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On Integrability Conditions of Derivation Equations in a Subspace of Asymmetric Affine Connection Space

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Abstract. In a space L_N of asymmetric affine connection by equations (1.1) a submanifold $X_M \subset L_N$ is defined. On X_M and on pseudonormal submanifold X^N_{N-M} asymmetric induced connections are defined. Because of asymmetry of induced connection it is possible to define four kinds of covariant derivative. In this work we are considering integrability conditions of derivational equations [4] obtained by help of the 1^{st} and the 2^{nd} kind of covariant derivative. The corresponding Gauss-Codazzi equations are obtained too.

1. Introduction

Consider a space L_N of asymmetric affine connection with a torsion (in local coordinates) $T^i_{jk} = L^i_{jk} - L^i_{kj}$. Spaces with asymmetric affine connection and their properties were studied by many authors [1, 11]. A submanifold $X_M \subset L_N$ is defined by equations

$$x^i = x^i(u^1, \dots, u^M) = x^i(u^\alpha), \quad i = \overline{1, N}. \tag{1.1}$$

Partial derivatives $B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}$ $(\text{rank}(B^i_\alpha) = M)$ define tangent vectors on X_M.

Consider $N - M$ contravariant vectors C^i_A $(A, B, C, \dots, \in \{M + 1, \dots, N\})$ defined on X_M and linearly independent, and let the matrix $\binom{B^\alpha_i}{C^A_i}$ be inverse for the matrix (B^i_α, C^i_A) provided that the following conditions are satisfied [10]:

$$\begin{aligned} &a)\ B^i_\alpha B^\beta_i = \delta^\beta_\alpha; \quad b)\ B^i_\alpha C^A_i = 0; \quad c)\ B^\alpha_i C^i_A = 0;\\ &d)\ C^i_A C^B_i = \delta^B_A; \quad e)\ B^i_\alpha B^\alpha_j + C^i_A C^A_j = \delta^i_j; \end{aligned} \tag{1.2}$$

The magnitudes B^i_α, B^α_i are **projection factors (tangent vectors)**, and the magnitudes C^i_A, C^A_i are **affine pseudonormals** [4, 10].

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The **induced connection** on X_M is [4, 10, 11]:

$$\widetilde{L}^{\alpha}_{\beta\gamma} = B^{\alpha}_{i}(B^{i}_{\beta,\gamma} + L^{i}_{jk}B^{j}_{\beta}B^{k}_{\gamma}), \tag{1.3}$$

where $B^{i}_{\beta,\gamma} = \partial B^{i}_{\beta}/\partial u^{\gamma} = \partial^2 x^i/\partial u^{\beta}\partial u^{\gamma}$. Because L is asymmetric by virtue of j,k, $\widetilde{L}$ is asymmetric in β,γ too. The submanifold X_M endowed with $\widetilde{L}$ becomes L_M and we write $L_M \subset L_N$.

The set of pseudonormals of the submanifold $X_M \subset L_N$ makes **a pseudonormal bundle** of X_M, and we note it X^{N}_{N-M}. We have defined in [4] **induced connection of pseudonormal bundle** with coefficients

$$\underset{\substack{1\\2}}{\overline{L}}{}^{A}_{B\mu} = C^{A}_{i}(C^{i}_{B,\mu} + \underset{kj}{L^{i}_{jk}}C^{j}_{B}B^{k}_{\mu}). \tag{1.4}$$

As the coefficients $L, \widetilde{L}, \overline{L}$ are generally asymmetric, we can define four kinds of covariant derivative for a tensor, defined in the points of X_M. For example:

$$t^{i\alpha A}_{\underset{\substack{1\\2\\3\\4}}{j\beta B|\mu}} = t^{i\alpha A}_{j\beta B,\mu} + \underset{\substack{mp\\pm\\mp}}{L^{i}_{pm}}t^{p\alpha A}_{j\beta B} - \underset{\substack{mj\\mj\\jm}}{L^{p}_{jm}}t^{i\alpha A}_{p\beta B} + \underset{\substack{\mu\pi\\\pi\mu\\\mu\pi}}{\widetilde{L}^{\alpha}_{\pi\mu}}t^{i\pi A}_{j\beta B} - \underset{\substack{\mu\beta\\\mu\beta\\\beta\mu}}{\widetilde{L}^{\pi}_{\beta\mu}}t^{i\alpha A}_{j\pi B} + \underset{\substack{1\\2\\1\\2}}{\overline{L}}{}^{A}_{P\mu}t^{i\alpha P}_{j\beta B} - \underset{\substack{1\\2\\2\\1}}{\overline{L}}{}^{P}_{\beta\mu}t^{i\alpha A}_{j\beta P}, \tag{1.5}$$

In this manner four connections $\underset{\theta}{\nabla}$ on $X_M \subset L_N$ are defined. We shall note the obtained structures ($X_M \subset L_N$, $\underset{\theta}{\nabla}, \theta \in \{1, \dots, 4\}$).

