ (k_1, k_2)) \circ ((j_1, j_2) \circ (k_1, k_2))$

Hence, λ_{ϖ} is a "fuzzy strong hyper p-ideal of $H \times H$".

Conversely, let λ_{ϖ} is a "fuzzy strong hyper p-ideal of $H \times H$". Then, we have

$$\inf_{(x,y)\in(i_1,i_2)\circ(i_1,i_2)}\ \lambda_{\varpi}(x, y) \geq \lambda_{\varpi}(i_1, i_2),\ \forall\ (i_1, i_2) \in H \times H$$
$$\Rightarrow \inf_{(x,y)\in(i_1\circ i_1, i_2\circ i_2)}\ [\min\ \{\varpi(x),\ \varpi(y)\}] \geq \min\ \{\varpi(i_1),\ \varpi(i_2)\}$$
$$\Rightarrow \min\ \{\inf_{x\in i_1\circ i_1}\ \varpi(x),\ \inf_{y\in i_2\circ i_2}\ \varpi(y)\} \geq \min\ \{\varpi(i_1),\ \varpi(i_2)\}$$
$$\Rightarrow \{\inf_{x\in i_1\circ i_1}\ \varpi(x),\ \inf_{y\in i_2\circ i_2}\ \varpi(y)\} \geq \{\varpi(i_1),\ \varpi(i_2)\}$$
$$\Leftrightarrow \inf_{x\in i_1\circ i_1}\ \varpi(x) \geq \varpi(i_1) \text{ and } \inf_{y\in i_2\circ i_2}\ \varpi(y) \geq \varpi(i_2),\ \forall\ i_1, i_2 \in H.$$

Hence the first condition for ϖ to be a "fuzzy strong hyper p-ideal" is satisfied.

Note that "being a fuzzy strong hyper p-ideal of $H \times H$, λ_{ϖ} is also a fuzzy weak hyper p-ideal of $H \times H$" (by Theorem 3.4), thus λ_{ϖ} satisfies

$$\lambda_{\varpi}(0, 0) \geq \lambda_{\varpi}(i, i),\ \forall\ (0, 0),\ (i, i) \in H \times H$$
$$\Rightarrow \min\ \{\varpi(0),\ \varpi(0)\} \geq \min\ \{\varpi(i),\ \varpi(i)\}$$
$$\Rightarrow \varpi(0) \geq \varpi(i),\ \forall\ i \in H$$

Now, for any, $(i_1, i_2),\ (j_1, j_2),\ (k_1, k_2)$ in $H \times H$, λ_{ϖ} satisfies

$$\Rightarrow \lambda_{\varpi}(i_1, i_2) \geq \min\ \{\sup\ \lambda_{\varpi}(e, f),\ \lambda_{\varpi}(j_1, j_2)\}$$

where

$$(e, f) \in ((i_1, i_2) \circ (k_1, k_2)) \circ ((j_1, j_2) \circ (k_1, k_2))$$
$= ((i_1 \circ k_1) \circ (j_1 \circ k_1),\ (i_2 \circ k_2) \circ (j_2 \circ k_2))$

$$\Rightarrow \min\ \{\varpi(i_1),\ \varpi(i_2)\} \geq \min\ [\sup\ \{\min\ \{\varpi(e),\ \varpi(f)\}\},\ \min\ \{\varpi(j_1),\ \varpi(j_2)\}]$$

where

$$(e, f) \in ((i_1 \circ k_1) \circ (j_1 \circ k_1),\ (i_2 \circ k_2) \circ (j_2 \circ k_2))$$

Putting $i_1 = j_1 = k_1 = 0$ we get

$$\Rightarrow \min\ \{\varpi(0),\ \varpi(i_2)\} \geq \min\ [\sup\ \{\min\ \{\varpi(0),\ \varpi(f)\}\}, \min\ \{\varpi(0),\ \varpi(j_2)\}]$$

Where

$$(e, f) \in (0, (i_2 \circ k_2) \circ (j_2 \circ k_2))$$

$\Rightarrow \varpi(i_2) \geq \min\ \{\sup_{f\in((i_2\circ k_2)\circ(j_2\circ k_2))}\ \varpi(f),\ \varpi(j_2)\}$, since $\varpi(0) \geq \varpi(i),\ \forall\ i \in H$

Similarly by putting $i_2 = j_2 = k_2 = 0$, we get,

$$\Rightarrow \varpi(i_1) \geq \min\ \{\sup_{e\in(i_1\circ k_1)\circ(j_1\circ k_1))}\ \varpi(e),\ \varpi(j_1)\}$$

Hence ϖ is a "fuzzy strong hyper p-ideal of H". □

Identically, as done above, we can corroborate the statement for the other two cases.

