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On Fixed Point Results for Matkowski Type of Mappings in G-Metric Spaces

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Abstract. The main result of this paper is a fixed point theorem for Matkowski type mapping with contractive iterate at a point in a class of G-metric spaces. Our result unifies, generalizes and complements some well known results in metric and G-metric spaces.

1. Introduction and preliminaries

In 1975 Matkowski introduced the following class of mappings:

Definition 1.1. *[9] Let T be a mapping on a metric space (X, d). Then T is called a weak contraction if there exists a function γ from $[0, \infty)$ to itself satisfying the following:*

i) *γ is nondecreasing,*

ii) $\lim_n \gamma^n(t) = 0$ *for all* $t > 0$,

iii) $d(Tx, Ty) \leq \gamma(d(x, y))$ *for all* $x, y \in X$.

In the same paper he proved the existence and uniqueness of a fixed point for such type of mappings. This result is significant because the concept of weak contraction of Matkowski type is independent of Meir-Keeler contraction [12], and it was generalized in different directions [10], [11], [17]. Matkowski generalized his own result proving a theorem of Segal- Guseman type [6].

Theorem 1.2. *[10] Let (X, d) be a complete metric space, $T : X \to X$, $\alpha : [0, \infty)^5 \to [0, \infty)$ and $\gamma(t) = \alpha(t, t, t, 2t, 2t)$ for $t \geq 0$. Suppose that*

1. *α is nondecreasing with respect to each variable,*
2. $\lim_{t\to\infty}(t - \gamma(t)) = \infty$,
3. $\lim_{t\to\infty} \gamma^n(t) = 0$, $t \geq 0$,

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4. *for every $x \in X$, there exists a positive integer $n = n(x)$ such that for all $y \in X$*

$$d(T^n x, T^n y) \le \alpha(d(x,y), d(x,T^n(x)), d(x,T^n(y)), d(T^n(x),y), d(T^n(y),y))$$

then T has a unique fixed point $a \in X$ and for each $x \in X$, $\lim_{k\to\infty} T^k(x) = a$.

The aim of this paper is to show that this result is valid in a more general class of spaces.

On 1963. S. Gähler introduced 2-metric spaces, but other authors proved that there is no relation between two distance functions and there is no easy relationship between results obtained in the two settings. B. C. Dhage introduced a new concept of the measure of nearness between three or more objects. But topological structure of so called D-metric spaces was incorrect. Finally, Z. Mustafa and B. Sims [13] introduced correct definition of a generalized metric space as follows.

Definition 1.3. *[13] Let X be a nonempty set, and let $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties*

(G1) $G(x,y,z) = 0$ *if* $x = y = z$;

(G2) $0 < G(x,x,y)$, *for all* $x, y \in X$, *with* $x \ne y$;

(G3) $G(x,x,y) \le G(x,y,z)$, *for all* $x,y,z \in X$, *with* $z \ne y$;

(G4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \ldots$, *(symmetry in all three variables);*

(G5) $G(x,y,z) \le G(x,a,a) + G(a,y,z)$, *for all* $x,y,z,a \in X$.

Then function G is called a generalized metric, abbreviated G-metric on X, and the pair (X,G) is called a G-metric space.

Clearly these properties are satisfied when $G(x,y,z)$ is the perimeter of the triangle in with vertices x, y and z. $\mathbb{R}^2$, moreover taking a in the interior of the triangle shows that (G5) is the best possible.

Example 1.1[13] Let (X,d) be an ordinary metric space, then (X,d) defines G-metrics on X by

$$G_s(x,y,z) = d(x,y) + d(y,z) + d(x,z),$$
$$G_m(x,y,z) = \max\{d(x,y), d(y,z), d(x,z)\}.$$

Example 1.2[13] Let $X = \{a,b\}$. Define G on $X \times X \times X$ by

$$G(a,a,a) = G(b,b,b) = 0, \quad G(a,a,b) = 1, \quad G(a,b,b) = 2,$$

and extend G to $X \times X \times X$ by using the symmetry in the variables. Then it is clear the (X,G) is a G-metric space.

The following useful properties of a G-metric are readily derived from the axioms.