2. Integrability conditions of derivational equations for tangents

2.0 From (2.17, 17′) in [4], we have a derivational equations for tangents

$$B^{i}_{\alpha\underset{\theta}{|}\mu} = \underset{\theta}{\Omega}{}^{P}_{\alpha\mu}C^{i}_{P}, \quad B^{\alpha}_{i\underset{\theta}{|}\mu} = \underset{\theta}{\widehat{\Omega}}{}^{\alpha}_{P\mu}C^{P}_{i}, \quad \theta \in \{1, 2\}, \tag{2.1}$$

and from (3.19, 19′) for pseudonormals

$$C^{i}_{A\underset{\theta}{|}\mu} = -\underset{\theta}{\widehat{\Omega}}{}^{\pi}_{A\mu}B^{i}_{\pi}, \quad C^{A}_{i\underset{\theta}{|}\mu} = -\underset{\theta}{\Omega}{}^{A}_{\pi\mu}B^{\pi}_{i}, \quad \theta \in \{1, 2\}. \tag{2.2}$$

So, one obtains

$$\begin{aligned} B^{i}_{\alpha\underset{\theta}{|}\mu\underset{\omega}{|}\nu} &= (\underset{\theta}{\Omega}{}^{P}_{\alpha\mu}C^{i}_{P})_{\underset{\omega}{|}\nu} = \underset{\theta}{\Omega}{}^{P}_{\alpha\mu\underset{\omega}{|}\nu}C^{i}_{P} + \underset{\theta}{\Omega}{}^{P}_{\alpha\mu}C^{i}_{P\underset{\omega}{|}\nu} \\ &\underset{(2.2)}{=} -\underset{\theta}{\Omega}{}^{P}_{\alpha\mu}\underset{\omega}{\widehat{\Omega}}{}^{\pi}_{P\nu}B^{i}_{\pi} + \underset{\theta}{\Omega}{}^{P}_{\alpha\mu\underset{\omega}{|}\nu}C^{i}_{P}, \end{aligned}$$

where " $\underset{(2.2)}{=}$ " signifies "equal with respect to (2.2)". In this manner

$$B^{i}_{\alpha\underset{\theta}{|}\mu\underset{\omega}{|}\nu} - B^{i}_{\alpha\underset{\omega}{|}\nu\underset{\theta}{|}\mu} = (\underset{\theta}{\widehat{\Omega}}{}^{\pi}_{P\mu}\underset{\omega}{\Omega}{}^{P}_{\alpha\nu} - \underset{\omega}{\widehat{\Omega}}{}^{\pi}_{P\nu}\underset{\theta}{\Omega}{}^{P}_{\alpha\mu})B^{i}_{\pi} + (\underset{\theta}{\Omega}{}^{P}_{\alpha\mu\underset{\omega}{|}\nu} - \underset{\theta}{\Omega}{}^{P}_{\alpha\nu\underset{\omega}{|}\mu})C^{i}_{P}, \tag{2.3}$$

and by analogous procedure:

$$B^{\alpha}_{i\underset{\theta}{|}\mu\underset{\omega}{|}\nu} - B^{\alpha}_{i\underset{\omega}{|}\nu\underset{\theta}{|}\mu} = (\underset{\theta}{\Omega}{}^{P}_{\pi\mu}\underset{\omega}{\widehat{\Omega}}{}^{\alpha}_{P\nu} - \underset{\omega}{\Omega}{}^{P}_{\pi\nu}\underset{\theta}{\widehat{\Omega}}{}^{\alpha}_{P\mu})B^{\pi}_{i} + (\underset{\theta}{\widehat{\Omega}}{}^{\alpha}_{P\mu\underset{\omega}{|}\nu} - \underset{\theta}{\widehat{\Omega}}{}^{\alpha}_{P\nu\underset{\omega}{|}\mu})C^{P}_{i}, \tag{2.3′}$$

where [4]

$$\underset{\substack{1\\2}}{\Omega}{}^{P}_{\alpha\mu} = C^{P}_{i}(B^{i}_{\alpha,\mu} + \underset{mp}{L^{i}_{pm}}B^{p}_{\alpha}B^{m}_{\mu}), \tag{2.4}$$

$$\widehat{\underset{\substack{1\\2}}{\Omega}}{}^{\alpha}_{P\mu} = C^i_P(B^\alpha_{i,\mu} - L^p_{\underset{mi}{im}} B^\alpha_p B^m_\mu). \qquad (2.4')$$

Based on the Ricci type identities [1, 2]

$$B^i_{\alpha\underset{\theta}{|}\mu\underset{\theta}{|}\nu} - B^i_{\alpha\underset{\theta}{|}\nu\underset{\theta}{|}\mu} = \underset{\theta}{R}{}^i_{pmn} B^p_\alpha B^m_\mu B^n_\nu - \underset{\theta}{\widetilde{R}}{}^\pi_{\alpha\mu\nu} B^i_\pi + (-1)^\theta \widetilde{T}^\pi_{\mu\nu} B^i_{\alpha\underset{\theta}{|}\pi}, \quad \theta \in \{1,2\}. \qquad (2.5)$$

where

$$\underset{1}{R}{}^i_{jmn} = L^i_{jm,n} - L^i_{jn,m} + L^p_{jm} L^i_{pn} - L^p_{jn} L^i_{pm}, \qquad (2.6)$$

$$\underset{2}{R}{}^i_{jmn} = L^i_{mj,n} - L^i_{nj,m} + L^p_{mj} L^i_{np} - L^p_{nj} L^i_{mp}, \qquad (2.7)$$

are curvature tensors of the 1st respectively the 2nd kind of the L_N and, analogously, $\underset{1}{\widetilde{R}}{}^\alpha_{\beta\mu\nu}$, $\underset{2}{\widetilde{R}}{}^\alpha_{\beta\mu\nu}$ are curvature tensors of $L_M \subset L_N$, obtained in the same manner by connection $\widetilde{L}^\alpha_{\beta\gamma}$.

