Filomat 29:7 (2015), 1515–1524
DOI 10.2298/FIL1507515Z

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Generalization of Hermite-Hadamard Inequalities Involving Hadamard Fractional Integrals

Zhi Zhang[a], Wei Wei[a], JinRong Wang[a,b]

[a]*Department of Mathematics, Guizhou University, Guiyang, Guizhou 550025, P.R. China*
[b]*School of Mathematics and Computer Science, Guizhou Normal College, Guiyang, Guizhou 550018, P.R. China*

Abstract. In this paper, we firstly give a general integral identity for once differentiable mapping involving Hadamard fractional integrals. Secondly, we use this integral identity to derive some new generalization of fractional Hermite-Hadamard inequalities through *GA*-convex functions via power means and integrals *GG*-convex functions via power means. Some applications to special means of real numbers are given.

1. Introduction

The concept of fractional calculus appeared in 1695 in the letter between L'Hospital and Leibniz. Since then, further development in this area has been explored by many mathematicians and we recommend to read the study of Riemann, Liouville, Caputo, and other famous mathematicians. Fractional calculus played an important role in various fields such as electricity, biology, economics and signal and image processing [1–4].

The classical Hermite-Hadamard inequality gives a lower and an upper estimations for both right-hand and left-hand integrals average of any convex function defined on a compact interval, involving the midpoint and the endpoints of the domain. For more information, see [5, 6] and the references therein. It seems that Set [7] firstly investigate fractional Ostrowski inequalities involving Riemann-Liouville fractional integrals. Then, Sarikaya et al. [8] studied Hermite-Hadamard type inequalities involving Riemann-Liouville fractional integrals. Inspired by the above interesting works, our group go on studying fractional version Hermite-Hadamard inequality involving Riemann-Liouville and Hadamard fractional integrals for all kinds of convex functions [9–12] and Işcan [13]. For other recent applications of fractional derivatives and fractional integrals, one can see [14–19].

Recently, Işcan [20] presented a general and interesting integral identity for twice differentiable functions involving Riemann-Liouville fractional integrals as follows:

2010 *Mathematics Subject Classification*. 26A33, 26A51, 26D15.
Keywords. Generalization of Hermite-Hadamard inequalities, Hadamard fractional integrals, *GA*-convex function, *GG*-convex function.
Received: 25 December 2013; Accepted: 06 April 2014
Communicated by Hari M. Srivastava
This work is partially supported by Guizhou Province Education Planning Project (2013A062), Key Support Subject (Applied Mathematics), Key Project on the Reforms of Teaching Contents and Course System of Guizhou Normal College and Doctor Project of Guizhou Normal College (13BS010).
Email addresses: zzstc520@gmail.com (Zhi Zhang), wwei@gzu.edu.cn (Wei Wei), wangjinrong@gzn c.edu.cn (JinRong Wang)

Lemma 1.1. *(see Lemma 2.1, [20]) Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° (the interior of I) such that $f'' \in L[a,b]$, where $a, b \in I$ with $a < b$. Then for all $x \in [a,b]$, $\lambda \in [0,1]$ and $\alpha > 0$ we have:*

$$\begin{aligned} R_f(x,\lambda,\alpha,a,b) &= \frac{(x-a)^{\alpha+2}}{(\alpha+1)(b-a)}\int_0^1 t\Big((\alpha+1)\lambda - t^\alpha\Big)f''(tx+(1-t)a)dt \\ &+\frac{(b-x)^{\alpha+2}}{(\alpha+1)(b-a)}\int_0^1 t\Big((\alpha+1) - t^\alpha\Big)f''(tx+(1-t)b)dt, \end{aligned}$$

where

$$\begin{aligned} & R_f(x,\lambda,\alpha,a,b) \\ = \ & (1-\lambda)\Big[\frac{(x-a)^\alpha + (b-x)^\alpha}{b-a}\Big]f(x) + \lambda\Big[\frac{(x-a)^\alpha f(a) + (b-x)^\alpha f(b)}{b-a}\Big] \\ & +\Big(\frac{1}{\alpha+1} - \lambda\Big)\Big[\frac{(b-x)^{\alpha+1} - (x-a)^{\alpha+1}}{b-a}\Big]f'(x) - \frac{\Gamma(\alpha+1)}{b-a}[J^\alpha_{x^-}f(a) + J^\alpha_{x^+}f(b)\Big], \end{aligned}$$

and Γ denotes Euler Gamma function.

By using of Lemma 1.1, the author established some new estimates on Hermite-Hadamard type and Simpson type inequalities for s-convex involving Riemann-Liouville fractional integrals.

However, to our knowledge, the interesting integral identity for once differentiable functions involving Hadamard fractional integrals have not been reported. To fix this gap, in the present paper, we firstly derive a general integral identity (see Lemma 2.1) for once differentiable mapping involving Hadamard fractional integrals. Then, we use this integral identity to derive some new generalization of fractional Hermite-Hadamard inequalities (see Theorems 3.4, 3.5, 3.6, and 3.7) through GA-convex functions via power means and integrals GG-convex functions via power means. Finally, some applications to special means of real numbers are given.

2. A General Integral Identity Involving Hadamard Fractional Integrals

In this section, we derive a general integral identity for once differentiable functions involving Hadamard fractional integrals.