Proposition 1.4. *[13] Let (X,G) be a G-metric space, then for any x,y,z and a from X it follows that:*

1. *if* $G(x,y,z) = 0$, *then* $x = y = z$,
2. $G(x,y,z) \le G(x,x,y) + G(x,x,z)$,
3. $G(x,x,y) \le 2G(y,y,x)$,
4. $G(x,y,z) \le G(x,a,z) + G(a,y,z)$,
5. $G(x,y,z) \le \frac{2}{3}(G(x,y,a) + G(x,a,z) + G(a,y,z))$,
6. $G(x,y,z) \le G(x,a,a) + G(y,a,a) + G(z,a,a)$.

Definition 1.5. *[13] Let (X, G) be a G-metric space, and let $\{x_n\}$ be a sequence of points of X. A point $x \in X$ is said to be the* limit *of the sequence $\{x_n\}$ if $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$, and one says that the sequence $\{x_n\}$ is G-convergent to x.*

Proposition 1.6. *[13] Let (X, G) be a G-metric space, then for a sequence $\{x_n\} \subseteq X$ and a point $x \in X$ the following are equivalent:*

1. *$\{x_n\}$ is G-convergent to x,*
2. *$G(x_n, x_n, x) \to 0$ as $n \to \infty$,*
3. *$G(x_n, x, x) \to 0$ as $n \to \infty$.*

Definition 1.7. *[13] Let (X, G) be a G-metric space, a sequence $\{x_n\}$ is called G-Cauchy if for every $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \epsilon$, for all $n, m, l \geq N$, that is, if $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.*

Proposition 1.8. *[13] In a G-metric space (X, G), the following are equivalent:*

1. *the sequence $\{x_n\}$ is G-Cauchy,*
2. *for every $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $n, m \geq n_0$.*

A G-metric space (X, G) is G-complete (or G is a complete G-metric), if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

Proposition 1.9. *[13] Let (X, G) be a G-metric space, then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Recently, Samet at all [15] and Jleli,Samet [7] observed that some fixed point theorems in context of G-metric space can be proved (by simple transformation) using related existing results in the setting of metric space. Namely, if the contraction condition of the fixed point theorem on G-metric space can be reduced to two variables, then one can construct an equivalent fixed point theorem in setting of usual metric space. This idea is not completely new, but it was not successfully used before (see [14]). Very recently, Karapinar and Agarwal suggest new contraction conditions in G-metric space in a way that the techniques in [15], [7] are not applicable. In this approach ([8]), contraction conditions can not be expressed in two variables. So, in some cases, as it is noticed even in Jleli-Samet paper [7], when the contraction condition is of nonlinear type, this strategy cannot be always successfully used. This is exactly the case in our paper.

For more fixed point results for mappings defined in G-metric spaces, we refer the reader to [1], [2], [3], [4] [5], [14] .

Definition 1.10. *Let X be a nonempty set and a function $\delta : X \times X \to [0, \infty)$ satisfies the following properties:*

1. *$\delta(x, y) = 0$ if and only if $x = y$,*
2. *$\delta(x, y) \leq \delta(x, z) + \delta(z, y)$ for any points $x, y, z \in X$,*

Then the pair (X, δ) is called a quasi -metric space. The sequence $\{x_n\} \subset X$ converges to $x \in X$, iff $\lim_n \delta(x_n, x) = \lim_n \delta(x, x_n) = 0$.

2. Main result

Let $\alpha : [0, \infty)^5 \to [0, \infty)$ be a nondecreasing function with respect to each variable and let $\gamma(t) = \alpha(t, t, 2t, 3t, 3t)$ for $t \geq 0$. Following Matkowski [10], let Γ be the set of all functions $\gamma : [0, \infty) \to [0, \infty)$ such that

1° $\lim \gamma^n(t) = 0,\ t > 0$;

2° $\lim_{t\to\infty} (t - \gamma(t)) = \infty$.

Simple examples of the function $\gamma \in \Gamma$ are: $\gamma(t) = q \cdot t,\ q \in (0,1),\ \ \gamma(t) = \frac{t}{1+t},\ \ \gamma(t) = \ln(1+t),\ \ t \in [0,\infty)$.

Remark 2.1. *It is obvious that γ is nondecreasing function and that consequences of the condition $\lim_{t\to\infty} \gamma^n(t) = 0,\ t \geq 0$, are:*

(a) $\gamma(t) < t,\ \ t > 0$;

(b) $\gamma(0) = 0$.