2.1 Taking (2.3) $\theta = \omega \in \{1,2\}$, equalizing the right sides of obtained equation and (2.5) and exchange $B^i_{\alpha\underset{\theta}{|}\pi}$ with respect of (2.1), we get **the 1st and the 2nd kind integrability condition of derivational equation** (2.1)**:**

$$\boxed{\begin{aligned}\underset{\theta}{R}{}^i_{pmn} B^p_\alpha B^m_\mu B^n_\nu = {} & (\underset{\theta}{\widetilde{R}}{}^\pi_{\alpha\mu\nu} + \underset{\theta}{\widehat{\Omega}}{}^\pi_{P\mu} \underset{\theta}{\Omega}{}^P_{\alpha\nu} - \underset{\theta}{\widehat{\Omega}}{}^\pi_{P\nu} \underset{\theta}{\Omega}{}^P_{\alpha\mu}) B^i_\pi \\ & + (\underset{\theta}{\Omega}{}^P_{\alpha\mu\underset{\theta}{|}\nu} - \underset{\theta}{\Omega}{}^P_{\alpha\nu\underset{\theta}{|}\mu} + (-1)^{\theta-1} \widetilde{T}^\pi_{\mu\nu} \underset{\theta}{\Omega}{}^P_{\alpha\pi}) C^i_P, \quad \theta \in \{1,2\}.\end{aligned}} \qquad (2.8)$$

a) Composing that equation with B^λ_i and taking into consideration (1.2), we obtain

$$\underset{\theta}{\widetilde{R}}{}^\lambda_{\alpha\mu\nu} = \underset{\theta}{R}{}^i_{pmn} B^\lambda_i B^p_\alpha B^m_\mu B^n_\nu + \underset{\theta}{\Omega}{}^P_{\alpha\mu} \underset{\theta}{\widehat{\Omega}}{}^\lambda_{P\nu} - \underset{\theta}{\Omega}{}^P_{\alpha\nu} \underset{\theta}{\widehat{\Omega}}{}^\lambda_{P\mu}, \ \theta \in \{1,2\}. \qquad (2.9)$$

which are **Gauss equation of the** 1st **and the** 2nd **kind** ($\theta = 1,2$) for $L_M \subset L_N$.
b) If one composes (2.8) with C^L_i, we get

$$\underset{\theta}{R}{}^i_{pmn} C^L_i B^p_\alpha B^m_\mu B^n_\nu = \underset{\theta}{\Omega}{}^L_{\alpha\mu\underset{\theta}{|}\nu} - \underset{\theta}{\Omega}{}^L_{\alpha\nu\underset{\theta}{|}\mu} + (-1)^{\theta-1} \widetilde{T}^\pi_{\mu\nu} \underset{\theta}{\Omega}{}^L_{\alpha\pi}, \quad \theta \in \{1,2\}. \qquad (2.10)$$

and that are **the** 1st **Codazzi equation of the** 1st **and the** 2nd **kind** ($\theta = 1,2$) for $L_M \subset L_N$.

2.1′ Starting from the Ricci type identities (2.3′) for $\theta \in \{1,2\}$

$$B^\alpha_{i\underset{\theta}{|}\mu\underset{\theta}{|}\nu} - B^\alpha_{i\underset{\theta}{|}\nu\underset{\theta}{|}\mu} = -\underset{\theta}{R}{}^p_{imn} B^\alpha_p B^m_\mu B^n_\nu + \underset{\theta}{\widetilde{R}}{}^\alpha_{\pi\mu\nu} B^\pi_i + (-1)^\theta \widetilde{T}^\pi_{\mu\nu} B^\alpha_{i\underset{\theta}{|}\pi}, \quad \theta \in \{1,2\}. \qquad (2.5')$$

substituting B^α_i into (2.5′) by virtue of (2.1′), we get **the** 1st **and the** 2nd **integrability condition of derivation equation** (2.1′)**:**