For $f \in L[a,b]$, the Hadamard fractional integrals [1] ${}_H J^\alpha_{a+}f$ and ${}_H J^\alpha_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$${}^H J^\alpha_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (\ln\frac{x}{t})^{\alpha-1}f(t)\frac{dt}{t},\ x > a,$$

and

$${}^H J^\alpha_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (\ln\frac{t}{x})^{\alpha-1}f(t)\frac{dt}{t},\ x < b,$$

respectively, where $\Gamma(\cdot)$ is the Gamma function.

Let $f : I \subseteq \mathbb{R}^+$ be a differentiable function on I° where I° denotes the interior of I. Denote

$$\begin{aligned} I_f(x,\lambda,\alpha,a,b) &= \lambda\Big[\frac{(\ln b - \ln x)^{\alpha-1}f(b) + (\ln x - \ln a)^{\alpha-1}f(a)}{\ln b - \ln a}\Big] \\ &-(1+\lambda)\Big[\frac{(\ln b - \ln x)^{\alpha-1} + (\ln x - \ln a)^{\alpha-1}}{\ln b - \ln a}\Big]f(x) \\ &+\frac{\Gamma(\alpha+1)}{\ln b - \ln a}\Big[{}^H J^\alpha_{x^+}f(b) +{}^H J^\alpha_{x^-}f(a)\Big], \end{aligned}$$

where $a, b \in I$ with $a < b$, $x \in [a,b]$ and $\lambda \in [0,1]$.

In order to prove our main results, we present the following general identity.

Lemma 2.1. *Let $a > 0$ and $f : [a,b] \to \mathbb{R}$ be a once differentiable mapping such that f' is integrable. For $0 \le \lambda \le 1$, the following equality for fractional integrals holds:*

$$I_f(x,\lambda,\alpha,a,b) = \frac{(\ln b - \ln x)^\alpha}{\ln b - \ln a}\int_0^1 (s^\alpha + \lambda) f'(e^{(1-s)\ln b + s\ln x}) e^{(1-s)\ln b + s\ln x} ds - \frac{(\ln x - \ln a)^\alpha}{\ln b - \ln a}\int_0^1 (s^\alpha + \lambda) f'(e^{(1-s)\ln a + s\ln x}) e^{(1-s)\ln a + s\ln x} ds.$$

Proof. Denote

$$\begin{aligned} U_f(x,\lambda,\alpha,a,b) &= \frac{(\ln b - \ln x)^\alpha}{\ln b - \ln a}\int_0^1 (s^\alpha + \lambda) f'(e^{(1-s)\ln b + s\ln x}) e^{(1-s)\ln b + s\ln x} ds, \\ V_f(x,\lambda,\alpha,a,b) &= -\frac{(\ln x - \ln a)^\alpha}{\ln b - \ln a}\int_0^1 (s^\alpha + \lambda) f'(e^{(1-s)\ln a + s\ln x}) e^{(1-s)\ln a + s\ln x} ds. \end{aligned}$$

By integration by parts twice and changing the variable, we have

$$\begin{aligned} & U_f(x,\lambda,\alpha,a,b) \\ = \; & \frac{(\ln b - \ln x)^{\alpha-1}}{\ln b - \ln a}\lambda f(b) - \frac{(\ln b - \ln x)^{\alpha-1}}{\ln b - \ln a}(1+\lambda) f(x) \\ & + \frac{(\ln b - \ln x)^{\alpha-1}}{\ln b - \ln a}\int_0^1 \alpha s^{\alpha-1} f(e^{(1-s)\ln b + s\ln x}) ds \\ = \; & \lambda \frac{(\ln b - \ln x)^{\alpha-1}}{\ln b - \ln a}\Big[f(b) - f(x)\Big] - \frac{(\ln b - \ln x)^{\alpha-1}}{\ln b - \ln a} f(x) \\ & + \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)(\ln b - \ln a)}\int_x^b \left(\ln \frac{b}{t}\right)^{\alpha-1} \frac{f(t)}{t} dt \\ = \; & \lambda \frac{(\ln b - \ln x)^{\alpha-1}}{\ln b - \ln a}\Big[f(b) - f(x)\Big] - \frac{(\ln b - \ln x)^{\alpha-1}}{\ln b - \ln a} f(x) + \frac{\Gamma(\alpha+1)}{\ln b - \ln a}\, {}^H J^\alpha_{x^+}(b). \end{aligned}$$

Similarly, we get

$$\begin{aligned} & V_f(x,\lambda,\alpha,a,b) \\ = \; & -\lambda \frac{(\ln x - \ln a)^{\alpha-1}}{\ln b - \ln a}\Big[f(x) - f(a)\Big] - \frac{(\ln x - \ln a)^{\alpha-1}}{\ln b - \ln a} f(x) + \frac{\Gamma(\alpha+1)}{\ln b - \ln a}\, {}^H J^\alpha_{x^-} f(a). \end{aligned}$$

From above two equalities, we have

$$I_f(x,\lambda,\alpha,a,b) = U_f(x,\lambda,\alpha,a,b) + V_f(x,\lambda,\alpha,a,b).$$

The proof is finished. □

3. Main Results

The following definitions and result will be used to derive the main results in this section.

Definition 3.1. *(see [21]) A function $f : [0,+\infty) \to \mathbb{R}$ is said to be GA-convex if $x, y \in [0,+\infty)$, $0 \le \alpha \le 1$,*

$$f(x^{1-\alpha} y^\alpha) \le (1-\alpha) f(x) + \alpha f(y).$$

Definition 3.2. *(see [22]) A function $f : [0,+\infty) \to \mathbb{R}$ is said to be GG-convex if $x, y \in [0,+\infty)$, $0 \le \alpha \le 1$,*

$$f(x^{1-\alpha} y^\alpha) \le f(x)^{1-\alpha} f(y)^\alpha.$$

Lemma 3.3. *(see Lemma 2.1, [11]) For $\alpha > 0$ and $k > 0$, we have*

$$I(\alpha) = \int_0^1 t^{\alpha-1}k^t dt = k\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln k)^{i-1}}{(\alpha)_i} < +\infty,$$

where $(\alpha)_i = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+i-1)$.