Definition 2.2. *If (X, G) be a G-metric space, $T : X \to X$, and for every $x \in X$, there exists a positive integer $n = n(x)$ such that for all $y \in X$*

$$G(T^{n(x)}x, T^{n(x)}x, T^{n(x)}y) \leq \alpha(G(x,x,y), G(x,x,T^{n(x)}y), G(x, T^{n(x)}x, T^{n(x)}x),$$

$$G(y, T^{n(x)}x, T^{n(x)}x), G(y,y,T^{n(x)}y), \tag{2.1}$$

then we say that T is a weak contraction in X.

Lemma 2.3. *If (X, G) is a G-metric space and $T : X \to X$ is a weak contraction in X, then for every $x \in X$, the orbit $\{T^k x\}_k$ is bounded.*

Proof. For any $x \in X$ and any integer s, $0 \leq s < n(x)$ we define the sequence

$$u_k(x,s) = u_k = G\left(x, x, Tx^{kn(x)+s}\right),\ \ k = 0,1,2,...$$

and the number

$$h(x,s) = h = \max\left\{G(x,x,T^{n(x)}x),\ G(x, T^{n(x)}x, T^{n(x)}x), G(x,x,T^s x)\right\}.$$

The property 2° of the function γ, implies that there exists a c, $c > h$, such that $t - \gamma(t) > h$, $t > c$. It is easy to see that $u_0 < c$.

Next, we show that $u_j < c$ for all $j = 0,1,2,\ldots$. The assumption that there exists a positive integer j such that $u_j \geq c$, but $u_i < c$ for $i < j$ will lead to contradiction.

Under the last assumption,

$$G\left(T^{n(x)}x, T^{n(x)}x, T^{(j-1)n(x)+s}x\right) \leq h + u_{j-1} < 2u_j,$$

$$G\left(T^{(j-1)n(x)+s}x,\ T^{(j-1)n(x)+s}x,\ T^{jn(x)+s}x\right) \leq 2u_{j-1} + u_j < 3u_j.$$

Now, since α is nondecreasing with respect to each variable and the mapping T is a weak contraction, we get

$$\begin{aligned} u_j &= G\left(x, x, T^{jn(x)+s}x\right) \\ &\leq G\left(x,x,T^{n(x)}x\right) + G\left(T^{n(x)}x,\ T^{n(x)}x,\ T^{jn(x)+s}x\right) \\ &\leq h + \alpha(u_j, u_j, u_j, 2u_j, 3u_j) \leq h + \gamma(u_j). \end{aligned}$$

The inequality $u_j - \gamma(u_j) \leq h$ contradicts to the choice of c. Hence, $u_j < c$ for all $j = 0,1,\ldots$. It completes the proof that for any fixed $x \in X$, $\sup_k G(x,x,T^k x) = M < \infty$, meaning that the orbit $\{T^k x\}_k$ is bounded. □

Lemma 2.4. *If (X, G) is a G-metric space and $T : X \to X$ is a weak contraction in X, then for every $x_0 \in X$, the sequence $x_{k+1} = T^{n(x_k)}x_k$, $k = 0,1,\ldots$, is a Cauchy sequence.*

Proof. It is easy to see that

$$x_{k+j} = T^{n(x_{k+j-1})+\ldots+n(x_k)}x_k,$$

for all $k, j \in \mathbb{N}$. Using the notation $s_0 = n(x_{k+j-1}) + \ldots + n(x_k)$, we get

$$\begin{aligned} G\left(x_k, x_k, x_{k+j}\right) &= G\left(x_k, x_k, T^{s_0}x_k\right) \\ &= G\left(T^{n(x_{k-1})}x_{k-1}, T^{n(x_{k-1})}x_{k-1}, T^{n(x_{k-1})}T^{s_0}x_{k-1}\right). \end{aligned}$$

Putting

$$t_{s_1} = \max\left\{G(x_{k-1}, x_{k-1}, T^i x_{k-1}) \ : i \in \{s_0, n(x_{k-1}), s_0 + n(x_{k-1})\}\right\},$$

we obtain

$$G\left(T^{n(x_{k-1})}x_{k-1}, T^{n(x_{k-1})}x_{k-1}, T^{s_0}x_{k-1}\right) \le 2t_{s_1} + t_{s_1} = 3t_{s_1}$$

and

$$G\left(T^{n(x_{k-1})+s_0}x_{k-1}, T^{s_0}x_{k-1}, T^{s_0}x_{k-1}\right) \le t_{s_1} + 2t_{s_1} = 3t_{s_1}.$$