$$\boxed{\begin{aligned}\underset{\theta}{R}{}^p_{imn} B^\alpha_p B^m_\mu B^n_\nu = {} & (\underset{\theta}{\widetilde{R}}{}^\alpha_{\pi\mu\nu} - \underset{\theta}{\Omega}{}^P_{\pi\mu} \underset{\theta}{\widehat{\Omega}}{}^\alpha_{P\nu} + \underset{\theta}{\Omega}{}^P_{\pi\nu} \underset{\theta}{\widehat{\Omega}}{}^\alpha_{P\mu}) B^\pi_i \\ & + (-\underset{\theta}{\widehat{\Omega}}{}^\alpha_{P\mu\underset{\theta}{|}\nu} + \underset{\theta}{\widehat{\Omega}}{}^\alpha_{P\nu\underset{\theta}{|}\mu} + (-1)^\theta \widetilde{T}^\pi_{\mu\nu} \underset{\theta}{\widehat{\Omega}}{}^\alpha_{P\pi}) C^P_i, \quad \theta \in \{1,2\}.\end{aligned}} \qquad (2.8')$$

a′) By composing the previous equation with B^i_λ one gets

$$\underset{\theta}{\widetilde{R}}{}^\alpha_{\lambda\mu\nu} = \underset{\theta}{R}{}^p_{imn} B^\alpha_p B^i_\lambda B^m_\mu B^n_\nu + \underset{\theta}{\Omega}{}^P_{\lambda\mu} \underset{\theta}{\widehat{\Omega}}{}^\alpha_{P\nu} - \underset{\theta}{\Omega}{}^P_{\lambda\nu} \underset{\theta}{\widehat{\Omega}}{}^\alpha_{P\mu}, \quad \theta \in \{1,2\},$$

and that equation by exchanges $\alpha \leftrightarrow \lambda$, $p \leftrightarrow i$ becomes (2.9).
b′) Composing (2.8′) with C^i_L, it follows that

$$\underset{\theta}{R}{}^p_{imn} C^i_L B^\alpha_p B^m_\mu B^n_\nu = -\underset{\theta}{\widehat{\Omega}}{}^\alpha_{L\mu\underset{\theta}{|}\nu} + \underset{\theta}{\widehat{\Omega}}{}^\alpha_{L\nu\underset{\theta}{|}\mu} + (-1)^\theta \widetilde{T}^\pi_{\mu\nu} \underset{\theta}{\widehat{\Omega}}{}^\alpha_{L\pi}, \ \theta \in \{1,2\}, \tag{2.10′}$$

which is the another form of **the** 1^{st} **Codazzi equation of the** 1^{st} **and the** 2^{nd} **kind for** $L_M \subset L_N$**.**

2.2 By application of the corresponding Ricci type identity [1, 2], one obtains

$$B^i_{\alpha\underset{1}{|}\mu\underset{2}{|}\nu} - B^i_{\alpha\underset{2}{|}\nu\underset{1}{|}\mu} = \underset{3}{R}{}^i_{p\mu\nu} B^p_\alpha - \underset{3}{\widetilde{R}}{}^\pi_{\alpha\mu\nu} B^i_\pi, \tag{2.11}$$

where

$$\underset{3}{R}{}^i_{j\mu\nu} = (L^i_{jm,n} - L^i_{nj,m} + L^p_{jm} L^i_{np} - L^p_{nj} L^i_{pm}) B^m_\mu B^n_\nu + T^i_{jm}(B^m_{\mu,\nu} - \widetilde{L}^\pi_{\nu\mu} B^m_\pi), \tag{2.12}$$

is **the** 3^{rd} **kind curvature tensor** of L_N relating to L_M, and

$$\underset{3}{\widetilde{R}}{}^\alpha_{\beta\mu\nu} = \widetilde{L}^\alpha_{\beta\mu,\nu} - \widetilde{L}^\alpha_{\nu\beta,\mu} + \widetilde{L}^\pi_{\beta\mu} \widetilde{L}^\alpha_{\nu\pi} - \widetilde{L}^\pi_{\nu\beta} \widetilde{L}^\alpha_{\pi\mu} + \widetilde{L}^\pi_{\nu\mu} \widetilde{T}^\alpha_{\pi\beta} \tag{2.13}$$

is **the** 3^{rd} **kind curvature tensor of the subspace** $L_M \subset L_N$.

On the other hand, putting at (2.3) $\theta = 1, \omega = 2$, and comparing the obtained equation with (2.11), we have **the** 3^{rd} **integrability condition of derivational equation** (2.1)**.**

$$\boxed{\underset{3}{R}{}^i_{p\mu\nu} B^p_\alpha - \underset{3}{\widetilde{R}}{}^\pi_{\alpha\mu\nu} B^i_\pi = (\underset{1}{\widehat{\Omega}}{}^\pi_{P\mu} \underset{2}{\Omega}{}^P_{\alpha\nu} - \underset{2}{\widehat{\Omega}}{}^\pi_{P\nu} \underset{1}{\Omega}{}^P_{\alpha\mu}) B^i_\pi + (\underset{1}{\Omega}{}^P_{\alpha\mu\underset{2}{|}\nu} - \underset{2}{\Omega}{}^P_{\alpha\nu\underset{1}{|}\mu}) C^i_P.} \tag{2.14}$$

a) By composing with B^λ_i, from here is obtained

$$\underset{3}{\widetilde{R}}{}^\lambda_{\alpha\mu\nu} = \underset{3}{R}{}^i_{p\mu\nu} B^\lambda_i B^p_\alpha + \underset{1}{\Omega}{}^P_{\alpha\mu} \underset{2}{\widehat{\Omega}}{}^\lambda_{P\nu} - \underset{2}{\Omega}{}^P_{\alpha\nu} \underset{1}{\widehat{\Omega}}{}^\lambda_{P\mu}, \tag{2.15}$$