Now we are ready to present our main results in this section.

Theorem 3.4. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $0 \le a < b$. If $|f'|$ is integrable and GA-convex on $[a,b]$, then for $0 \le \lambda \le 1$, $x \in (a,b)$ and $\alpha > 0$, the following inequality holds:*

$$\begin{aligned}
&|I_f(x,\lambda,\alpha,a,b)| \\
\le\ & \frac{(\ln b-\ln x)^\alpha}{\ln b-\ln a}\Bigg|x|f'(b)|\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln\frac{x}{b})^{i-1}}{(\alpha+1)_i} - x[|f'(b)|-|f'(x)|]\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln\frac{x}{b})^{i-1}}{(\alpha+2)_i} \\
&+\frac{\lambda}{\ln x-\ln b}\Big[x|f'(x)|-2\lambda|f'(b)|+b|f'(x)|\Big]\Bigg| \\
&+\frac{(\ln x-\ln a)^\alpha}{\ln b-\ln a}\Bigg|x|f'(a)|\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln\frac{x}{a})^{i-1}}{(\alpha+1)_i} - x[|f'(a)|-|f'(x)|]\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln\frac{x}{a})^{i-1}}{(\alpha+2)_i} \\
&+\frac{\lambda}{\ln x-\ln a}\Big[x|f'(x)|-2\lambda|f'(a)|+a|f'(x)|\Big]\Bigg|.
\end{aligned}$$

Proof. By using Definition 3.1 and Lemma 3.3, we have

$$\begin{aligned}
&|I_f(x,\lambda,\alpha,a,b)| \\
\le\ & |U_f(x,\lambda,\alpha,a,b)| + |V_f(x,\lambda,\alpha,a,b)| \\
\le\ & \frac{(\ln b-\ln x)^\alpha}{\ln b-\ln a}\Bigg|b|f'(b)|\int_0^1 s^\alpha(\frac{x}{b})^s ds - b|f'(b)|\int_0^1 s^{\alpha+1}(\frac{x}{b})^s ds \\
&+b|f'(x)|\int_0^1 s^{\alpha+1}(\frac{x}{b})^s ds + \lambda b|f'(b)|\int_0^1 (\frac{x}{b})^s ds - \lambda b|f'(b)|\int_0^1 s(\frac{x}{b})^s ds \\
&+\lambda b|f'(x)|\int_0^1 s(\frac{x}{b})^s ds\Bigg| \\
&+\frac{(\ln x-\ln a)^\alpha}{\ln b-\ln a}\Bigg|a|f'(a)|\int_0^1 s^\alpha(\frac{x}{a})^s ds - a|f'(a)|\int_0^1 s^{\alpha+1}(\frac{x}{a})^s ds \\
&+a|f'(x)|\int_0^1 s^{\alpha+1}(\frac{x}{a})^s ds + \lambda a|f'(a)|\int_0^1 (\frac{x}{a})^s ds - \lambda a|f'(a)|\int_0^1 s(\frac{x}{a})^s ds \\
&+\lambda a|f'(x)|\int_0^1 s(\frac{x}{a})^s ds\Bigg| \\
\le\ & \frac{(\ln b-\ln x)^\alpha}{\ln b-\ln a}\Bigg|x|f'(b)|\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln\frac{x}{b})^{i-1}}{(\alpha+1)_i} - x[|f'(b)|-|f'(x)|]\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln\frac{x}{b})^{i-1}}{(\alpha+2)_i} \\
&+\lambda\frac{1}{\ln\frac{x}{b}}\Big[x|f'(x)|-2\lambda|f'(b)|+b|f'(x)|\Big]\Bigg| \\
&+\frac{(\ln x-\ln a)^\alpha}{\ln b-\ln a}\Bigg|x|f'(a)|\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln\frac{x}{a})^{i-1}}{(\alpha+1)_i} - x[|f'(a)|-|f'(x)|]\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln\frac{x}{a})^{i-1}}{(\alpha+2)_i} \\
&+\lambda\frac{1}{\ln\frac{x}{a}}\Big[x|f'(x)|-2\lambda|f'(a)|+a|f'(x)|\Big]\Bigg|.
\end{aligned}$$

The proof is done. □

Theorem 3.5. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $0 \le a < b$. If $|f'|^q, q > 1$ is integrable and GA-convex on $[a,b]$, then for $0 \le \lambda \le 1, x \in (a,b)$ the following inequality holds:*