The property that the mapping T is a weak contraction with respect to nondecreasing function α implies

$$G\left(x_k, x_k, T^{s_0}x_k\right) \le \alpha(t_{s_1}, t_{s_1}, 2t_{s_1}, 3t_{s_1}, 3t_{s_1}) = \gamma(t_{s_1}).$$

Repeating this procedure, one concludes that there exists an $s_j \in \mathbb{N}$, $j = 1, 2, \ldots, k-1$ such that

$$G\left(x_{k-j}, x_{k-j}, T^{s_j}x_{k-j}\right) \le \gamma\left(G(x_{k-j-1}, x_{k-j-1}, T^{s_{j+1}}x_{k-j-1})\right).$$

The mapping γ is nondecreasing which yields

$$G\left(x_k, x_k, x_{k+j}\right) \le \gamma^k\left(G\left(x_0, x_0, T^{s_k}x_0\right)\right) \le \gamma^k(M).$$

Since $\gamma \in \Gamma$, we see that $\lim \gamma^n(M) = 0$. This is precisely the assertion of the lemma, that is, $\{x_k\}_k$ is a Cauchy sequence. □

Theorem 2.5. *If (X, G) is a complete G-metric space and $T : X \to X$ is a weak contraction in X, then T has a unique fixed point $a \in X$, for every $x \in X$, $\lim_k T^k x = a$ and $T^{n(a)}$ is continuous at a.*

Proof. By completeness of X, the (Cauchy) sequence $\{x_k\}_k$ defined in the above lemma is convergent, i.e. $\lim_k x_k = a \in X$.

The proof falls naturally into six consecutive parts in which we prove that:

1. $T^{n(a)}a = a$,
2. a is a unique fixed point of $T^{n(a)}$,
3. $Ta = a$,
4. a is a unique fixed point of T,
5. for every $x \in X$, $\lim_k T^k x = a$,
6. $T^{n(a)}$ is continuous at a

1. Applying the same reasoning as in the last lemma, we can show that $\lim_k G(x_k, x_k, T^{n(a)}x_k) = 0$. It means that for every $\varepsilon > 0$ there exists a $k_1(\varepsilon) \in \mathbb{N}$ such that for $k \ge k_1(\varepsilon)$,

$$G(x_k, x_k, T^{n(a)}x_k) < \frac{1}{8}(\varepsilon - \gamma(\varepsilon)).$$

Since $\lim_k G(x_k, x_k, a) = 0$, for every $\varepsilon > 0$ there exists a $k_2(\varepsilon) \in \mathbb{N}$ such that for $k \geq k_2(\varepsilon)$,

$$G(a, x_k, x_k) < \frac{1}{8}((\varepsilon - \gamma(\varepsilon))$$

Next, we claim that $T^{n(a)}a = a$. Indeed, if we suppose the opposite, i.e. if we suppose that there exists an $\varepsilon > 0$ such that $G(T^{n(a)}a, T^{n(a)}a, a) = \varepsilon$, then for $k \geq \max\{k_1(\varepsilon),\ k_2(\varepsilon)\}$

$$\begin{aligned} \varepsilon = G(T^{n(a)}a, T^{n(a)}a, a) &\leq G(T^{n(a)}a, T^{n(a)}a, T^{n(a)}x_k) \\ &\quad + G(T^{n(a)}x_k, T^{n(a)}x_k, x_k) + G(x_k, x_k, a) \\ &\leq \alpha\Big(G(a, a, x_k), G(a, a, T^{n(a)}x_k), \\ &\quad G(a, T^{n(a)}a, T^{n(a)}a), G(x_k, T^{n(a)}a, T^{n(a)}a), \\ &\quad G(x_k, x_k, T^{n(a)}x_k)\Big) + \frac{1}{2}(\varepsilon - \gamma(\varepsilon)). \end{aligned}$$