which is **Gauss equation of the** 3^{rd} **kind** for $L_M \subset L_N$.
b) Composing (2.14) with C^L_i, one gets

$$\underset{3}{R}{}^i_{p\mu\nu} C^L_i B^p_\alpha = \underset{1}{\Omega}{}^L_{\alpha\mu\underset{2}{|}\nu} - \underset{2}{\Omega}{}^L_{\alpha\mu\underset{1}{|}\nu}, \tag{2.16}$$

and this is 1^{st} **Codazzi equation of the** 3^{rd} **kind** for $L_M \subset L_N$.

2.2′ For B^α_i, using the Ricci type identity [1, 2], it is

$$B^\alpha_{i\underset{1}{|}\mu\underset{2}{|}\nu} - B^\alpha_{i\underset{2}{|}\nu\underset{1}{|}\mu} = -\underset{3}{R}{}^p_{i\mu\nu} B^\alpha_p + \underset{3}{\widetilde{R}}{}^\alpha_{\pi\mu\nu} B^\pi_i, \tag{2.11′}$$

Putting into (2.3′) $\theta = 1, \omega = 2$, by comparing the obtained equation and (2.11′), we obtain the 3^{rd} integrability condition of derivational equation (2.1′).

$$\boxed{-\underset{3}{R}{}^p_{i\mu\nu} B^\alpha_p + \underset{3}{\widetilde{R}}{}^\alpha_{\pi\mu\nu} B^\pi_i = (\underset{1}{\Omega}{}^P_{\pi\mu} \underset{2}{\widehat{\Omega}}{}^\alpha_{P\nu} - \underset{2}{\Omega}{}^P_{\pi\nu} \underset{1}{\widehat{\Omega}}{}^\alpha_{P\mu}) B^\pi_i + (\underset{1}{\widehat{\Omega}}{}^\alpha_{P\mu\underset{2}{|}\nu} - \underset{2}{\widehat{\Omega}}{}^\alpha_{P\nu\underset{1}{|}\mu}) C^P_i.} \tag{2.14′}$$

Therefrom, analogously to previous case, we have

$$\underset{3}{\widetilde{R}}{}^\alpha_{\lambda\mu\nu} = \underset{3}{R}{}^p_{i\mu\nu} B^\alpha_p B^i_\lambda + \underset{1}{\Omega}{}^P_{\lambda\mu} \underset{2}{\widehat{\Omega}}{}^\alpha_{P\nu} - \underset{2}{\Omega}{}^P_{\lambda\nu} \underset{1}{\widehat{\Omega}}{}^\alpha_{P\mu}.$$

Doing an exchange of indices $\alpha \leftrightarrow \lambda, p \leftrightarrow i$, we see that this equation reduce to (2.15).

By composing (2.14′) with C^i_α, one obtains

$$\underset{3}{R}^{p}_{i\mu\nu} C^i_L B^\alpha_p = -\underset{1}{\widehat{\Omega}}{}^{\alpha}_{L\mu|\underset{2}{\nu}} - \underset{2}{\widehat{\Omega}}{}^{\alpha}_{L\mu|\underset{1}{\nu}}, \tag{2.16′}$$

and this is **another form of the** 1^{st} **Codazzi equation of the** 3^{rd} **kind** for $L_M \subset L_N$.

Based on exposed, the following theorems are valid.

Theorem 2.1. *The* 1^{st} *and* 2^{nd} *kind integrability conditions of derivational equations* (2.1) *and* (2.1′) *for submanifold* $X_M \subset L_N$ *with the structure* $(X_M \subset L_N, \underset{\theta}{\nabla}, \theta \in \{1,2\})$, *where the connections* $\underset{\theta}{\nabla}$ *are defined in* (1.5), *are given in* (2.8) *and* (2.8′) *respectively. The* 3^{rd} *kind integrability conditions for the mentioned equations are* (2.14) *and* (2.14′).

Theorem 2.2. *Gauss equations of the* 1^{st} *and the* 2^{nd} *kind are given in* (2.9), *and of the* 3^{rd} *one in* (2.15). *The* 1^{st} *Codazzi equations of the* 1^{st} *and the* 2^{nd} *kind are given in* (2.10), *and of the* 3^{rd} *kind in* (2.16). *The equations* (2.10′), (2.16′) *are another forms of* (2.10) *and* (2.16) *respectively.*