$$\begin{aligned}
|I_f(x,\lambda,\alpha,a,b)| \le\ & \frac{(\ln b - \ln x)^\alpha}{\ln b - \ln a}\Bigg|\Big(x^q \sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln(\frac{x}{b})^q)^{i-1}}{(q\alpha+1)_i}\Big)^{\frac{1}{q}}\Big(\frac{|f'(b)|+|f'(x)|^p}{2}\Big)^{\frac{1}{p}} \\
&+\Big((\lambda b)^q \frac{1}{q\ln\frac{x}{b}}(\frac{x}{b})^q - (\lambda b)^q\frac{1}{q\ln\frac{x}{b}}\Big)^{\frac{1}{q}}\Big(\frac{|f'(b)|+|f'(x)|^p}{2}\Big)^{\frac{1}{p}}\Bigg| \\
&+\frac{(\ln x - \ln a)^\alpha}{\ln b - \ln a}\Bigg|\Big(x^q \sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln(\frac{x}{a})^q)^{i-1}}{(q\alpha+1)_i}\Big)^{\frac{1}{q}}\Big(\frac{|f'(a)|+|f'(x)|^p}{2}\Big)^{\frac{1}{p}} \\
&+\Big((\lambda a)^q \frac{1}{q\ln\frac{x}{a}}(\frac{x}{a})^q - (\lambda a)^q\frac{1}{q\ln\frac{x}{a}}\Big)^{\frac{1}{q}}\Big(\frac{|f'(a)|+|f'(x)|^p}{2}\Big)^{\frac{1}{p}}\Bigg|.
\end{aligned}$$

Proof. By using Definition 3.1 and Lemma 3.3 and Hölder inequality, we have

$$\begin{aligned}
& |I_f(x,\lambda,\alpha,a,b)| \\
\le\ & |U_f(x,\lambda,\alpha,a,b)| + |V_f(x,\lambda,\alpha,a,b)| \\
\le\ & \frac{(\ln b - \ln x)^\alpha}{\ln b - \ln a}\Bigg|\Big(\int_0^1 b^q s^{q\alpha}(\frac{x}{b})^{qs}ds\Big)^{\frac{1}{q}}\Big(\int_0^1 (1-s)|f'(b)|^p ds \\
& + \int_0^1 s|f'(x)|^p ds\Big)^{\frac{1}{p}} + \Big(\int_0^1 (\lambda b)^q(\frac{x}{b})^{qs}ds\Big)^{\frac{1}{q}} \\
& \Big(\int_0^1 (1-s)|f'(b)|^p ds + \int_0^1 s|f'(x)|^p ds\Big)^{\frac{1}{p}}\Bigg| \\
& +\frac{(\ln x - \ln a)^\alpha}{\ln b - \ln a}\Bigg|\Big(\int_0^1 a^q s^{q\alpha}(\frac{x}{a})^{qs}ds\Big)^{\frac{1}{q}}\Big(\int_0^1 (1-s)|f'(a)|^p ds \\
& + \int_0^1 s|f'(x)|^p ds\Big)^{\frac{1}{p}} + \Big(\int_0^1 (\lambda a)^q(\frac{x}{a})^{qs}ds\Big)^{\frac{1}{q}} \\
& \Big(\int_0^1 (1-s)|f'(a)|^p ds + \int_0^1 s|f'(x)|^p ds\Big)^{\frac{1}{p}}\Bigg| \\
\le\ & \frac{(\ln b - \ln x)^\alpha}{\ln b - \ln a}\Bigg|\Big(x^q \sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln(\frac{x}{b})^q)^{i-1}}{(q\alpha+1)_i}\Big)^{\frac{1}{q}}\Big(\frac{|f'(b)|+|f'(x)|^p}{2}\Big)^{\frac{1}{p}} \\
& +\Big((\lambda b)^q \frac{1}{q\ln(\frac{x}{b})}(\frac{x}{b})^q - (\lambda b)^q\frac{1}{q\ln\frac{x}{b}}\Big)^{\frac{1}{q}}\Big(\frac{|f'(b)|+|f'(x)|^p}{2}\Big)^{\frac{1}{p}}\Bigg| \\
& +\frac{(\ln x - \ln a)^\alpha}{\ln b - \ln a}\Bigg|\Big(x^q \sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln(\frac{x}{a})^q)^{i-1}}{(q\alpha+1)_i}\Big)^{\frac{1}{q}}\Big(\frac{|f'(a)|+|f'(x)|^p}{2}\Big)^{\frac{1}{p}} \\
& +\Big((\lambda a)^q \frac{1}{q\ln(\frac{x}{a})}(\frac{x}{a})^q - (\lambda a)^q\frac{1}{q\ln\frac{x}{a}}\Big)^{\frac{1}{q}}\Big(\frac{|f'(a)|+|f'(x)|^p}{2}\Big)^{\frac{1}{p}}\Bigg|.
\end{aligned}$$

The proof is done. □

Theorem 3.6. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $0 \le a < b$. If $|f'|$ is integrable and*

GG-convex on $[a,b]$*, then for* $0 \le \lambda \le 1, x \in (a,b)$ *the following inequality holds:*

$$\begin{aligned}
&|I_f(x,\lambda,\alpha,a,b)| \\
\le\ & \frac{(\ln b-\ln x)^\alpha}{\ln b-\ln a}\left|x|f'(x)|\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln\frac{x|f'(x)|}{b|f'(b)|})^{i-1}}{(\alpha+1)_i}+\lambda\frac{1}{\ln\frac{x|f'(x)|}{b|f'(b)|}}\Big(x|f'(x)|-b|f'(b)|\Big)\right| \\
&+\frac{(\ln x-\ln a)^\alpha}{\ln b-\ln a}\left|x|f'(x)|\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln\frac{x|f'(x)|}{a|f'(a)|})^{i-1}}{(\alpha+1)_i}+\lambda\frac{1}{\ln\frac{x|f'(x)|}{a|f'(a)|}}\Big(x|f'(x)|-a|f'(a)|\Big)\right|.
\end{aligned}$$