The next two relations

$$G(a, a, T^{n(a)}x_k) \leq G(a, a, x_k) + G(x_k, x_k, T^{n(a)}x_k) < \frac{1}{4}(\varepsilon - \gamma(\varepsilon)),$$

$$G(x_k, T^{n(a)}a, T^{n(a)}a) \leq G(x_k, a, a) + G(a, T^{n(a)}a, T^{n(a)}a) < 2\varepsilon,$$

imply

$$\varepsilon \leq \alpha(\varepsilon, \varepsilon, \varepsilon, 2\varepsilon, \varepsilon) + \frac{1}{2}(\varepsilon - \gamma(\varepsilon)) < \frac{1}{2}(\varepsilon + \gamma(\varepsilon)) < \varepsilon,$$

which is a contradiction, meaning that the assumption that $T^{n(a)}a \neq a$ is not correct.

2. The proof that a is the only fixed point of the mapping $T^{n(a)}$ is once again by contradiction. If $b \in X$ would be another fixed point of $T^{n(a)}$, then

$$\begin{aligned} G(a, a, b) &= G(T^{n(a)}a, T^{n(a)}a, T^{n(a)}b) \\ &\leq \alpha(G(a, a, b), G(a, a, b), 0, G(b, a, a), 0) \\ &\leq \gamma(G(a, a, b)) < G(a, a, b), \end{aligned}$$

which establishes our claim that $a = b$.

3. The equality

$$T\,a = T\,T^{n(a)}a = T^{n(a)}T\,a$$

implies that $T\,a = a$.

4. From 2. and 3. we conclude that a is the only point from X such that $T\,a = a$.

5. Next we prove that $\lim_k T^k x = a$, for each $x \in X$. For each $x \in X$ and each $s \in \mathbb{N}$, $0 \leq s < n(a)$, define the sequence

$$a_k = G(a, a, T^{kn(a)+s}x), \quad k = 0, 1, 2, \cdots.$$

If for some $k \in \mathbb{N}$, $a_k > a_{k-1}$, then

$$\begin{aligned} a_k &= G(T^{n(a)}a, T^{n(a)}a, T^{n(a)}T^{(k-1)n(a)+s}x) \\ &\leq \alpha\Big(G\left(a, a, T^{(k-1)n(a)+s}x\right), G\left(a, a, T^{kn(a)+s}x\right), \\ &\quad G\left(a, T^{n(a)}a, T^{n(a)}a\right), G\left(T^{kn(a)+s}x, T^{n(a)}a, T^{n(a)}a\right), \\ &\quad G\left(T^{(k-1)n(a)+s}x, T^{(k-1)n(a)+s}x, T^{kn(a)+s}x\right)\Big) \\ &\leq \alpha(a_{k-1}, a_k, 0, a_{k-1}, a_k + 2a_{k-1}) \leq \gamma(a_k) < a_k. \end{aligned}$$

By the last contradiction, we deduce that $a_k \leq a_{k-1}$ for all $k \in \mathbb{N}$. Hence,

$$a_k \leq \alpha(a_{k-1}, a_{k-1}, a_{k-1}, a_{k-1}, 3a_{k-1}) \leq \gamma(a_{k-1}) \leq \cdots \leq \gamma^k(a_0).$$

Letting $k \to \infty$ in the last relation, $\lim_k a_k = \lim_k \gamma^k(a_0) = 0$, and consequently $\lim_k T^k x = a$.

6. Finally, we finish the proof by showing the continuity of $T^{n(a)}$ at a.

We consider a sequence $\{y_m\}_m \subset X$ converging to a. For any $m \in \mathbb{N}$

$$\begin{aligned} G\left(a,a,T^{n(a)}y_m\right) &= G(T^{n(a)}a, T^{n(a)}a, T^{n(a)}y_m) \\ &\le \alpha\Big(G(a,a,y_m), G\left(a,a,T^{n(a)}y_m\right), \\ &\quad G\left(a,T^{n(a)}a,T^{n(a)}a\right), G\left(y_m,T^{n(a)}a,T^{n(a)}a\right), \\ &\quad G\left(y_m,y_m,T^{n(a)}y_m\right)\Big) \\ &= \alpha\Big(G(a,a,y_m), G\left(a,a,T^{n(a)}y_m\right), \\ &\quad G(a,a,a), G(y_m,a,a), G\left(y_m,y_m,T^{n(a)}y_m\right)\Big) \\ &\le \alpha\Big(G(a,a,y_m), G\left(a,a,T^{n(a)}y_m\right),\ 0, \\ &\quad G(y_m,a,a), G(y_m,y_m,a) + G\left(a,a,T^{n(a)}y_m\right)\Big). \end{aligned}$$