3. Integrability conditions of derivational equations for pseudonormals

3.0 In order to obtain integrability conditions for derivational equations of pseudonormals, we treat analogously to the case of tangents. From (2.2,1) one obtains

$$C^i_{A|\underset{\theta}{\mu}|\underset{\omega}{\nu}} - C^i_{A|\underset{\omega}{\nu}|\underset{\theta}{\mu}} = -(\underset{\theta}{\widehat{\Omega}}{}^{\pi}_{A\mu|\underset{\omega}{\nu}} - \underset{\omega}{\widehat{\Omega}}{}^{\pi}_{A\nu|\underset{\theta}{\mu}})B^i_\pi - (\underset{\theta}{\widehat{\Omega}}{}^{\pi}_{A\mu}\underset{\omega}{\Omega}{}^{P}_{\pi\nu} - \underset{\omega}{\widehat{\Omega}}{}^{\pi}_{A\nu}\underset{\theta}{\Omega}{}^{P}_{\pi\mu})C^i_P \tag{3.1}$$

and from (2.2′, 1′):

$$C^A_{i|\underset{\theta}{\mu}|\underset{\omega}{\nu}} - C^A_{i|\underset{\omega}{\nu}|\underset{\theta}{\mu}} = -(\underset{\theta}{\Omega}{}^{A}_{\pi\mu|\underset{\omega}{\nu}} - \underset{\omega}{\Omega}{}^{A}_{\pi\nu|\underset{\theta}{\mu}})B^\pi_i - (\underset{\theta}{\Omega}{}^{A}_{\pi\mu}\underset{\omega}{\widehat{\Omega}}{}^{\pi}_{P\nu} - \underset{\omega}{\Omega}{}^{A}_{\pi\nu}\underset{\theta}{\widehat{\Omega}}{}^{\pi}_{P\mu})C^P_i. \tag{3.1′}$$

By virtue of the Ricci type identity from [2] is

$$C^i_{A|\underset{\theta}{\mu}|\underset{\theta}{\nu}} - C^i_{A|\underset{\theta}{\nu}|\underset{\theta}{\mu}} = \underset{\theta}{R}{}^{i}_{pmn} C^p_A B^m_\mu B^n_\nu - \underset{\theta}{\overline{R}}{}^{P}_{A\mu\nu} C^i_P + (-1)^\theta \widetilde{T}^\pi_{\mu\nu} C^i_{A|\underset{\theta}{\pi}}, \quad \theta \in \{1,2\}, \tag{3.2}$$

and also one can prove based on (1.5) that

$$C^A_{i|\underset{\theta}{\mu}|\underset{\theta}{\nu}} - C^A_{i|\underset{\theta}{\nu}|\underset{\theta}{\mu}} = -\underset{\theta}{R}{}^{p}_{imn} C^A_p B^m_\mu B^n_\nu + \underset{\theta}{\overline{R}}{}^{A}_{P\mu\nu} C^P_i + (-1)^\theta \widetilde{T}^\pi_{\mu\nu} C^A_{i|\underset{\theta}{\pi}}, \quad \theta \in \{1,2\}, \tag{3.2′}$$

where [1]

$$\underset{\theta}{\overline{R}}{}^{A}_{B\mu\nu} = \underset{\theta}{\overline{L}}{}^{A}_{B\mu,\nu} - \underset{\theta}{\overline{L}}{}^{A}_{B\nu,\mu} + \underset{\theta}{\overline{L}}{}^{P}_{B\mu}\underset{\theta}{\overline{L}}{}^{A}_{P\nu} - \underset{\theta}{\overline{L}}{}^{P}_{B\mu}\underset{\theta}{\overline{L}}{}^{A}_{P\nu}, \quad \theta \in \{1,2\} \tag{3.3}$$

are **the** 1^{st} **and the** 2^{nd} **kind curvature tensors** of L_N with respect to the pseudonormal submanifold X^N_{N-M}.

3.1 If one substitutes $\theta = \omega \in \{1,2\}$ into (3.1) and equalizes the right sides of obtained equation and (3.2), exchanging previously $C^i_{A|\underset{\theta}{\pi}}$ with help of (2.2), one obtains

$$\boxed{\begin{aligned} \underset{\theta}{R}{}^{i}_{pmn} C^p_A B^m_\mu B^n_\nu &= [(-1)^\theta \widetilde{T}^\sigma_{\mu\nu}\underset{\theta}{\widehat{\Omega}}{}^{\pi}_{A\sigma} - \underset{\theta}{\widehat{\Omega}}{}^{\pi}_{A\mu|\underset{\theta}{\nu}} + \underset{\theta}{\widehat{\Omega}}{}^{\pi}_{A\nu|\underset{\theta}{\mu}}]B^i_\pi \\ &\quad + (\underset{\theta}{\overline{R}}{}^{P}_{A\mu\nu} - \underset{\theta}{\widehat{\Omega}}{}^{\pi}_{A\mu}\underset{\theta}{\Omega}{}^{P}_{\pi\nu} + \underset{\theta}{\widehat{\Omega}}{}^{\pi}_{A\nu}\underset{\theta}{\Omega}{}^{P}_{\pi\mu})C^i_P, \quad \theta \in \{1,2\}, \end{aligned}} \tag{3.4}$$

and this are **the** 1^{st} **and** 2^{nd} **kind integrability conditions for pseudonormals of derivational equation (2.2).**

a) If one composes (3.4) with B^λ_i, it is obtained that by virtue of (1.2):

$$\underset{\theta}{R}{}^i{}_{pmn} B^\lambda_i C^p_A B^m_\mu B^n_\nu = (-1)^\theta \widetilde{T}^\sigma_{\mu\nu}\underset{\theta}{\widehat{\Omega}}{}^\lambda_{A\sigma} - \underset{\theta}{\widehat{\Omega}}{}^\lambda_{A\mu\underset{\theta}{|}\nu} + \underset{\theta}{\widehat{\Omega}}{}^\lambda_{A\nu\underset{\theta}{|}\mu}.$$