Proof. By using Definition 3.2 and Lemma 3.3, we have

$$\begin{aligned}
&\Bigg|\frac{(\ln b-\ln x)^\alpha}{\ln b-\ln a}\int_0^1(s^\alpha+\lambda)f'(e^{(1-s)\ln b+s\ln x})e^{(1-s)\ln b+s\ln x}ds \\
&-\frac{(\ln x-\ln a)^\alpha}{\ln b-\ln a}\int_0^1(s^\alpha+\lambda)f'(e^{(1-s)\ln a+s\ln x})e^{(1-s)\ln a+s\ln x}ds\Bigg| \\
\le\ & |U_f(x,\lambda,\alpha,a,b)|+|V_f(x,\lambda,\alpha,a,b)| \\
\le\ & \frac{(\ln b-\ln x)^\alpha}{\ln b-\ln a}\left|b|f'(b)|\int_0^1 s^\alpha\Big(\frac{x|f'(x)|}{b|f'(b)|}\Big)^s ds+\lambda b|f'(b)|\int_0^1\Big(\frac{x|f'(x)|}{b|f'(b)|}\Big)^s ds\right| \\
&+\frac{(\ln x-\ln a)^\alpha}{\ln b-\ln a}\left|a|f'(a)|\int_0^1 s^\alpha\Big(\frac{x|f'(x)|}{a|f'(a)|}\Big)^s ds+\lambda a|f'(a)|\int_0^1\Big(\frac{x|f'(x)|}{a|f'(a)|}\Big)^s ds\right| \\
\le\ & \frac{(\ln b-\ln x)^\alpha}{\ln b-\ln a}\Bigg|x|f'(x)|\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln\frac{x|f'(x)|}{b|f'(b)|})^{i-1}}{(\alpha+1)_i} \\
&+\lambda\frac{1}{\ln\frac{x|f'(x)|}{b|f'(b)|}}\Big(x|f'(x)|-b|f'(b)|\Big)\Bigg| \\
&+\frac{(\ln x-\ln a)^\alpha}{\ln b-\ln a}\Bigg|x|f'(x)|\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln\frac{x|f'(x)|}{a|f'(a)|})^{i-1}}{(\alpha+1)_i} \\
&+\lambda\frac{1}{\ln\frac{x|f'(x)|}{a|f'(a)|}}\Big(x|f'(x)|-a|f'(a)|\Big)\Bigg|.
\end{aligned}$$

The proof is done. □

Theorem 3.7. *Let* $f:[a,b]\to\mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $0\le a<b$*. If* $|f'|^p, p>1$ *is integrable and GG-convex on* $[a,b]$*, then for* $0\le\lambda\le1, x\in(a,b)$ *the following inequality holds:*

$$\begin{aligned}
&|I_f(x,\lambda,\alpha,a,b)| \\
\le\ & \frac{(\ln b-\ln x)^\alpha}{\ln b-\ln a}\Bigg|\Big(\frac{1}{p}\frac{|f'(x)|^p-|f'(b)|^p}{\ln|f'(x)|-\ln|f'(b)|}\Big)^{\frac{1}{p}}\Bigg[\Big(x^q\sum_{i=1}^{\infty}(-1)^{i-1}\frac{[\ln(\frac{x}{b})^q]^{i-1}}{(q\alpha+1)_i}\Big)^{\frac{1}{q}} \\
&+\Big(\frac{\lambda^q}{q}\frac{x^q-b^q}{\ln x-\ln b}\Big)^{\frac{1}{q}}\Bigg]\Bigg| \\
&+\frac{(\ln x-\ln a)^\alpha}{\ln b-\ln a}\Bigg|\Big(\frac{1}{p}\frac{|f'(x)|^p-|f'(a)|^p}{\ln|f'(x)|-\ln|f'(a)|}\Big)^{\frac{1}{p}}\Bigg[\Big(x^q\sum_{i=1}^{\infty}(-1)^{i-1}\frac{[\ln(\frac{x}{b})^q]^{i-1}}{(q\alpha+1)_i}\Big)^{\frac{1}{q}} \\
&+\Big(\frac{\lambda^q}{q}\frac{x^q-a^q}{\ln x-\ln a}\Big)^{\frac{1}{q}}\Bigg]\Bigg|.
\end{aligned}$$