Theorem 3.11. *Let, "$f : X \to Y$ be an onto hyper BCK-algebras from a hyper BCK-algebra X to a hyper BCK-algebra Y". If, "ν is a "fuzzy strong hyper p-ideal of Y then the hyper homomorphic pre-image ϖ of ν under f is a fuzzy strong hyper p-ideal of X".*

Proof. Let, ν is a "fuzzy strong hyper p-ideal of Y". Since, ϖ is a "hyper homomorphic pre-image" of ν under f, so ϖ is defined by $\varpi = \nu \circ f$ that is $\varpi(i) = \nu(f(i))$, $\forall\ i \in X$. Since ν satisfies

$$\inf_{f(x)\in f(i)\circ f(i)=f(i\circ i)}\ \nu(f(x)) \geq \nu(f(i)),\ \forall\ i \in X \text{ and } f(i) \in Y$$

$$\Rightarrow \inf_{x\in i\circ i}\ \varpi(x) \geq \varpi(i),\ \forall\ i \in X$$

Now for any $i, j, k \in X$ consider

$$\varpi(i) = \nu(f(i)) \geq \min\ \{\sup_{f(y)\in(f(i)\circ k')\circ(j'\circ k')}\ \nu(f(y)),\ \nu(j')\}$$

where $j', k' \in Y$. Since $f : X \to Y$ is an onto "hyper BCK-algebras", so for $j', k' \in Y$, $\exists\ j, k \in X$ such that $f(j) = j'$ and $f(k) = k'$. Hence we get

$$\varpi(i) \geq \min\ \{\sup_{f(y)\in(f(i)\circ f(k))\circ(f(j)\circ f(k))=f((i\circ k)\circ(j\circ k))}\ \nu(f(y)),\ \nu(f(j))\}$$

$$\Rightarrow \varpi(i) \geq \min\ \{\sup_{y\in(i\circ k)\circ(j\circ k)}\ \varpi(y),\ \varpi(j)\},\ \forall\ i, j, k \in X$$

Hence proved. □

Correspondingly, as done above, we can corroborate the statement for "fuzzy (weak) hyper p-ideals". Lastly, we confer the product of two fuzzy hyper p-ideals. One may consult [3], for basic material on the "product of fuzzy hyper BCK-ideals".

Theorem 3.12. *A fuzzy set $\varpi = \varpi_1 \times \varpi_2$ is a "fuzzy (weak, strong) hyper p-ideal" of $H = H_1 \times H_2$ iff ϖ_1 and ϖ_2 are "fuzzy (weak, strong) hyper p-ideals" of H_1 and H_2 respectively.*

Proof. Let $\varpi = \varpi_1 \times \varpi_2$ be a "fuzzy hyper p-ideal" of $H = H_1 \times H_2$ and let $i_1 \ll i_2$ for some $i_1, i_2 \in H_1$. Then $(i_1, 0) \ll (i_2, 0)$ which implies $\varpi((i_1, 0)) = \varpi_1(i_1) \geq \varpi((i_2, 0)) = \varpi_1(i_2)$, that is, $\varpi_1(i_1) \geq \varpi_1(i_2)$

Moreover for any $i_1, j_1, k_1 \in H_1$, let $t = \min\{\inf_{a\in(i_1\circ k_1)\circ(j_1\circ k_1)}\ \varpi_1(a),\ \varpi_1(j_1)\}$

Then, $\forall\ b \in (i_1 \circ k_1) \circ (j_1 \circ k_1)$, $\varpi_1(b) \geq \inf_{a\in(i_1\circ k_1)\circ(j_1\circ k_1)}\ \varpi_1(a) \geq t$ and $\varpi_1(j_1) \geq t$

$\Rightarrow \varpi((b, 0)) \geq t$ and $\varpi((j_1, 0)) \geq t$, $\forall\ (b, 0) \in ((i_1, 0) \circ (k_1, 0)) \circ ((j_1, 0) \circ (k_1, 0))$

$\Rightarrow (b, 0) \in \varpi_t$ and $(j_1, 0) \in \varpi_t$,

$$\Rightarrow ((i_1, 0) \circ (k_1, 0)) \circ ((j_1, 0) \circ (k_1, 0)) \subseteq \varpi_t \text{ and } (j_1, 0) \in \varpi_t$$

$$\Rightarrow ((i_1, 0) \circ (k_1, 0)) \circ ((j_1, 0) \circ (k_1, 0)) \ll \varpi_t \text{ and } (j_1, 0) \in \varpi_t$$

$\Rightarrow (i_1, 0) \in \varpi_t$, "since ϖ_t is a hyper p-ideal" (by Theorem 3.6).