Exchanging here $i \leftrightarrow p$, $A \leftrightarrow L$, $\lambda \leftrightarrow \alpha$, $\sigma \leftrightarrow \pi$, we obtain (2.10′).
b) Composing (3.4) with C^L_i and using (1.2) it follows that

$$\underset{\theta}{R}{}^i{}_{pmn} C^L_i C^p_A B^m_\mu B^n_\nu = \underset{\theta}{\overline{R}}{}^L_{A\mu\nu} - \underset{\theta}{\widehat{\Omega}}{}^\pi_{A\mu}\underset{\theta}{\Omega}{}^L_{\pi\nu} + \underset{\theta}{\widehat{\Omega}}{}^\pi_{A\nu}\underset{\theta}{\Omega}{}^L_{\pi\mu}, \quad \theta \in \{1,2\}. \tag{3.5}$$

The equation (3.5) is **the** 2^{nd} **Codazzi equation of the** 1^{st} **and the** 2^{nd} **kind** for $X_M \subset L_N$.

3.1′ Taking $\theta = \omega \in \{1,2\}$ in (3.1′), substituting in (3.2′) the corresponding value of $C^A_{i\underset{\theta}{|}\pi}$ and equalizing the right sides of mentioned equations we have

$$\boxed{\begin{aligned}\underset{\theta}{R}{}^p{}_{imn} C^A_p B^m_\mu B^n_\nu = {} & [-(-1)^\theta \widetilde{T}^\sigma_{\mu\nu}\underset{\theta}{\Omega}{}^A_{\pi\sigma} + \underset{\theta}{\Omega}{}^A_{\pi\mu\underset{\theta}{|}\nu} - \underset{\theta}{\Omega}{}^A_{\pi\nu\underset{\theta}{|}\mu}]B^\pi_i \\ & + (\underset{\theta}{\overline{R}}{}^A_{P\mu\nu} + \underset{\theta}{\Omega}{}^A_{\pi\mu}\underset{\theta}{\widehat{\Omega}}{}^\pi_{P\nu} - \underset{\theta}{\Omega}{}^A_{\pi\nu}\underset{\theta}{\widehat{\Omega}}{}^\pi_{P\mu})C^P_i, \quad \theta \in \{1,2\},\end{aligned}} \tag{3.4′}$$

which is **the** 1^{st} **and the** 2^{nd} **kind integrability conditions** ($\theta = 1,2$) **of derivational equation** (2.2′)**.**
a) Composing the previous equation with B^i_λ we get

$$\underset{\theta}{R}{}^p{}_{imn} C^A_p B^i_\lambda B^m_\mu B^n_\nu = -(-1)^\theta \widetilde{T}^\sigma_{\mu\nu}\underset{\theta}{\Omega}{}^A_{\lambda\sigma} + \underset{\theta}{\Omega}{}^A_{\lambda\mu\underset{\theta}{|}\nu} - \underset{\theta}{\Omega}{}^A_{\lambda\nu\underset{\theta}{|}\mu}$$

and that is, exchanging some indices, the equation (2.10).
b) If one composes (3.4′) with C^i_L, it follows that

$$\underset{\theta}{R}{}^p{}_{imn} C^A_p C^i_L B^m_\mu B^n_\nu = \underset{\theta}{\overline{R}}{}^A_{L\mu\nu} + \underset{\theta}{\Omega}{}^A_{\pi\mu}\underset{\theta}{\widehat{\Omega}}{}^\pi_{L\nu} - \underset{\theta}{\Omega}{}^A_{\pi\nu}\underset{\theta}{\widehat{\Omega}}{}^\pi_{L\mu},$$

and that is (3.5).

3.2 With respect of (3.9) in [11] is

$$C^i_{A\underset{1}{|}\mu\underset{2}{|}\nu} - C^i_{A\underset{2}{|}\nu\underset{1}{|}\mu} = \underset{3}{R}{}^i{}_{p\mu\nu} C^p_A - \underset{3}{\overline{R}}{}^P_{A\mu\nu} C^i_P, \tag{3.6}$$

where $\underset{3}{R}$ is given in (2.12) and ([11], the equation (3.10)):

$$\underset{3}{\overline{R}}{}^A_{B\mu\nu} = \underset{1}{\overline{L}}{}^A_{B\mu,\nu} - \underset{2}{\overline{L}}{}^A_{B\nu,\mu} + \underset{1}{\overline{L}}{}^P_{B\mu}\underset{2}{\overline{L}}{}^A_{P\nu} - \underset{2}{\overline{L}}{}^P_{B\nu}\underset{1}{\overline{L}}{}^A_{P\mu} + \overline{L}^\pi_{\nu\mu}(\underset{2}{\overline{L}}{}^A_{B\pi} - \underset{1}{\overline{L}}{}^A_{B\pi}) \tag{3.7}$$

is **the third kind curvature tensor of the space** L_N **with respect of** X^N_{N-M}**.**