Proof. By using Definition 3.2, Lemma 3.3 and Hölder inequality, we have

$$
\begin{aligned}
&\Big|\frac{(\ln b-\ln x)^{\alpha}}{\ln b-\ln a}\int_0^1 (s^{\alpha}+\lambda)f'(e^{(1-s)\ln b+s\ln x})e^{(1-s)\ln b+s\ln x}ds\\
&-\frac{(\ln x-\ln a)^{\alpha}}{\ln b-\ln a}\int_0^1 (s^{\alpha}+\lambda)f'(e^{(1-s)\ln a+s\ln x})e^{(1-s)\ln a+s\ln x}ds\Big|\\
\le\ & |U_f(x,\lambda,\alpha,a,b)|+|V_f(x,\lambda,\alpha,a,b)|\\
\le\ & \frac{(\ln b-\ln x)^{\alpha}}{\ln b-\ln a}\Big|\Big(b^q\int_0^1 s^{q\alpha}(\frac{x}{b})^{qs}ds\Big)^{\frac{1}{q}}\Big(|f'(b)|^p\int_0^1(\frac{|f'(x)|}{|f'(b)|})^{ps}ds\Big)^{\frac{1}{p}}\\
&+\Big((\lambda b)^q\int_0^1(\frac{x}{b})^{qs}ds\Big)^{\frac{1}{q}}\Big(|f'(b)|^p\int_0^1(\frac{|f'(x)|}{|f'(b)|})^{ps}ds\Big)^{\frac{1}{p}}\Big|\\
&+\frac{(\ln x-\ln a)^{\alpha}}{\ln b-\ln a}\Big|\Big(a^q\int_0^1 s^{q\alpha}(\frac{x}{a})^{qs}ds\Big)^{\frac{1}{q}}\Big(|f'(a)|^p\int_0^1(\frac{|f'(x)|}{|f'(a)|})^{ps}ds\Big)^{\frac{1}{p}}\\
&+\Big((\lambda a)^q\int_0^1(\frac{x}{a})^{qs}ds\Big)^{\frac{1}{q}}\Big(|f'(a)|^p\int_0^1(\frac{|f'(x)|}{|f'(a)|})^{ps}ds\Big)^{\frac{1}{p}}\Big|\\
\le\ & \frac{(\ln b-\ln x)^{\alpha}}{\ln b-\ln a}\Big|\Big(\frac{1}{p}\frac{|f'(x)|^p-|f'(b)|^p}{\ln|f'(x)|-\ln|f'(b)|}\Big)^{\frac{1}{p}}\Big[\Big(x^q\sum_{i=1}^{\infty}(-1)^{i-1}\frac{[\ln(\frac{x}{b})^q]^{i-1}}{(q\alpha+1)_i}\Big)^{\frac{1}{q}}\\
&+\Big(\frac{\lambda^q}{q}\frac{x^q-b^q}{\ln x-\ln b}\Big)^{\frac{1}{q}}\Big]\Big|\\
&+\frac{(\ln x-\ln a)^{\alpha}}{\ln b-\ln a}\Big|\Big(\frac{1}{p}\frac{|f'(x)|^p-|f'(a)|^p}{\ln|f'(x)|-\ln|f'(a)|}\Big)^{\frac{1}{p}}\Big[\Big(x^q\sum_{i=1}^{\infty}(-1)^{i-1}\frac{[\ln(\frac{x}{b})^q]^{i-1}}{(q\alpha+1)_i}\Big)^{\frac{1}{q}}\\
&+\Big(\frac{\lambda^q}{q}\frac{x^q-a^q}{\ln x-\ln a}\Big)^{\frac{1}{q}}\Big]\Big|.
\end{aligned}
$$

The proof is done. □

4. Applications to Special Means

Consider the following special means (see [23]) for arbitrary real numbers $x, y, x \neq y$ as follows:

(M_1) $A(x,y) = x + y, x, y \in \mathbb{R}$.

(M_2) $H(x,y) = \frac{1}{\ln|x|-\ln|y|}, x, y \in \mathbb{R}\setminus\{0\}$.

(M_3) $K(x,y) = \frac{1}{\frac{1}{|x|}-\frac{1}{|y|}}, x, y \in \mathbb{R}\setminus\{0\}$.

(M_4) $G(x,y) = \sqrt{xy}$.

(M_5) $L(x,y) = \frac{y-x}{\ln|y|-\ln|x|}, |x| \neq |y|, xy \neq 0$.

We give some applications to special means of real numbers.

Proposition 4.1. *Let $a, b \in \mathbb{R}^+\setminus\{0\}, 0 \le a < b, x \in [0,b]$. Then*

$$(i) \quad \left|\lambda A(b,a)H(b,a) - (1+\lambda)H(b,a)G(x,x) + 2L(b,a)\right|$$
$$\leq \frac{\ln b - \ln x}{\ln b - \ln a}\left||f'(b)|\frac{x(\ln x - \ln b) - x + b}{(\ln x - \ln b)^2} + \lambda\frac{1-b}{\ln x - \ln b}\right|$$
$$+\frac{\ln x - \ln a}{\ln b - \ln a}\left||f'(a)|\frac{x(\ln x - \ln a) - x + a}{(\ln x - \ln a)^2} + \lambda\frac{1-a}{\ln x - \ln a}\right|;$$

$$(ii) \quad \left|\lambda A(b,a)H(b,a) - (1+\lambda)H(b,a)G(x,x) + 2L(b,a)\right|$$
$$\leq \frac{\ln b - \ln x}{\ln b - \ln a}\left|\left[b^q I(q+1)\right]^{\frac{1}{q}} + \left[\frac{(\lambda x)^q - (\lambda b)^q}{q(\ln x - \ln b)}\right]^{\frac{1}{q}}\right|$$
$$+\frac{\ln x - \ln a}{\ln b - \ln a}\left|\left[a^q I(q+1)\right]^{\frac{1}{q}} + \left[\frac{(\lambda x)^q - (\lambda a)^q}{q(\ln x - \ln a)}\right]^{\frac{1}{q}}\right|;$$

$$(iii) \quad \left|\lambda A(b,a)H(b,a) - (1+\lambda)H(b,a)G(x,x) + 2L(b,a)\right|$$
$$\leq \frac{\ln b - \ln x}{\ln b - \ln a}\left||f'(b)|\frac{x(\ln x - \ln b) - x + b}{(\ln x - \ln b)^2} + \lambda\frac{x-b}{\ln x - \ln b}\right|$$
$$+\frac{\ln x - \ln a}{\ln b - \ln a}\left||f'(a)|\frac{x(\ln x - \ln a) - x + a}{(\ln x - \ln a)^2} + \lambda\frac{x-a}{\ln x - \ln a}\right|;$$

$$(iv) \quad \left|\lambda A(b,a)H(b,a) - (1+\lambda)H(b,a)G(x,x) + 2L(b,a)\right|$$
$$\leq \frac{\ln b - \ln x}{\ln b - \ln a}\left|\left[b^q I(q+1)\right]^{\frac{1}{q}} + \left[\frac{(\lambda x)^q - (\lambda b)^q}{q(\ln x - \ln b)}\right]^{\frac{1}{q}}\right|$$
$$+\frac{\ln x - \ln a}{\ln b - \ln a}\left|\left[a^q I(q+1)\right]^{\frac{1}{q}} + \left[\frac{(\lambda x)^q - (\lambda a)^q}{q(\ln x - \ln a)}\right]^{\frac{1}{q}}\right|.$$