Therefore, $\varpi((i_1, 0)) \geq t$. Thus

$\varpi_1(i_1) \geq t = \min\{\inf_{a\in(i_1\circ k_1)\circ(j_1\circ k_1)}\ \varpi_1(a),\ \varpi_1(j_1)\}$, which is our required condition.

Likewise, it can be proved that, ϖ_2 is a "fuzzy hyper p-ideal" of H_2. Conversely suppose that ϖ_1 and ϖ_2 are "fuzzy hyper p-ideals of H_1 and H_2" respectively.

For any $(i, l), (j, m) \in H = H_1 \times H_2$, where $i, j \in H_1$ and $l, m \in H_2$, let $(i, l) \ll (j, m)$.

Since $(i,l) \ll (j,m)$ imply $i \ll j$ and $l \ll m$

$$\Rightarrow \varpi_1(i) \geq \varpi_1(j) \text{ and } \varpi_2(l) \geq \varpi_2(m)$$
$$\Rightarrow \min\ \{\varpi_1(i),\ \varpi_2(l)\} \geq \min\{\varpi_1(j),\ \varpi_2(m)\}$$
$$\Rightarrow (\varpi_1 \times \varpi_2)((i,l)) \geq (\varpi_1 \times \varpi_2)((j,m))$$
$$\Rightarrow \varpi((i,l)) \geq \varpi((j,m))$$
$$\text{Thus } (i,l) \ll (j,m) \Rightarrow \varpi((i,l)) \geq \varpi((j,m))$$

Moreover for any $(i,l),\ (j,m),\ (k,n) \in H$, where $i,j,k \in H_1$ and $l,m,n \in H_2$,

$$\varpi((i,l)) = (\varpi_1 \times \varpi_2)((i,l)) = \min\{\varpi_1(i),\ \varpi_2(l)\}$$
$$\geq \min[\min\{\inf_{c\in(i\circ k)\circ(j\circ k)}\ \varpi_1(c),\ \varpi_1(j)\},\ \min\{\inf_{d\in(l\circ n)\circ(m\circ n)}\ \varpi_2(d),\ \varpi_2(m)\}]$$
$$= \min[\min\{\inf_{c\in(i\circ k)\circ(j\circ k)}\ \varpi_1(c),\ \inf_{d\in(l\circ n)\circ(m\circ n)}\ \varpi_2(d)\},\ \min\{\ \varpi_1(j),\ \varpi_2(m)\}]$$
$$= \min[\inf_{c\in(i\circ k)\circ(j\circ k),\ d\in(l\circ n)\circ(m\circ n)}\ \{\min\{\varpi_1(c),\ \varpi_2(d)\}\},\ \min\{\ \varpi_1(j),\ \varpi_2(m)\}]$$
$$= \min\{\inf_{(c,d)\in((i\circ k)\circ(j\circ k),\ (l\circ n)\circ(m\circ n))}\ (\varpi_1 \times \varpi_2)(c,d),\ (\varpi_1 \times \varpi_2)((j,m))\}$$
$$= \min\{\inf_{(c,d)\in((i\circ k)\circ(j\circ k),\ (l\circ n)\circ(m\circ n))}\ \varpi((c,d)),\ \varpi((j,m))\}$$
$$\Rightarrow \varpi((i,l)) \geq \min\{\inf_{(c,d)\in(((i,l)\circ(k,n))\circ((j,m)\circ(k,n)))}\ \varpi((c,d)),\ \varpi((j,m))\}$$

Hence proved. □

Correspondingly, as done above, we can corroborate the statement for the other two cases.

4. Conclusion

From our above discussion we can conclude that:

- a "(fuzzy) strong hyper p-ideal" is a "(fuzzy) hyper p-ideal" and a "(fuzzy) hyper p-ideal" is a "(fuzzy) weak hyper p-ideal".
- λ_ϖ, "the strongest fuzzy relation" on a "hyper BCK-algebra", is a "fuzzy (weak, strong) hyper p-ideal" in case, ϖ is a "fuzzy (weak, strong) hyper p-ideal".
- "Hyper homomorphic pre-image", defined on an "onto hyper homomorphism", of a "fuzzy (weak, strong) hyper p-ideal" is also a "fuzzy (weak, strong) hyper p-ideal".
- The product of two "fuzzy (weak, strong) hyper p-ideal" is again a "fuzzy (weak, strong) hyper p-ideal".

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