On the other hand, putting at (3.1) $\theta = 1, \omega = 2$, and equalizing the right sides of the obtained equation and (3.6), we get

$$\boxed{\underset{3}{R}{}^i{}_{p\mu\nu} C^p_A - \underset{3}{\overline{R}}{}^P_{A\mu\nu} C^i_P = -(\underset{1}{\widehat{\Omega}}{}^\pi_{A\mu\underset{2}{|}\nu} - \underset{2}{\widehat{\Omega}}{}^\pi_{A\nu\underset{1}{|}\mu})B^i_\pi - (\underset{1}{\widehat{\Omega}}{}^\pi_{A\mu}\underset{2}{\Omega}{}^P_{\pi\nu} - \underset{2}{\widehat{\Omega}}{}^\pi_{A\nu}\underset{1}{\Omega}{}^P_{\pi\mu})C^i_P,} \tag{3.8}$$

and that is the 3^{rd} **kind integrability condition for pseudonormals of derivational equation** (2.2)**.**
a) By composing (3.8) with B^λ_i, we get

$$\underset{3}{R}{}^i{}_{p\mu\nu} C^p_A B^\lambda_i = -(\underset{1}{\widehat{\Omega}}{}^\pi_{A\mu\underset{2}{|}\nu} - \underset{2}{\widehat{\Omega}}{}^\pi_{A\nu\underset{1}{|}\mu}), \tag{3.9}$$

and by substitution $p \leftrightarrow i$, $\lambda \leftrightarrow \alpha$, one obtains (2.16′).
b) Composing (3.8) with C^L_i, we have

$$\boxed{\underset{3}{\overline{R}}{}^L_{A\mu\nu} = \underset{3}{R}{}^i_{p\mu\nu} C^L_i C^p_A + \underset{1}{\widehat{\Omega}}{}^\pi_{A\mu} \underset{2}{\Omega}{}^L_{\pi\nu} - \underset{2}{\widehat{\Omega}}{}^\pi_{A\nu} \underset{1}{\Omega}{}^L_{\pi\mu}} \tag{3.10}$$

what, comparing with (3.5) we call **the** 2^{nd} **Codazzi equation of the** 3^{rd} **kind.**

3.2′ Analogically to (3.6), can be proved that the Ricci-type identity

$$C^A_{i|\underset{1}{\mu}|\underset{2}{\nu}} - C^A_{i|\underset{2}{\nu}|\underset{1}{\mu}} = \underset{3}{\overline{R}}{}^A_{P\mu\nu} C^P_i - \underset{3}{R}{}^p_{i\mu\nu} C^A_p, \tag{3.6′}$$

is valid, where $\underset{3}{\overline{R}}, \underset{3}{R}$ are given in (3.7) and (2.12) respectively.

If one takes $\theta = 1, \omega = 2$ in (3.1′) and compares the obtained equations with (3.6′), one obtains

$$\boxed{\underset{3}{R}{}^p_{i\mu\nu} C^A_p - \underset{3}{\overline{R}}{}^A_{P\mu\nu} C^P_i = (\underset{1}{\Omega}{}^A_{\pi\mu|\underset{2}{\nu}} - \underset{2}{\Omega}{}^A_{\pi\nu|\underset{2}{\mu}}) B^\pi_i + (\underset{1}{\Omega}{}^A_{\pi\mu} \underset{2}{\widehat{\Omega}}{}^\pi_{P\nu} - \underset{2}{\Omega}{}^A_{\pi\nu} \underset{1}{\widehat{\Omega}}{}^\pi_{P\mu}) C^P_i} \tag{3.8′}$$

which is 3^{rd} **kind integrability condition of derivational equation** (2.2′).

As in previous cases, from (3.8′) one gets

$$\underset{3}{R}{}^p_{i\mu\nu} C^A_p B^i_\lambda = \underset{1}{\Omega}{}^A_{\lambda\mu|\underset{2}{\nu}} - \underset{2}{\Omega}{}^A_{\lambda\nu|\underset{1}{\mu}},$$

Further, from (3.8′), we obtain (3.10).

Based on exposed in this section, we have following theorems:

Theorem 3.1. *The* 1^{st} *and the* 2^{nd} *kind integrability conditions of derivational equations* (2.2)*, and* (2.2′) *for pseudonormals of a submanifold* $X_M \subset L_N$ *with structure* $(X_M \subset L_N, \underset{\theta}{\nabla}, \theta \in \{1,2\})$*, where the connections* $\underset{\theta}{\nabla}$ *are defined in* (1.5)*, are given in* (3.4) *and* (3.4′) *respectively. The* 3^{rd} *kind integrability conditions for these equations are* (3.8)*,* (3.8′)*.*

Theorem 3.2. *The* 2^{nd} *Codazzi equation of the* 1^{st} *and the* 2^{nd} *kind is given by* (3.5)*, and of the* 3^{rd} *one by* (3.10)*.*

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