Proof. Applying Theorems 3.4, 3.5, 3.6, and 3.7, for $f(x) = x$, $\lambda \in [0,1]$ and $\alpha = 1$, one can obtain the results immediately. □ wei

Proposition 4.2. *Let* $a, b \in \mathbb{R}^+ \setminus \{0\}, 0 \leq a < b, x \in [0,b]$. *Then*

$$(i) \quad \left|\lambda A(a^{-1}, b^{-1})H(b,a) - (1+\lambda)H(b,a)G(x,x) + 2H(b,a)K^{-1}(a,b)\right|$$
$$\leq \frac{\ln b - \ln x}{\ln b - \ln a}\left|\frac{x(\ln x - \ln b - x) + b}{b^2(\ln x - \ln b)^2} + (\lambda - 2)(\frac{1}{x^2} - \frac{1}{b^2})\frac{x(\ln x - \ln b - x) + b}{(\ln x - \ln b)^2}\right.$$
$$\left.+x(\frac{1}{x^2} - \frac{1}{b^2}) + \frac{\lambda}{b^2}\frac{(x-b)}{\ln x - \ln b}\right| + \frac{\ln x - \ln a}{\ln b - \ln a}\left|\frac{x(\ln x - \ln a - x) + a}{a^2(\ln x - \ln a)^2}\right.$$
$$\left.+(\lambda - 2)(\frac{1}{x^2} - \frac{1}{a^2})\frac{x(\ln x - \ln a - x) + a}{(\ln x - \ln a)^2} + x(\frac{1}{x^2} - \frac{1}{a^2}) + \frac{\lambda}{a^2}\frac{(x-a)}{\ln x - \ln a}\right|;$$

$$(ii)\qquad \Big|\lambda A(a^{-1},b^{-1})H(b,a)-(1+\lambda)H(b,a)G(x,x)+2H(b,a)K^{-1}(a,b)\Big|$$

$$\le \frac{\ln b-\ln x}{\ln b-\ln a}\Big|\frac{I^{\frac{1}{q}}(q-1)}{bx^2}(\frac{x^{2p}+b^{2p}}{2})^{\frac{1}{p}}+\frac{\lambda(x^q-b^q)^{\frac{1}{q}}}{(\ln x-\ln b)b^2x^2}(\frac{x^{2p}+b^{2p}}{2})^{\frac{1}{p}}\Big|$$

$$+\frac{\ln x-\ln a}{\ln b-\ln a}\Big|\frac{I^{\frac{1}{q}}(q-1)}{ax^2}(\frac{x^{2p}+a^{2p}}{2})^{\frac{1}{p}}+\frac{\lambda(x^q-a^q)^{\frac{1}{q}}}{(\ln x-\ln a)a^2x^2}(\frac{x^{2p}+a^{2p}}{2})^{\frac{1}{p}}\Big|;$$

$$(iii)\qquad \Big|\lambda A(a^{-1},b^{-1})H(b,a)-(1+\lambda)H(b,a)G(x,x)+2H(b,a)K^{-1}(a,b)\Big|$$

$$\le \frac{\ln b-\ln x}{b(\ln b-\ln a)\ln\frac{x|f'(x)|}{b|f'(b)|}}\Big|(1+\lambda-\frac{1}{\ln\frac{x|f'(x)|}{b|f'(b|}})\frac{x|f'(x)|}{b|f'(b)|}+\frac{1}{\ln\frac{x|f'(x)|}{b|f'(b)|}}-\lambda\Big|$$

$$+\frac{\ln x-\ln a}{a(\ln b-\ln a)\ln\frac{x|f'(x)|}{a|f'(a)|}}\Big|(1+\lambda-\frac{1}{\ln\frac{x|f'(x)|}{a|f'(a)|}})\frac{x|f'(x)|}{a|f'(a)|}+\frac{1}{\ln\frac{x|f'(x)|}{a|f'(a)|}}-\lambda\Big|;$$

$$(iv)\qquad \Big|\lambda A(a^{-1},b^{-1})H(b,a)-(1+\lambda)H(b,a)G(x,x)+2H(b,a)K^{-1}(a,b)\Big|$$

$$\le \frac{\ln b-\ln x}{\ln b-\ln a}\Big|bI^{\frac{1}{q}}(q-1)\Big[\frac{1}{b^{2p}\ln\frac{|f'(x)|}{|f'(b)|}}\Big(|\frac{f'(x)}{f'(b)}|^p-1\Big)\Big]^{\frac{1}{p}}$$

$$+\Big[\frac{\lambda}{\ln(\frac{x}{b})^q}\Big(x^q-b^q\Big)\Big]^{\frac{1}{q}}\Big[\frac{1}{b^{2p}\ln\frac{|f'(x)|}{|f'(b)|}}\Big(|\frac{f'(x)}{f'(b)}|^p-1\Big)\Big]^{\frac{1}{p}}\Big|$$