If $\lim_m G\left(a,a,T^{n(a)}y_m\right) \neq 0$, then there exists $\varepsilon > 0$ such that for some $m_1 \in \mathbb{N}$, $G\left(a,a,T^{n(a)}y_m\right) > \varepsilon$ for all $m > m_1$. On the other hand, since $\lim_k y_m = a$, for given ε there exist $m_2 \in \mathbb{N}$ and $m_3 \in \mathbb{N}$ such that

$$m > m_2 \quad\Rightarrow\quad G(a,a,y_m) < \varepsilon < G\left(a,a,T^{n(a)}y_m\right),$$

and

$$m > m_3 \quad\Rightarrow\quad G(y_m,y_m,a) < \varepsilon < G\left(a,a,T^{n(a)}y_m\right).$$

Putting $m_0 = \max\{m_1, m_2, m_3\}$, for all $m > m_0$, we get

$$\begin{aligned} G\left(a,a,T^{n(a)}y_m\right) &\le \alpha\Big(G\left(a,a,T^{n(a)}y_m\right), G\left(a,a,T^{n(a)}y_m\right),\ 0, \\ &\quad G\left(a,a,T^{n(a)}y_m\right), 2G\left(a,a,T^{n(a)}y_m\right)\Big) \\ &\le \gamma\Big(G\left(a,a,T^{n(a)}y_m\right)\Big) < G\left(a,a,T^{n(a)}y_m\right). \end{aligned}$$

Obviously, the assumption $\lim_m G\left(a,a,T^{n(a)}y_m\right) \neq 0$ induces the contradiction in the last relation. Hence, $T^{n(a)}$ is continuous at a. □

Remark 2.6. *In a symmetric G-metric space, one can put $\gamma(t) = \alpha(t,t,t,2t,2t)$ (as it is done in Matkovski paper [10]), but Jleli-Samet technique can not be applied.*

In [7] it was shown that if (X,G) is a G-metric space, putting $\delta(x,y) = G(x,y,y)$, (X,δ) is a quasi metric space (δ is not symmetric). Simple replacement G with δ in (2.1), defines the weak contraction T in (X,δ)

$$\delta(T^{n(x)}x, T^{n(x)}y) \le \alpha(\delta(x,y), \delta(x,T^{n(x)}y), \delta(T^{n(x)}x,x),$$
$$\delta(T^{n(x)}x,y), \delta(y,T^{n(x)}y)).$$

The following result is an immediate consequence of above definitions and relations.

Theorem 2.7. *If (X,δ) is a complete quasi-metric space such that $\delta(x,y) \le 2\delta(y,x)$ for all $x,y \in X$ and $T : X \to X$ is a weak contraction in X, then T has a unique fixed point $a \in X$, for every $x \in X$, $\lim_k T^k x = a$ and $T^{n(a)}$ is continuous at a.*

Proof. It is obvious that (X,δ) has the same topological structure as any G-metric space (X,G) where G satisfies equality $\delta(x,y) = G(x,y,y)$. Now, repeating the proof of Theorem 2.5, we conclude that the assertion of theorem is true. □

The next few corollaries are also a consequence of Theorem 2.5.

Corollary 2.8. *[5] Let (X, G) be a complete G-metric space, $T : X \to X$, $\gamma \in \Gamma$ and for each $x \in X$ there exists a positive integer $n = n(x)$ such that*

$$G\left(T^{n(x)}x, T^{n(x)}x, T^{n(x)}y\right) \leq \gamma\left(G(x, x, y)\right), \tag{2.2}$$

for all $y \in X$. Then T has a unique fixed point $a \in X$. Moreover, for each $x \in X$, $\lim\limits_{k} T^k x = a$ and $T^{n(a)}$ is continuous at a.

Corollary 2.9. *[3] Let (X, G) be a complete G-metric space, $T : X \to X$ and for each $x \in X$ there exists a positive integer $n = n(x)$ such that*

$$G\left(T^{n(x)}x, T^{n(x)}x, T^{n(x)}y\right) \leq q\, G(x, x, y)$$

for all $y \in X$ and some $q \in (0, 1)$. Then T has a unique fixed point $a \in X$. Moreover, for each $x \in X$, $\lim\limits_{k} T^k x = a$ and $T^{n(a)}$ is continuous at a.