$$+\frac{\ln x-\ln a}{\ln b-\ln a}\Big|aI^{\frac{1}{q}}(q-1)\Big[\frac{1}{a^{2p}\ln\frac{|f'(x)|}{|f'(a)|}}\Big(|\frac{f'(x)}{f'(a)}|^p-1\Big)\Big]^{\frac{1}{p}}$$

$$+\Big[\frac{\lambda}{\ln(\frac{x}{a})^q}\Big(x^q-a^q\Big)\Big]^{\frac{1}{q}}\Big[\frac{1}{a^{2p}\ln\frac{|f'(x)|}{|f'(a)|}}\Big(|\frac{f'(x)}{f'(a)}|^p-1\Big)\Big]^{\frac{1}{p}}\Big|.$$

Proof. Applying Theorems 3.4, 3.5, 3.6, and 3.7, for $f(x)=x^{-1}$, $\lambda\in[0,1]$ and $\alpha=1$, one can obtain the results immediately. □

5. Acknowledgements

We acknowledge the valuable comments and suggestions of the anonymous referee and of the editor.

References

[1] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and applications of fractional differential equations, Elsevier Science B.V., 2006.
[2] K. Diethelm, The analysis of fractional differential equations, Lecture Notes in Mathematics, 2010.
[3] D. Baleanu, J. A. T. Machado, A. C. J. Luo, Fractional dynamics and control, Springer, 2012.
[4] V. E. Tarasov, Fractional dynamics: Application of fractional calculus to dynamics of particles, fields and media, Springer, HEP, 2011.
[5] R. Bai, F. Qi, B. Xi, Hermite-Hadamard type inequalities for the m- and (α,m)-logarithmically convex functions, Filomat, 27(2013), 1-7.
[6] W. H. Li, F. Qi, Some Hermite-Hadamard type inequalities for functions whose n-th derivatives are (α,m)-convex, Filomat, 27(2013), 1575-1582.
[7] E. Set, New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals, Comput. Math. Appl., 63(2012), 1147-1154.
[8] M. Z. Sarikaya, E. Set, H. Yaldiz, N. Başak, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, Math. Comput. Model., 57(2013), 2403-2407.

[9] C. Zhu, M. Fečkan, J. Wang, Fractional integral inequalities for differentiable convex mappings and applications to special means and a midpoint formula, J. Appl. Math. Statistics Inform., 8(2012), 21-28.
[10] J. Wang, X. Li, C. Zhu, Refinements of Hermite-Hadamard type inequalities involving fractional integrals, Bull. Belg. Math. Soc. Simon Stevin, 20(2013), 655-666.
[11] J. Wang, J. Deng, M. Fečkan, Hermite-Hadamard type inequalities for *r*-convex functions via Riemann-Liouville fractional integrals, Ukrainian Math. J., 65(2013), 193-211.
[12] J. Wang, X. Li, M. Fečkan, Y. Zhou, Hermite-Hadamard-type inequalities for Riemann-Liouville fractional integrals via two kinds of convexity, Appl. Anal., 92(2013), 2241-2253.
[13] I. Işcan, New general integral inequalities for quasi-geometrically convex functions via fractional integrals, J. Inequal. Appl., 2013(2013):491, 1-15.
[14] H. M. Srivastava and P. Agarwal, Certain fractional integral operators and the generalized incomplete hypergeometric functions, Appl. Appl. Math., 8(2013), 333-345.
[15] X. J. Yang, D. Baleanu, H. M. Srivastava, J. A. T. Machado, On local fractional continuous wavelet transform, Abstr. Appl. Anal. 2013 (2013), Art. ID 725416, 1-5.
[16] A. M. Yang, Z. S. Chen, H. M. Srivastava, X. J. Yang, Application of the local fractional series method and the variational iteration method to the Helmhotz equation involving local fractional derivative operators, Abstr. Appl. Anal., 2013(2013), Art. ID 259125, 1-6.
[17] X. J. Ma, H. M. Srivastava, D. Baleanu, X. J. Yang, A new Neumann series method for solving a family of local fractional Fredholm and Volterra integral equations, Math. Probl. Eng., 2013(2013), Art. ID 325121, 1-6.
[18] Y. J. Hao, H. M. Srivastava, H. Jafari, X. J. Yang, Helmholtz and diffusion equations associated with local fractional derivative operators involving the Cantorian and Cantor-type cylindrical coordinates, Adv. Math. Phys., 2013(2013), Art. ID 754248, 1-5.
[19] X. J. Yang, H. M. Srivastava, J. H. He, D. Baleanu, Cantor-type cylindrical-coordinate method for differential equations with local fractional derivatives, Phys. Lett. A, 377(2013), 1696-1700.
[20] I. Işcan, Generalization of different type integral inequalities for *s*-convex functions via fractional integrals, Appl. Anal., 93 (2014), 1846–1862.
[21] R. A. Santnoianu, Improved *GA*-convexity inequalities, J. Inequal. Pure Appl. Math. 3(2002), Art. 82.
[22] M. Z. Sarikaya, N. Aktan, Convexity according to the according to the geometric mean, Math. Inequal. Appl., (2000), 155-167.
[23] C. E. M. Pearce, J. E. Pečarić, Inequalities for differentiable mappings with application to special means and quadrature formula, Appl. Math. Lett., 13(2000), 51-55.