Proof. Function $\gamma(t) = q\, t$, $t \in [0, \infty)$, belongs to Γ, so corollary is a consequence of Theorem 2.5. □

Remark 2.10. *Taking $\gamma(t) = q \cdot t,\ 0 < q < 1$, we obtain the fixed point result from [2] or [3], so Theorem 2.5. is also a generalization of Guseman fixed point result from [6].*

Corollary 2.11. *Let (X, G) be a complete G-metric space, $T : X \to X$, and for each $x \in X$ there exists a positive integer $n = n(x)$ such that*

$$G\left(T^{n(x)}x, T^{n(x)}x, T^{n(x)}y\right) \leq \frac{G(x, x, y)}{1 + G(x, x, y)},$$

for all $y \in X$. Then T has a unique fixed point $a \in X$. Moreover, for each $x \in X$, $\lim\limits_{k} T^k x = a$ and $T^{n(a)}$ is continuous at a.

Proof. Since the function $\gamma = \frac{t}{1+t}$, $t \in [0, \infty)$, belongs to the family Γ, we can apply Theorem 2.5. □

If $n(x) = 1$ ($n(x) = m \in \mathbb{N}$), for each $x \in X$, we are going to prove that condition 3° can be omitted so, in that case we will have an improvement and another proof of Theorem 3.1 (Corollary 3.2) from [16].

Proposition 2.12. *Let (X, G) be a complete G-metric space, $\gamma : [0, \infty) \to [0, \infty)$ be a nondecreasing function with $\lim\limits_{k\to\infty} \gamma^k(t) = 0$ for $t > 0$. If $T : X \to X$ satisfies*

$$G(Tx, Tx, Ty) \leq \gamma(G(x, x, y)), \tag{2.3}$$

for all $x, y \in X$, then T has a unique fixed point $a \in X$. Moreover, for each $x \in X$, $\lim\limits_{k} T^k x = a$ and $T^{n(a)}$ is continuous at a.

Proof. We prove, by induction, that for any $x_0 = x$, $x \in X$, the orbit $\{T^k x_0\}_k$ is bounded. By the properties of the function γ, $\lim\limits_{k} \gamma^k(G(x_0, x_0, x_1)) = 0$ and $\gamma(t) < t$, $t > 0$, we get that there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$

$$\gamma^k(G(x_0, x_0, x_1)) < 1 - \gamma(1).$$

Hence

$$G(Tx_k, Tx_k, Tx_{k+1}) < 1 - \gamma(1), \quad k \geq k_0. \tag{2.4}$$

For $j > k_0$, $j \in \mathbb{N}$, we claim that

$$G\left(x_{k_0}, x_{k_0}, x_j\right) \leq 1. \tag{2.5}$$

Indeed, for $j = k_0 + 1$, the last relation is true by (2.4). Assuming inequality (2.5) holds for $j = i$, $i > k_0$, we will prove it holds for $j = i + 1$. Really, letting $j = i + 1$, we get

$$G\left(x_{k_0}, x_{k_0}, x_{i+1}\right) \leq G\left(x_{k_0}, x_{k_0}, x_{k_0+1}\right) + G\left(x_{k_0+1}, x_{k_0+1}, x_{i+1}\right) \leq$$

$$\leq G\left(x_{k_0}, x_{k_0}, x_{k_0+1}\right) + \gamma\left(G\left(x_{k_0}, x_{k_0}, x_i\right)\right) \leq$$

$$\leq 1 - \gamma(1) + \gamma(1) = 1.$$

Finally, for any $k \in \mathbb{N}$

$$G\left(x_0, x_0, T^k x_0\right) \leq G\left(x_0, x_0, x_{k_0}\right) + G\left(x_{k_0}, x_{k_0}, x_k\right) \leq$$

$$G\left(x_0, x_0, x_{k_0}\right) + \max\{1, G\left(x_{k_0}, x_{k_0}, x_1\right), \cdots, G\left(x_{k_0}, x_{k_0}, x_{k_0-1}\right) = M < \infty.$$

Now, the rest of the proof runs as in Theorem 2.1. □

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